JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 48, 1–16 (1974)

On the Asymptotic Integration of Linear Differential Systems

W. A. HARRIS, JR.*

Department of Mathematics, University of Southern California, Los Angeles, California 90007

AND

D. A. LUTZ†

Department of Mathematics, University of Wisconsin—Milwaukee, Milwaukee, Wisconsin 53201

Submitted by N. Levinson

1. INTRODUCTION

The stability and asymptotic behavior of solutions of an autonomous linear differential system $x' = Ax$ are determined by the spectrum of the constant matrix A. If $B(t)$ is a suitably small perturbation, then the stability and asymptotic behavior of solutions of $x' = [A + B(t)]\, x$ are still determined by the limiting system $x' = Ax$. For example, if $P^{-1}AP = \Lambda$ is a diagonal matrix, $B(t)$ is continuous for $t \geqslant t_0$, and $\int_{t_0}^{\infty} | B(t)|\, dt < \infty$, then $x' = [A + B(t)]\, x$ has a fundamental matrix $X(t)$ satisfying, as $t \to \infty$,

$$X(t) = [P + o(1)] \exp(\Lambda t).$$

The classical theorems of Levinson [8] and Hartman–Wintner [7] are deeper results of this nature which describe the asymptotic behavior of solutions of the nonautonomous linear differential system $x' = A(t)\, x$ in terms of the eigenvalues of the matrix $A(t)$.

In this note we apply two preparatory lemmas to extend the validity of the fundamental results of Levinson and Hartman–Wintner to a wider class of systems $x' = A(t)\, x$. In addition, we discuss the interrelation between the basic results and give examples to delineate them.

Asymptotic integration, of interest in itself, is also a useful tool, for example, in the study of stability and boundary value problems. Our interest in the

* Supported in part by the United States Army under contract DA-ARO-D-31-124-71-G176.

† Supported in part by the National Science Foundation under Grant GP-28149.

problem was motivated by recent results of Devinatz [3, 4] and Fedoryuk [5] in which extensions of Levinson's basic theorem are presented and utilized to determine the deficiency index of certain differential operators.

2. Preparatory Lemmas

We are concerned with the linear differential system $y' = A(t)\,y$, in which the matrix $A(t)$ has the form $\Lambda + V(t)$, where Λ is a constant diagonal matrix with distinct eigenvalues and $V(t) \to 0$ as $t \to \infty$. For t sufficiently large, $\Lambda + V(t)$ has distinct eigenvalues and there exists a matrix $S(t)$ for which

$$S^{-1}(t)\,[\Lambda + V(t)]\,S(t) = \Lambda(t),$$

where $\Lambda(t)$ is a diagonal matrix whose elements are the eigenvalues of $A(t)$ and $\Lambda(t) \to \Lambda$ as $t \to \infty$. Utilizing $S(t)$ as a transformation, $y = S(t)\,u$, the linear differential system $y' = A(t)\,y$ becomes $u' = B(t)\,u$, where

$$B(t) = S^{-1}(t)\,[\Lambda + V(t)]\,S(t) - S^{-1}(t)\,S'(t) = \Lambda(t) + \hat{V}(t).$$

If $\hat{V}(t)$ is "sufficiently regular" (in the sense of integrability[1]), then we can determine the asymptotic integration of $u' = B(t)\,u$ and hence also that of the original system $y' = A(t)\,y$.

In order to determine the regularity of $\hat{V}(t)$, we must determine the regularity of $S(t)$, $S^{-1}(t)$, and $S'(t)$ in terms of the regularity of $V(t)$. Our basic preparatory lemma provides a means for studying this problem.

Lemma 1. *Let Λ be a constant diagonal matrix with distinct eigenvalues and $V(t)$ a continuous matrix for $t \geqslant t_0$ such that $V(t) \to 0$ as $t \to \infty$. Then there exists for $t \geqslant t_0' \geqslant t_0$ a matrix $Q(t)$, $Q(t) \to 0$ as $t \to \infty$, such that*

$$[I + Q(t)]^{-1}\,[\Lambda + V(t)]\,[I + Q(t)] = \Lambda(t),$$

where $\Lambda(t)$ is a diagonal matrix whose elements are the eigenvalues of $\Lambda + V(t)$. Furthermore, $Q(t)$ may be chosen so that diag $Q(t) \equiv 0$ *and so that $Q(t)$ has the same regularity properties as $V(t)$, i.e., $Q(t) = O(|V(t)|)$, $Q'(t) = O(|V'(t)|)$, etc., as $t \to \infty$.*

Remark 2.1. The existence of and an explicit representation for the matrix $Q(t)$ can be established through the use of projection matrices of the form

$$(2\pi i)^{-1} \int_\gamma [\lambda I - A(t)]^{-1}\,d\lambda,$$

[1] $|V(t)| \in L_p(t_0, \infty)$ means that $\int_{t_0}^{\infty} |V(t)|^p\,dt < \infty$, where any convenient matrix norm may be used.

where γ is a suitable contour in the complex λ-plane. In particular, we may determine the matrix $I + Q(t)$ in the form

$$I + Q(t) = \prod_{i=1}^{n} [I + Q_i(t)],$$

where the $Q_i(t)$ are defined inductively in the following manner.

$$Q_i(t) = [P_i(t) - P_i(\infty)]\,[2P_i(\infty) - I],$$
$$P_i(t) = (2\pi i)^{-1} \int_{\gamma_i} [\lambda I - A_i(t)]^{-1}\, d\lambda,$$

where

$$A_1(t) = A(t), \qquad A_{i+1}(t) = [I + Q_i(t)]^{-1} A_i(t)\,[I + Q_i(t)],$$

and γ_i is a circle not passing through any eigenvalue of $A(\infty)$ and containing exactly one such eigenvalue, say λ_i, in its interior.

