Exponential stability criteria of uncertain systems with multiple time delays

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Abstract

The exponential stability (with convergence rate \( \alpha \)) of uncertain linear systems with multiple time delays is studied in this paper. Using the characteristic function of linear time-delay system, stability criteria are derived to guarantee \( \alpha \)-stability. Sufficient conditions are also obtained for exponential stability of uncertain parametric systems with multiple time delays. For two-dimensional time-invariant system with multiple time delays, the proposed stability criteria are shown to be less conservative than those in the literature. Numerical examples are given to illustrate the validity of our new stability criteria.

Keywords: Multiple time delays; Exponential stability; Robustness; Uncertain parameters

1. Introduction

The system stability and convergence properties are strongly affected by time delays, which are often encountered in various engineering systems due to measurement and computational delays, transmission and transport lags. Since the existence of time delays
are frequently the sources of instability, the stability analysis of time-delay systems with or without uncertainties has been an active area of research for the past decades.

There are many different methods presented to deal with the stability problem of the time-delay systems in the literature [1,2]. Some stability criteria are directly derived from the characteristic equation, involving the determination of eigenvalues, measures and norms of matrices, or matrix conditions in terms of Hurwitz matrices [3–9]. For time-varying delay systems, stability and robust stability criteria are given in terms of the Lyapunov–Razumikhin theorem and the solution of either a Lyapunov or Riccati equation [10–18]. Based on the linear matrix inequality (LMI) approach [19], robust stability and stabilisation conditions have been developed without tuning of a scaling parameter and/or a positive definite matrix [20–26]. Sun and Hsieh [27] proposed exponential stability criteria for nonlinear systems with multiple time delays. Both delay-independent and delay-dependent criteria have also been addressed for robust exponential stability of nonlinear systems with time-varying delays in [28].

This paper deals with exponential stability (with convergence rate $\alpha$) of linear systems with multiple time delays. Based on the characteristic functions, stability criteria are derived to guarantee $\alpha$-stability of linear systems with multiple time delays. Scalar inequalities involving eigenvalues, spectral radius, and matrix measures constitute the mathematical foundations of our approach. The results obtained are extended to treat the exponential stability of uncertain parametric systems with multiple time delays. For two-dimensional time-invariant linear system, the proposed stability criteria are shown to be less conservative comparing to the criteria derived in [27]. Numerical examples are given to demonstrate the validity of our new criteria and to compare them with the existing ones.

2. System description and mathematical lemmas

Throughout this article the following conventions are used:

- $\mathbb{R}$ ($\mathbb{C}$) the set of all real (complex) numbers;
- $\mathbb{R}^n$ the $n$-dimensional real space;
- $\mathbb{R}^{n \times n}$ ($\mathbb{C}^{n \times n}$) the set of all real (complex) $n$ by $n$ matrices;
- $I$ the unit matrix;
- $\lambda_j(A)$ the $j$th eigenvalue of the matrix $A$;
- $\lambda_{\text{max}}(A)$ the maximum eigenvalue of the Hermitian matrix $A$;
- $|s|$ the modulus of the complex number $s$;
- $A^T$ the transpose of the matrix $A$;
- $A^*$ the conjugate transpose of the matrix $A$;
- $\det(A)$ the determinant of the matrix $A$;
- $\text{Re}(s)$ the real part of the complex number $s$;
- $\rho(A)$ the spectral radius of the matrix $A$;
- $\|A\|$ the spectral norm of the matrix $A$; $\|A\| = \sqrt{\lambda_{\text{max}}(A^*A)}$;
- $\mu(A)$ the matrix measure of the matrix $A$; $\mu(A) = \frac{1}{2} \lambda_{\text{max}}(A + A^*)$. 
For any given integers \(i\) and \(j\) \((j > i)\), define
\[
A_j^i = \{i, i+1, \ldots, j-1, j\}.
\]
(1)

Consider the following linear time-invariant system with multiple time delays
\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + \sum_{j=1}^{m} A_j x(t-\tau_j), \quad t \geq 0, \\
x(t) &= \theta(t), \quad t \in [-\tau, 0],
\end{align*}
\]
(2)
where \(x(t) \in \mathbb{C}^{n \times 1}\) is the state vector, \(A_0 \in \mathbb{C}^{n \times n}\) is assumed to be a Hurwitz matrix, \(\theta(t) \in \mathbb{C}^{n \times 1}\) is a given continuous function, the constant parameters \(\tau_j \geq 0\) \((\forall j \in \Lambda^m)\) with \(\tau = \max\{\tau_j, j \in \Lambda^m\}\) represent the delay arguments, and \(A_j \in \mathbb{C}^{n \times n}\) \((\forall j \in \Lambda^m)\).