Although this representation is not practicable from a computational standpoint, it is convenient for determining how the regularity properties of $Q(t)$ are inherited from those of $A(t)$. For example,

$$Q'(t) = \sum_{k=1}^{n} \prod_{i=1}^{k-1} [I + Q_i(t)]\, Q_k'(t) \prod_{i=k+1}^{n} [I + Q_i(t)],$$

$$Q_i'(t) = (2\pi i)^{-1} \int_{\gamma_i} [\lambda I - A_i(t)]^{-1} A_i'(t)\,[\lambda I - A_i(t)]^{-1}\, d\lambda [2P_i(\infty) - I],$$

and so $Q'(t) = O(|V'(t)|)$ as $t \to \infty$ (see, e.g. [2, pp. 111–113] for a discussion of this method).

The matrix $Q(t)$ thus constructed will not in general have $\operatorname{diag} Q(t) \equiv 0$. However, $I + Q(t)$ is uniquely determined up to post multiplication by a nonsingular diagonal matrix. Hence, writing

$$I + Q(t) = [I + \hat{Q}(t)]\,[I + \operatorname{diag} Q(t)],$$

or

$$\hat{Q}(t) = [Q(t) - \operatorname{diag} Q(t)]\,[I + \operatorname{diag} Q(t)]^{-1},$$

we can determine an appropriate matrix $\hat{Q}(t)$ satisfying all the conclusions of the lemma. The usefulness of this normalization in applications will be shown later. We note that aside from this normalization, this lemma was used by Levinson [8].

We provide now an elementary proof of this lemma which has other important ramifications.

Proof of Lemma 1. Write

$$\begin{aligned} \Lambda &= \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}, \\ \Lambda(t) &= \Lambda + \operatorname{diag}\{d_{11}(t), d_{22}(t), \ldots, d_{nn}(t)\} = \Lambda + D(t), \\ Q(t) &= (q_{ij}(t)), \qquad 1 \leqslant i, j \leqslant n, \qquad q_{ii}(t) \equiv 0. \end{aligned}$$

The existence of a matrix $Q(t) = o(1)$ as $t \to \infty$ for which

$$[I + Q(t)]^{-1} [\Lambda + V(t)] [I + Q(t)] = \Lambda(t) \tag{2.1}$$

is equivalent to the existence of a solution $Q(t) = o(1)$ of the equation

$$\Lambda Q(t) - Q(t) \Lambda + V(t) Q(t) - Q(t) D(t) + V(t) - D(t) = 0,$$

or in component form,

$$d_{ii} - v_{ii} - \sum_{i \neq k} v_{ik} q_{ki} = 0, \tag{2.2}$$

$$(\lambda_i - \lambda_k) q_{ik} + \sum_{\alpha \neq k} v_{i\alpha} q_{\alpha k} - q_{ik} d_{kk} + v_{ik} = 0, \tag{2.3}$$

where $d_{ii} = d_{ii}(t)$, $v_{ii} = v_{ik}(t)$, $q_{ik} = q_{ik}(t)$. Solving Eq. (2.2) for d_{ii} and substituting the result into Eq. (2.3), we obtain the nonlinear system of equations

$$(\lambda_i - \lambda_k) q_{ik} + \sum_{\alpha \neq k} v_{i\alpha} q_{\alpha k} - q_{ik} \left(v_{kk} + \sum_{\beta \neq k} v_{k\beta} q_{\beta k} \right) + v_{ik} = 0. \tag{2.4}$$

Conversely, a solution to the nonlinear system of equations (2.4) together with Eqs. (2.2) for the definition of the d_{ii} will yield a solution of the equation

$$[\Lambda + V(t)] [I + Q(t)] = [I + Q(t)] [\Lambda + D(t)].$$

Hence if $Q(t) = o(1)$ as $t \to \infty$, $I + Q(t)$ will be nonsingular for t sufficiently large and Eq. (2.1) will be satisfied. The nonlinear system of equations (2.4) is of the (vector) form $f(t, q) = 0$, where $f(\infty, 0) = 0$ and $f_q(\infty, 0)$ is nonsingular; hence the standard implicit function theorem guarantees the existence of a unique solution to this equation, $q = o(1)$, which inherits the regularity properties of $V(t)$.

The proof of Lemma 1 provides a method for computing an approximation to $\Lambda(t)$ and $Q(t)$ of order $o(|\, V(t)|^2)$ $(t \to \infty)$ by solving the linear systems

$$(\lambda_i - \lambda_k) \hat{q}_{ik} + \sum_{\alpha \neq k} v_{i\alpha} \hat{q}_{\alpha k} - v_{kk} \hat{q}_{ik} + v_{ik} = 0, \qquad \hat{d}_{ii} = v_{ii} + \sum_{i \neq k} v_{ik} \hat{q}_{ik} \,.$$

In particular, and approximation of order $O(|V(t)|^2)$ is

$$\tilde{q}_{ik} = (\lambda_k - \lambda_i)^{-1} v_{ik} + O(|V(t)|^2),$$
$$\tilde{d}_{ii} = v_{ii} + O(|V(t)|^2).$$

We formalize this remark as

LEMMA 2. *Let Λ be a constant diagonal matrix with distinct eigenvalues and let $V(t)$ be a continuous matrix for $t \geqslant t_0$ such that $V(t) \to 0$ as $t \to \infty$. Then there exists for $t \geqslant t_0' \geqslant t_0$ a matrix $T(t)$,* diag *$T(t) \equiv 0$, $T(t) \to 0$ as $t \to \infty$, such that*

$$[I + T(t)]^{-1} [\Lambda + V(t)] [I + T(t)] = \{\Lambda + \operatorname{diag} V(t)\} + \hat{V}(t),$$

where $\hat{V}(t) = O(|V(t)|^2)$, $T(t) = O(|V(t)|)$, and $T(t)$ has the same regularity properties as $V(t)$.

Proof of Lemma 2. Choose $t_{ik}(t) = (\lambda_k - \lambda_i)^{-1} v_{ik}(t)$, $i \neq k$, $t_{kk}(t) \equiv 0$, and perform the computation as in the proof of Lemma 1.

Remark 2.2. This lemma shows that if Λ is a diagonal matrix with distinct eigenvalues and $V(t)$ is a continuous matrix such that $V(t) \to 0$ as $t \to \infty$, then the eigenvalues of $\Lambda + V(t)$ are the eigenvalues of

$$\Lambda + \operatorname{diag} V(t) + O(|V(t)|^2)$$

as $t \to \infty$.

Remark 2.3. A direct proof of Lemma 2 may be effected by utilizing the identity

$$[I + T(t)]^{-1} = I - T(t) + T^2(t) [I + T(t)]^{-1}.$$

This transformation has been used to advantage in similar problems by Fedoryuk [5] and Devinatz [3].

Remark 2.4. If $|V(t)| \in L_2(t_0, \infty)$ and Λ is a diagonal matrix with distinct eigenvalues, then the eigenvalues of $\Lambda + V(t)$ are the diagonal entries of $\Lambda + V(t)$ to within integrable (L_1) terms. This remark allows the results of Hartman–Wintner to be stated in terms of the eigenvalues of $\Lambda + V(t)$ instead of in the customary manner involving the diagonal entries of $\Lambda + V(t)$.

Remark 2.5. In some cases it is convenient for applications to assume that $V(t)$ has the form $V(t) = V_1(t) + V_2(t)$ and make assumptions on $|V_1(t)|$, $|V_2(t)|$, $|V_1(t)|\,|V_2(t)|$, $|V_1(t)|\,|V_2'(t)|$, etc. The proof of the lemmas provides a method of determining how such properties are inherited by $Q(t)$.

In particular, it follows directly that the resultant Q's are "linear" with respect to $V(t)$, i.e.,

$$Q(t) = Q_1(t) + Q_2(t) + O(|V(t)|^2), \qquad Q_i(t) = O(|V_i(t)|),$$
$$Q'(t) = Q_1'(t) + Q_2'(t) + O(|V(t)|\,|V'(t)|), \qquad Q_i'(t) = O(|V_i'(t)|),$$
$$i = 1, 2, \text{ etc.}$$

Remark 2.6. The preceding remark shows that Lemmas 1 and 2 remain valid if the constant diagonal matrix Λ is replaced by the diagonal matrix $\Lambda(t) = \Lambda + \hat{\Lambda}(t)$, where $\hat{\Lambda}(t) \to 0$ as $t \to \infty$ and shares the same regularity properties as $V(t)$. This allows us to make repeated application of the lemmas.

3. Main Results and Applications

In this section we use the preparatory Lemmas 1 and 2 to transform a given linear differential system so that the basic results of Levinson and Hartman–Wintner are applicable. For the convenience of the reader, we now state these basic results.

THEOREM A (Levinson [8; 1, pp. 92–95]). *Let*

$$\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$$

be continuous for $t \geqslant t_0$ and assume for each index pair $j \neq k$ that either

$$\text{(i)} \quad \int_{t_0}^{t} \operatorname{Re}(\lambda_k(s) - \lambda_j(s))\, ds \to \infty \quad \textit{as} \quad t \to \infty$$

and

$$\int_{s}^{t} \operatorname{Re}(\lambda_k(s) - \lambda_j(s))\, ds > -K \quad \textit{for all } t_0 \leqslant s \leqslant t,$$

or

$$\text{(ii)} \quad \int_{s}^{t} \operatorname{Re}(\lambda_k(s) - \lambda_j(s))\, ds < K \quad \textit{for all } t_0 \leqslant s \leqslant t.$$

Furthermore, assume that $R(t)$ is continuous for $t \geqslant t_0$ and $|R(t)| \in L_1(t_0, \infty)$. Then the linear differential system $x' = (\Lambda(t) + R(t))\,x$ has a fundamental matrix satisfying as $t \to \infty$

$$X(t) = [I + o(1)] \exp\left(\int_{t_0}^{t} \Lambda(s)\, ds\right).$$

THEOREM B (Hartman–Wintner [7, pp. 71–72]). *Let*

$$\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$$

be continuous for $t \geqslant t_0$ *and assume that for each index pair* $j \neq k$,

$$|\operatorname{Re}(\lambda_j(t) - \lambda_k(t))| \geqslant \mu > 0.$$

Furthermore, assume that $V(t)$ *is continuous and* $|V(t)| \in L_2(t_0, \infty)$. *Then the linear differential system* $x' = (\Lambda(t) + V(t))\,x$ *has a fundamental matrix satisfying as* $t \to \infty$

$$X(t) = [I + o(1)] \exp\left(\int_{t_0}^{t} [\Lambda(s) + \operatorname{diag} V(s)]\, ds\right).$$

Actually, Levinson considered the linear differential system

$$y' = (A + V(t) + R(t))\,y \tag{3.1}$$

and made a preliminary transformation $y = S(t)\,x$ (essentially Lemma 1) to obtain

$$x' = (\Lambda(t) - S^{-1}(t)\,S'(t) + S^{-1}(t)\,R(t)\,S(t))\,x.$$

Under the assumption $|V'(t)| \in L_1(t_0, \infty)$, Levinson applied Theorem A to obtain the following.