**Definition 1.** System (2) is said to be \(\alpha\)-stable, with \(\alpha > 0\), if there exists a function \(q(\cdot)\) such that, for each \(\theta(t) \in \mathbb{C}^{n \times 1}\), the solution \(\Phi(t, \theta(t))\) of system (2) satisfies
\[
\|\Phi(t, \theta(t))\| \leq q(\|\theta(0)\|_s) \exp(-\alpha t), \quad \forall t \geq 0,
\]
(3)
where \(\|\theta(0)\|_s = \sup_{-\tau \leq r \leq 0} \|\theta(r)\|\).

The following lemmas will be used in the proof of our main results.

**Lemma 1** [29]. If all the roots of the characteristic equation
\[
\Omega(s) = \det\left(sI - A_0 - \sum_{j=1}^{m} A_j \exp(-\tau_j s)\right) = 0
\]
(4)
lie in the open left-half plane \(\text{Re}(s) < -\alpha < 0\), then there exists a constant \(M > 0\) such that for each \(\theta(t) \in \mathbb{C}^{n \times 1}\), the solution \(\Phi(t, \theta(t))\) of system (2) satisfies
\[
\|\Phi(t, \theta(t))\| \leq M \|\theta(0)\|_s \exp(-\alpha t), \quad \forall t \geq 0,
\]
(5)
i.e., the system (2) is \(\alpha\)-stable.

**Lemma 2** [30]. Let \(A, B \in \mathbb{C}^{n \times n}\). Then we have
(a) \(\text{Re}[\lambda_j(A)] \leq \mu(A), \forall j \in \Lambda^p\);
(b) \(\mu(A+B) \leq \mu(A) + \mu(B)\);
(c) \(\mu(A) \leq \|A\|\).

For any matrix \(E \in \mathbb{C}^{n \times n}\), define
\[
E_s = \frac{1}{2}(E + E^*), \quad E_u = \frac{1}{2}(E - E^*) \quad \text{and} \quad \phi(E) = \sqrt{\rho^2(E_s) + \rho^2(E_u)}.
\]
(6)

**Lemma 3.** For a given matrix \(E \in \mathbb{C}^{n \times n}\) and positive constants \(\alpha\) and \(\tau\), the following inequality holds:
\[
\mu(E e^{-\tau s}) \leq \phi(E) e^{\tau \alpha}, \quad \forall \text{Re}(s) \geq -\alpha.
\]
(7)
Proof. Let \( s = a + ib \) (\( i = \sqrt{-1} \) and
\[
G = \frac{1}{2}\left[ Ee^{-ts} + (Ee^{-ts})^* \right].
\]
Then, \( a \geq -\alpha \) (\( \forall \text{Re}(s) \geq -\alpha \)) and
\[
G = \frac{1}{2}e^{-\tau a}\left[ Ee^{-\tau b} + E^*e^{\tau b} \right] = \frac{1}{2}e^{-\tau a}\left[ (E + E^*) \cos \tau b + i(E^* - E) \sin \tau b \right]
= e^{-\tau a}[Ee \cos \tau b - iE_*e \sin \tau b].
\]
Since \((iE_*)^* = -i(-E_*) = iE_*, \) \( iE_* \) is a Hermitian matrix. Therefore,
\[
\lambda_{\text{max}}(G) \leq e^{\tau a}\left( \lambda_{\text{max}}(Ee \cos \tau b) + \lambda_{\text{max}}(-iE_* \sin \tau b) \right) .
\]
From \( |\lambda_j(iE_*)| = |\bar{\lambda}_j(E_*)| = |\lambda_j(E_*)| \) (\( \forall j \in \Lambda_i^a \)) we have
\[
\lambda_{\text{max}}(G) \leq e^{\tau a}\left( \rho(E) |\cos \tau b| + \rho(E_*) |\sin \tau b| \right)
\leq e^{\tau a}\phi(E) \left( |\cos \tau b| \cdot \frac{\rho(E)}{\phi(E)} + |\sin \tau b| \cdot \frac{\rho(E_*)}{\phi(E)} \right).
\]
It follows from the Cauchy–Schwarz inequality that
\[
0 \leq |\cos \tau b| \cdot \frac{\rho(E)}{\phi(E)} + |\sin \tau b| \cdot \frac{\rho(E_*)}{\phi(E)} \leq 1.
\]
Thus
\[
\mu(Ee^{-ts}) = \lambda_{\text{max}}(G) \leq \phi(E)e^{\tau a}, \quad \forall \text{Re}(s) \geq -\alpha.
\]
The proof is completed. \( \Box \)