THEOREM C (Levinson [1, pp. 92–93]). *Let* A *be a constant matrix with distinct eigenvalues,* $V'(t)$ *be continuous for* $t \geqslant t_0$, $V(t) \to 0$ *as* $t \to \infty$, $|V'(t)| \in L_1(t_0, \infty)$, $R(t)$ *be continuous for* $t \geqslant t_0$, $|R(t)| \in L_1(t_0, \infty)$, *and assume that the eigenvalues of the matrix* $A + V(t)$ *satisfy condition* (i) *or* (ii) *of Theorem* A. *Then the linear differential system* $x' = (A + V(t) + R(t))\,x$ *has a fundamental matrix satisfying for* $t \to \infty$

$$X(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \Lambda(s)\, ds\right),$$

$t_1 \geqslant t_0$, *where* $P^{-1}AP$ *is a diagonal matrix and* $\Lambda(t)$ *is a diagonal matrix with components the eigenvalues of* $A + V(t)$.

If we assume that A is a diagonal matrix and $V(t) \to 0$ as $t \to \infty$, then we may utilize the transformation $y = [I + Q(t)]\,u$ of Lemma 1 to transform the linear differential system (3.1) into

$$u' = (\Lambda(t) + \hat{V}(t) + \hat{R}(t))\,u, \tag{3.2}$$

where

$$\hat{V}(t) = -[I + Q(t)]^{-1}\,Q'(t) \qquad \text{and} \qquad \hat{R}(t) = [I + Q(t)]^{-1}\,R(t)\,[I + Q(t)].$$

If $\hat{V}(t) \notin L_1$, but $\hat{V}(t) \to 0$ as $t \to \infty$, then we may utilize the transformation $u = [I + T(t)]\, v$ of Lemma 2 to transform the linear differential system (3.2) into

$$v' = (\Lambda(t) + \tilde{V}(t) + \tilde{R}(t))\, v, \tag{3.3}$$

where

$$\tilde{V}(t) = \operatorname{diag} \hat{V}(t) - [I + T(t)]^{-1}\, T'(t) + O(|\hat{V}(t)|^2)$$

and

$$\tilde{R}(t) = [I + T(t)]^{-1}\, \hat{R}(t)\, [I + T(t)].$$

Since

$$Q(t) = O(|V(t)|), \qquad Q'(t) = O(|V'(t)|), \qquad Q''(t) = O(|V''(t)|),$$
$$T(t) = O(|\hat{V}(t)|) = O(|V'(t)|),$$
$$T'(t) = O(|\hat{V}'(t)|) = O(|V''(t)| + |V'(t)|^2),$$

then

$$\tilde{V}(t) = O(|V''(t)| + |V'(t)|^2 + |V(t)|\,|V'(t)|), \qquad \text{and} \qquad \tilde{R}(t) = O(|R(t)|).$$

Hence, if $|V''(t)| \in L_1$, $|V'(t)| \in L_2$ (which together imply that $\hat{V}(t) \to 0$ as $t \to \infty$), then $|\tilde{V}(t)| \in L_1$, $|\tilde{R}(t)| \in L_1$, and we may apply Theorem A to obtain the following result.

THEOREM D (Devinatz [4, p. 354]). *Let A be a constant matrix with distinct eigenvalues, $V''(t)$ and $R(t)$ be continuous for $t \geqslant t_0$, $V(t) \to 0$ as $t \to \infty$, $|V'(t)|^2$, $|V(t)|\,|V'(t)|$, $|V''(t)|$, $|R(t)| \in L_1(t_0, \infty)$, and assume that the eigenvalues of $A + V(t)$ satisfy condition* (i) *or* (ii) *of Theorem* A. *Then the linear differential system $x' = (A + V(t) + R(t))\, x$ has a fundamental matrix satisfying for $t \to \infty$*

$$X(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \Lambda(s)\, ds\right)$$

where $t_1 \geqslant t_0$, $P^{-1}AP$ is a diagonal matrix, and $\Lambda(t)$ is a diagonal matrix with components the eigenvalues of $A + V(t)$.

Remark 3.1. In Devinatz's statement of this theorem, it is assumed that $V(t) = V_1(t) + V_2(t)$, where $V_1(t) \to 0$, $V_2(t) \to 0$ as $t \to \infty$ and $|V_1'(t)|$, $|V_2'(t)|^2$, $|V_2'(t)|\,|V(t)|$, $|V_2''(t)| \in L_1(t_0, \infty)$. To obtain this result we use Remark 2.5 to modify the preceding proof by now defining in system (3.2),

$$\hat{V}(t) = Q_2'(t),$$
$$\hat{V}(t) + \hat{R}(t) = -[I + Q(t)]^{-1} Q'(t) + [I + Q(t)]^{-1} R(t)\, [I + Q(t)],$$

and noting that $\hat{V}(t) = O(|V_2'(t)|)$ and $|\hat{R}(t)| \in L_1$.

Remark 3.2. An alternative proof of Theorem D is obtained by applying Lemma 1 twice. Since in system (3.2),

$$\hat{V}(t) = -[I + Q(t)]^{-1} Q'(t), \qquad Q'(t) = O(|\, V'(t)|),$$

and $\hat{V}(t) \to 0$ as $t \to \infty$, there exists a transformation $u = [I + \hat{Q}(t)]\, w$ by Lemma 1 which yields the linear differential system

$$w' = (\hat{A}(t) + R_1(t))\, w,$$

where $\hat{A}(t)$ is a diagonal matrix whose components are the eigenvalues of $A(t) - [I + Q(t)]^{-1} Q'(t)$. But $|\, V''(t)| \in L_1$ implies that $|\, R_1(t)| \in L_1$; by Remark 2.2,

$$\hat{A}(t) = A(t) - \operatorname{diag}\{[I + Q(t)]^{-1} Q'(t)\} + O(|\, V'(t)|^2);$$

$|\, V''(t)|$ and $|\, V(t)|\,|\, V'(t)| \in L_1$ imply that $|\operatorname{diag}\{[I + Q(t)]^{-1} Q'(t)\}| \in L_1$. Hence, if $|\, V''(t)|$, $|\, V'(t)|^2$ and $|\, V(t)|\,|\, V'(t)| \in L_1$, then

$$\hat{A}(t) + R_1(t) = A(t) + R_2(t),$$

where $|\, R_2(t)| \in L_1$ and we may apply Theorem A to obtain Theorem D.

Devinatz's Theorem is a special case of general results that can be obtained by repeated application of Lemma 1 and/or Lemma 2 as indicated above. For simplicity, we state these theorems in terms of a single matrix $V(t)$, keeping in mind the modifications indicated by Remarks 2.4 and 3.1.

THEOREM 1. *Let A be a constant matrix with distinct eigenvalues; for some positive integer k let $V^{(k)}(t)$ and $R(t)$ be continuous for $t \geqslant t_0$; $V^{(i)}(t) \to 0$ as $t \to \infty$, $0 \leqslant i \leqslant k - 2$; $|\, V^{(j)}(t)| \in L_2(t_0, \infty)$, $1 \leqslant j \leqslant k - 1$; $|\, V(t)|\,|\, V'(t)|$, $|\, V^{(k)}(t)|$, and $|\, R(t)| \in L_1(t_0, \infty)$; and assume that the eigenvalues of the matrix $A + V(t)$ satisfy condition* (i) *or* (ii) *of Theorem* A. *Then the linear differential system $x' = (A + V(t) + R(t))\, x$ has a fundamental matrix satisfying as $t \to \infty$*

$$X(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \Lambda(s)\, ds\right),$$

$t_1 \geqslant t_0$, where $P^{-1}AP$ is a diagonal matrix and $\Lambda(t)$ is a diagonal matrix whose components are the eigenvalues of $A + V(t)$.

This result may be viewed not only as an extension of Levinson's Theorem C, but also (under the additional assumption that $|\, V(t)| \in L_2(t_0, \infty)$) as an extension of Hartman–Wintner's Theorem B in the sense that the restriction on the eigenvalues of $A + V(t)$ has been weakened at the expense of strengthening the assumptions on $V(t)$. We note that $|V^{(i)}(t)|$, $|V^{(i+1)}(t)| \in L_2$ implies $|V^{(i)}(t)|\,|V^{(i+1)}(t)| \in L_1$, $1 \leqslant i \leqslant k-2$, and Remark 2.2 implies that applications of Lemmas 1 and 2 modify the eigenvalues of $A + V(t)$ to within integrable terms. In particular, the eigenvalues of $A + V(t)$ may be computed to within integrable terms by

$$\Lambda(t) = \operatorname{diag}\{P^{-1}[A + V(t)]\,P\},$$

where $P^{-1}AP$ is a diagonal matrix.

The preparatory lemmas can also be used to obtain extensions of Hartman–Wintner's Theorem B by relaxation of the assumption that $|V(t)| \in L_2$. For example, if $V(t) \to 0$ as $t \to \infty$, $|V'(t)| \in L_2$, and the eigenvalues of A have distinct real parts, then by utilizing the transformation $y = P[I + Q(t)]\,u$ of Lemma 1, $P^{-1}AP$ diagonal, the system $y' = (Q + V(t))\,y$ becomes $u' = (\hat{\Lambda}(t) + \hat{V}(t))\,u$, where

$$\hat{\Lambda}(t) = \Lambda(t) - \operatorname{diag}\{[I + Q(t)]^{-1}\,Q'(t)\}$$

and

$$\hat{\Lambda}(t) + \hat{V}(t) = \Lambda(t) - [I + Q(t)]^{-1}\,Q'(t).$$

Clearly, Theorem B applies to yield a fundamental matrix satisfying as $t \to \infty$

$$\hat{Y}(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \hat{\Lambda}(s)\,ds\right), \qquad t_1 \geqslant t_0 .$$

If, in addition, $|V(t)|\,|V'(t)| \in L_1$, then $|\operatorname{diag}\{[I + Q(t)]^{-1}\,Q'(t)\}| \in L_1$ and we have a fundamental matrix satisfying, as $t \to \infty$,

$$Y(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \Lambda(s)\,ds\right).$$

We formalize this result as follows.