Lemma 4. For any real matrix \( E \in \mathbb{R}^{2 \times 2}, \) \( \phi(E) \leq \|E\|. \) Moreover, \( \phi(E) = \|E\| \) if and only if \( E \) is a symmetric matrix or \( E = aI + U \) where \( a \in \mathbb{R} \) and \( U^T = -U. \)

Proof. In the case of \( E \in \mathbb{R}^{2 \times 2}, \) we take \( E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) where \( a, b, \) and \( c \) and \( d \in \mathbb{R}. \) Then
\[
E_s = \begin{bmatrix} a & \frac{1}{2}(b + c) \\ \frac{1}{2}(b + c) & d \end{bmatrix}, \quad E_u = \begin{bmatrix} 0 & \frac{1}{2}(b - c) \\ \frac{1}{2}(b - c) & 0 \end{bmatrix}.
\]
Thus, \( \rho(E_s) = \frac{1}{2}[(a + d) + \sqrt{(a - d)^2 + (b + c)^2}] \) and \( \rho(E_u) = \frac{1}{2}|b - c|. \) Hence
\[
\phi^2(E) = \frac{1}{2}\left( a^2 + b^2 + c^2 + d^2 + a + d\sqrt{(a - d)^2 + (b + c)^2} \right).
\]
On the other hand,
\[
\|E\|^2 = \lambda_{\text{max}}(E^TE)
= \frac{1}{2}\left( a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 - b^2 + c^2 - d^2)^2 + 4(ab + cd)^2} \right).
\]
Taking notice of
\[ (a^2 - b^2 + c^2 - d^2)^2 + 4(ab + cd)^2 - (a + d)^2[(a - d)^2 + (b + c)^2] \\
= (c^2 - b^2)^2 + 2(a^2 - d^2)(c^2 - b^2) \\
+ [2(ab + cd) + (a + d)(b + c)](a - d)(b - c) \\
= (c^2 - b^2)^2 + (a - d)(c - b)[(a + d)(b + c) - 2ab - 2cd] \\
= (c^2 - b^2)^2 + (a - d)^2(c - b)^2 \geq 0, \quad (15) \]

we have \( \phi(E) \leq \|E\| \) from (13) and (14). Moreover, \((c^2 - b^2)^2 + (a - d)^2(c - b)^2 = 0\) if and only if \(c = b\) or \(a = d\) and \(c = -b\). Thus, \( \phi(E) = \|E\| \) if and only if \(E\) is a symmetric matrix or \(E = aI + U\) \((U^T = -U)\). This completes the proof. \(\square\)

**Lemma 5.** For any real matrix \(E \in \mathbb{R}^{n \times n}\), \(\phi(E) = \|E\|\) if \(E\) is a symmetric matrix or \(E = aI + U\) where \(a \in \mathbb{R}\) and \(U \in \mathbb{R}^{n \times n}\) \((U^T = -U)\).