THEOREM 2. *Let A be a constant matrix with eigenvalues having distinct real parts, $V(t)$ be a continuous matrix for $t \geqslant t_0$ such that $V(t) \to 0$ as $t \to \infty$, $V'(t)$ be continuous for $t \geqslant t_0$ and $|V'(t)|^2$, $|V(t)|\,|V'(t)| \in L_1(t_0, \infty)$. Then the linear differential system $x' = (A + V(t))\,x$ has a fundamental matrix satisfying, as $t \to \infty$,*

$$X(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \Lambda(s)\,ds\right), \qquad t_1 \geqslant t_0 ,$$

where $P^{-1}AP$ is a diagonal matrix and $\Lambda(t)$ is a diagonal matrix whose components are the eigenvalues of $A + V(t)$.

We emphasize that $|V(t)|^2$ is not necessarily integrable and hence $\Lambda(t)$ is not necessarily $\operatorname{diag}\{P^{-1}[A + V(t)]P\}$.

The assumption that $|V(t)|\,|V'(t)| \in L_1$ has allowed us to state the asymptotic integration in Theorems 1 and 2 in terms of the eigenvalues of $A + V(t)$. By modifying the eigenvalues of $A + V(t)$ by functions which are also computable, we can obtain the following result.

THEOREM 3. *Let A be a constant matrix with distinct eigenvalues; for some positive integer k, let $V^{(k)}(t)$ and $R(t)$ be continuous for $t \geqslant t_0$; let $V^{(i)}(t) \to 0$ as $t \to \infty$, $0 \leqslant i \leqslant k-2$, and $|V^{(i)}(t)| \in L_2$, $1 \leqslant i \leqslant k-1$; and let $|V^{(k)}(t)|$, $|R(t)| \in L_1(t_0, \infty)$. Let $P^{-1}AP$ be a diagonal matrix and let $Q(t)$ be the unique matrix (of Lemma 1) for which*

$$[I + Q(t)]^{-1} P^{-1}[A + V(t)]\, P[I + Q(t)] = \Lambda(t),$$

$\operatorname{diag} Q(t) \equiv 0$, *and let the eigenvalues of $\hat{\Lambda}(t) = \Lambda(t) - \operatorname{diag}\{[I + Q(t)]^{-1} Q'(t)\}$ satisfy condition* (i) *or* (ii) *of Theorem* A. *Then the linear differential system $x' = (A + V(t) + R(t))\,x$ has a fundamental matrix satisfying as $t \to \infty$*

$$X(t) = [P + o(1)] \exp\left(\int_{t_1}^{t} \hat{\Lambda}(s)\, ds\right), \qquad t_1 \geqslant t_0 .$$

We note that, for example, if $|V(t)|^2\, |V'(t)| \in L_1(t_0, \infty)$, then

$$\operatorname{diag}\{[I + Q(t)]^{-1} Q'(t)\} = -\operatorname{diag} Q(t)\, Q'(t)$$

to within integrable terms.

As a final application of the preparatory lemmas, we deduce some sufficient conditions for the uniform stability of a linear differential system $x' = (A + V(t))\,x$, $V(t) \to 0$ as $t \to \infty$, which are extensions of results due to Conti and Cesari (see, e.g. [2, p. 114]).

THEOREM 4. *Assume that the eigenvalues of $A + V(t)$ have nonpositive real parts for t sufficiently large; that the eigenvalues of A with zero real part are simple; and that for some positive integer k, $V^{(k)}(t)$ is continuous for $t \geqslant t_0$, $V^{(i)}(t) \to 0$ as $t \to \infty$, $0 \leqslant i \leqslant k-2$, $|V^{(i)}(t)| \in L_2(t_0, \infty)$, $1 \leqslant i \leqslant k-1$, $|V(t)|\,|V'(t)| \in L_1(t_0, \infty)$, and $|V^{(k)}(t)| \in L_1(t_0, \infty)$. Then the zero solution of $x' = (A + V(t))\,x$ is uniformly stable for $t \geqslant t_0$.*

Proof. We follow the method utilized by Coppel [2, pp. 113–114] for the case $k = 1$. Without loss of generality, we assume that A has block diagonal form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where Λ_1 is diagonal and contains the eigenvalues of A with zero real part. Utilizing transformations corresponding to the block analog of Lemma 1, followed by $(k - 2)$ applications of the block analog of Lemma 2, we obtain (for t sufficiently large) the linear differential system

$$u' = (\hat{A}(t) + R(t))\, u,$$

where

$$| R(t)| \in L_1 \qquad \text{and} \qquad \hat{A}(t) = \begin{pmatrix} \Lambda_1(t) & 0 \\ 0 & \hat{A}_2(t) \end{pmatrix},$$

where $\Lambda_1(t)$ is a diagonal matrix whose components are the eigenvalues of $A + V(t)$ which tend to Λ_1 as $t \to \infty$, and $\hat{A}_2(t) \to A_2$ as $t \to \infty$.

Since $| R(t)| \in L_1$, $u' = (\hat{A}(t) + R(t))\, u$ and $u' = \hat{A}(t)\, u$ are uniformly stable simultaneously [2, p. 65], and $u' = \hat{A}(t)\, u$ is uniformly stable [2, p. 114].

4. Examples

In this section we give some examples illustrating the type of results that can be obtained using Theorems 1–3, some examples which indicate the sharpness of the assumptions which are necessary for their applicability, as well as some examples which demonstrate the independence of the various hypotheses.

Example 1. Let

$$A(t) = \Lambda + V(t) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} + t^{-\alpha} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $0 < \alpha \leqslant 1$. The eigenvalues of $A(t)$ are $\lambda(t) = 1 \pm (1 + t^{-2\alpha})^{1/2}$, i.e.,

$$\lambda_1(t) = -\tfrac{1}{2}t^{-2\alpha} + \tfrac{1}{4}t^{-4\alpha} + \cdots, \qquad \lambda_2(t) = 2 + \tfrac{1}{2}t^{-2\alpha} - \tfrac{1}{4}t^{-4\alpha} + \cdots,$$

and

$$\operatorname{Re}(\lambda_2(t) - \lambda_1(t)) = 2 + O(t^{-2\alpha}).$$

Also

$$V'(t) = -\alpha t^{-\alpha-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad | V'(t)| \in L_1(1, \infty);$$

hence, Levinson's Theorem C applies. However, after naively applying Hartman–Wintner's basic result, the system $y' = A(t)\, y$ would have a fundamental matrix of the form

$$Y(t) = [I + o(1)] \begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix} \qquad \text{as} \qquad t \to \infty,$$

which is only valid if $\alpha > \frac{1}{2}$ (i.e., $|V(t)| \in L_2$). Whereas, for example, the correct asymptotic integration obtained from Theorem C for $\alpha = \frac{1}{2}$ is

$$Y(t) = [I + o(1)] \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2}e^{2t} \end{pmatrix} \quad \text{as} \quad t \to \infty.$$

EXAMPLE 2. To illustrate the independence of the various hypotheses on $V(t)$ and its derivatives, consider the (scalar) function $v(t) = t^{-\beta} \sin t^{1-\alpha}$.

(i) If $\frac{1}{2} < \beta < 1$ and $(1-\beta)/(k+1) < \alpha \leqslant (1-\beta)/k$, then $|v^{(i)}(t)| \in L_2(1, \infty)$, $|v^{(i)}(t)| \notin L_1(1, \infty)$, $0 \leqslant i \leqslant k-1$, $|v^{(k)}(t)| \in L_1(1, \infty)$, and $|v(t)|\,|v'(t)| \in L_1(1, \infty)$. Hence for the system $y' = A(t)\,y$, where

$$A(t) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + v(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we may apply Lemma 2, k times to yield a system to which Levinson's basic result applies. This illustrates how Theorem 1 provides an extension of Levinson's and Devinatz's Theorems. In addition, as we remarked in Section 3, Theorem 1 may be viewed as an extension of Hartman–Wintner's basic result, which does not apply in this case since

$$\operatorname{Re}(\lambda_1(t) - \lambda_2(t)) \equiv 0.$$

(ii) If $\beta = \frac{1}{2}$ and $\alpha > 0$, then $|v(t)| \notin L_2(1, \infty)$; however

$$|v'(t)| \in L_2(1, \infty) \qquad \text{and} \qquad |v(t)|\,|v'(t)| \in L_1(1, \infty).$$

Hence for the system $y' = A(t)\,y$, where

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + v(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we may apply Lemma 1 once to yield a system to which Hartman–Wintner's basic result applies. This case is covered by Theorem 2.

EXAMPLE 3. The important and well-studied differential equation $w'' - f(x)\,w = 0$, where f is a positive, twice continuously differentiable function, $\int^x f^{1/2} \to \infty$, can be written in the form $y' = A(t)\,y$, where

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{4} f^{-3/2} f' \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

by setting

$$y_1 = y' + f^{1/2} y, \qquad y_2 = y' - f^{1/2} y, \qquad \text{and} \qquad t = \int^x f^{1/2}.$$

For such second-order equations, the transformations involved in the preparatory lemmas can be explicitly constructed and special results can be obtained due to this explicit determination of the resultant system. We do not pursue this matter further, since there are also special techniques available for these second-order equations (see e.g. [6, pp. 369–384; 10; 2, pp. 118–128]).

EXAMPLE 4. Let

$$A(t) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} 0 & -t^{1/4}\sin t^{1/2} \\ t^{-1/4}\cos t^{1/2} & 0 \end{pmatrix}.$$

The eigenvalues of $A(t)$ to within integrable terms are

$$\lambda_{1,2}(t) = \pm\, i(1 + t^{-1/2}\sin 2t^{1/2} - \tfrac{1}{4}t^{-1}\sin^2 2t^{1/2}).$$

Note that $|V(t)| \notin L_2(1, \infty)$ and $|V(t)|\,|V'(t)| \notin L_1(1, \infty)$. However, since $|V''(t)| \in L_1(1, \infty)$ and $|V'(t)| \in L_2(1, \infty)$, an application of Lemmas 1 and 2 yields the linear differential system

$$v' = (A(t) + \mathrm{diag}\{Q(t)\,Q'(t)\} + R(t))\,v,$$

where $|R(t)| \in L_1$. A straight forward computation yields, for this example (to within integrable terms)

$$Q(t)\,Q'(t) = \begin{pmatrix} (8t)^{-1}\sin^2 t^{1/2} & 0 \\ 0 & -(8t)^{-1}\cos^2 t^{1/2} \end{pmatrix}.$$

Thus defining $\hat{A}(t) = \mathrm{diag}\{\hat{\lambda}_1(t), \hat{\lambda}_2(t)\}$, where

$$\hat{\lambda}_1(t) = i(1 + t^{-1/2}\sin 2t^{1/2} - \tfrac{1}{4}t^{-1}\sin^2 2t^{1/2}) + (8t)^{-1}\sin^2 t^{1/2}$$

and

$$\hat{\lambda}_2(t) = -i(1 + t^{-1/2}\sin 2t^{1/2} - \tfrac{1}{4}t^{-1}\sin^2 2t^{1/2}) - (8t)^{-1}\cos^2 t^{1/2},$$

the resultant system has the form $v' = (\hat{A}(t) + \hat{R}(t))\,v$. Since

$$\mathrm{Re}(\hat{\lambda}_1(t) - \hat{\lambda}_2(t)) = (8t)^{-1} \qquad \text{and} \qquad |\hat{R}(t)| \in L_1,$$

Theorem A is applicable. We note that the modification is real although the original eigenvalues of $A(t)$ are purely imaginary and thus this modification is necessary to obtain the true character of the solutions. This example is covered by Theorem 3.

EXAMPLE 5. Consider the linear differential system $y' = A(t)\,y$, where $A(t)$ has an asymptotic expansion of the form

$$A(t) \sim A_0 + A_1 t^{-1} + A_2 t^{-2} + \cdots, \qquad t \to \infty,$$

and A_0 has distinct eigenvalues. Clearly $A(t) = A_0 + V(t) + R(t)$, $V(t) = A_1 t^{-1}$ and Levinson's Theorem applies, namely there exists a fundamental matrix of the form

$$Y(t) = [P + o(1)] \exp\left(\int_{t_0}^{t} \Lambda(s)\, ds\right),$$

where the elements of the diagonal matrix $\Lambda(t)$ are the eigenvalues of $A_0 + V(t)$ and $P^{-1}A_0P$ is a diagonal matrix. In fact, since $|V(t)| \in L_2$,

$$\Lambda(t) = \Lambda + \operatorname{diag}\{P^{-1}V(t)\, P\}.$$

Without difficulty, we may obtain a similar result for the case

$$A(t) \sim t^r(A_0 + A_1 t^{-1} + \cdots), \qquad t \to \infty,$$

where A_0 has distinct eigenvalues and r is a positive integer. Making the change of variable $s = t^{r+1}$, applying Levinson's Theorem, and changing the variable back to t, we obtain a fundamental matrix satisfying as $t \to \infty$,

$$Y(t) = [P + o(1)] \exp\left(\int_{t_0}^{t} \Lambda(s)\, ds\right),$$

where the elements of the diagonal matrix $\Lambda(t)$ are the eigenvalues of $t^r(A_0 + A_1 t^{-1} + \cdots + t^{-r-1}A_{r+1})$ and $P^{-1}A_0P$ is a diagonal matrix. This is the beginning of the standard asymptotic expansion up to the indicated terms [1, pp. 142–143, 160–161].

EXAMPLE 6. We close our discussion with the differential equation of an adiabatic oscillator, $w'' + (1 + at^{-1} \sin \lambda t)\, w = 0$, where a, λ are real parameters. This example has many interesting features.

(i) Levinson's basic result does not apply since $|V'(t)| \notin L_1$.

(ii) Hartman–Wintner's basic result does not apply since

$$\operatorname{Re}(\lambda_1 - \lambda_2) = 0.$$

(iii) When naively applied, Levinson and Hartman–Wintner yield the same result since $|V(t)| \in L_2$.

(iv) Lemma 1 and Lemma 2 may be applied as often as desired and yield systems to which neither Levinson's nor Hartman–Wintner's basic result applies, however the asymptotic integration indicated by these theorems remains invariant.

(v) The asymptotic integration obtained by naively applying Levinson's or Hartman–Wintner's Theorems can be either correct or incorrect, depending upon the value of the parameter λ.

We write this example in the vector form $y' = A(t)\,y$, i.e., $y_1 = w + iw'$, $y_2 = w - iw'$, where

$$A(t) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{ia \sin \lambda t}{2t} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

For $\lambda \neq \pm 2$, Atkinson [2, pp. 113–124] and Wintner [11] have proven general (second-order) results which imply the existence of a fundamental matrix satisfying as $t \to \infty$

$$Y(t) = [I + o(1)] \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix};$$

but for $\lambda = \pm 2$, the results of Atkinson [2, pp. 125–128] and Wintner [9, 10] can be used to show the existence of a fundamental matrix satisfying, as $t \to \infty$,

$$Y(t) = \left[\begin{pmatrix} 1 & -ie^{2it} \\ -ie^{-2it} & 1 \end{pmatrix} + o(1)\right] \begin{pmatrix} t^{a/4}\, e^{it} & 0 \\ 0 & t^{-a/4}\, e^{-it} \end{pmatrix}.$$

References

1. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw–Hill, New York, 1955.
2. W. Coppel, "Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, 1965.
3. A. Devinatz, An asymptotic theorem for systems of linear differential equations, *Trans. Amer. Math. Soc.* **160** (1971), 353–363.
4. A. Devinatz, The deficiency index of a certain class of ordinary self-adjoint differential operators, *Adv. in Math.* **8** (1972), 434–473.
5. M. Fedoryuk, Asymptotic methods in the theory of one-dimensional singular differential operators, *Trans. Moskow Math. Soc.* (1966), 333–386.
6. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
7. P. Hartman and A. Wintner, Asymptotic integrations of linear differential equations, *Amer. J. Math.* **77** (1955), 45–86 and 932.
8. N. Levinson, The asymptotic nature of solutions of linear differential equations, *Duke Math. J.* **15** (1948), 111–126.
9. A. Wintner, The adiabatic linear oscillator, *Amer. J. Math.* **68** (1946), 385–397.
10. A. Wintner, Asymptotic integrations of the adiabatic oscillator, *Amer. J. Math.* **69** (1947), 251–272.
11. A. Wintner, Addenda to the paper on Böcher's Theorem, *Amer. J. Math.* **78** (1956), 895–897.