**Proof.** It is evident that \(\phi(E) = \rho(E) = \|E\|\) for a symmetric matrix \(E\). When \(E = aI + U\) with \(U^T = -U\),

\[
\phi(E)^2 = \rho^2(E) + \rho^2(E) = a^2 + \rho^2(U). \quad (16)
\]

Taking notice of \(E^TE = a^2I + U^TU\) we have

\[
\|E\|^2 = \lambda_{\text{max}}(E^TE) = a^2 + \lambda_{\text{max}}(U^TU) = a^2 + \|U\|^2. \quad (17)
\]

Since \(\rho(U) = \|U\|\) when \(U^T = -U\), (16) and (17) yield \(\phi(E) = \|E\|\). This completes the proof. \(\square\)

3. Main results

**Theorem 1.** The linear system (2) is \(\alpha\)-stable for some \(\alpha > 0\) if there exists an invertible matrix \(P \in \mathbb{C}^{n \times n}\) such that the following inequality is satisfied:

\[
\xi_1 \triangleq \mu(P^{-1}A_0P) + \sum_{j=1}^{m} \phi(\tilde{A}_j) < 0, \quad (18)
\]

where \(\phi(\tilde{A}_j) = \sqrt{\rho^2(\tilde{A}_{jj}) + \rho^2(\tilde{A}_{jj})}\), and

\[
\tilde{A}_j = P^{-1}A_jP, \quad \tilde{A}_{jj} = \frac{1}{2}(\tilde{A}_j + \tilde{A}_j^*), \quad \tilde{A}_{jj} = \frac{1}{2}(\tilde{A}_j - \tilde{A}_j^*). \quad (19)
\]

The convergence rate \(\alpha\) is given by \(\alpha = \beta - \epsilon\) where \(\beta > 0\) is the unique positive solution of the following equation

\[
\mu(P^{-1}A_0P) + \beta + \sum_{j=1}^{m} \phi(\tilde{A}_j) \exp(\tau_j \beta) = 0 \quad (20)
\]

and \(\epsilon\) is any positive number such that \(\epsilon < \beta\).
Proof. According to Lemma 1, the system (2) is $\alpha$-stable if and only if
\[
\Omega(s) = \det \left[ sI - A_0 - \sum_{j=1}^{m} A_j \exp(-\tau_j s) \right] \neq 0, \quad \forall \text{Re}(s) \geq -\alpha. \quad (21)
\]
This is equivalent to
\[
s \neq \lambda_i \left[ A_0 + \sum_{j=1}^{m} A_j \exp(-\tau_j s) \right], \quad \forall \text{Re}(s) \geq -\alpha, \quad \forall i \in \Lambda_n^1. \quad (22)
\]
Define
\[
f(z) \triangleq \mu \left( P^{-1} A_0 P + \sum_{j=1}^{m} \phi(\tilde{A}_j) \exp(\tau_j z) \right), \quad z \in [0, \infty). \quad (23)
\]
Then, $f$ is a strictly increasing function of $z$ and $f(z) \to \infty$ as $z \to \infty$. It follows from condition (18) that $f(0) = \xi_1 < 0$. Hence, there exists a unique positive constant $\beta > 0$ such that (20) holds and $f(\alpha) = f(\beta - \epsilon) < 0$. According to Lemma 2, it can be deduced that, for any invertible matrix $P$,
\[
\max_{i \in \Lambda_n^1} \left\{ \text{Re} \left[ \lambda_i \left( A_0 + \sum_{j=1}^{m} A_j \exp(-\tau_j s) \right) \right] \right\}
\leq \mu \left( P^{-1} A_0 P + \sum_{j=1}^{m} \phi(\tilde{A}_j) \exp(\tau_j \alpha) \right), \quad \forall \text{Re}(s) \geq -\alpha. \quad (24)
\]
It follows from Lemma 3 that
\[
\mu \left( \tilde{A}_j \exp(-\tau_j s) \right) \leq \phi(\tilde{A}_j) \exp(\tau_j \alpha), \quad \forall \text{Re}(s) \geq -\alpha. \quad (25)
\]
Therefore
\[
\max_{i \in \Lambda_n^1} \left\{ \text{Re} \left[ \lambda_i \left( A_0 + \sum_{j=1}^{m} A_j \exp(-\tau_j s) \right) \right] \right\}
\leq \mu \left( P^{-1} A_0 P + \sum_{j=1}^{m} \phi(\tilde{A}_j) \exp(\tau_j \alpha) \right) = f(\alpha) - \alpha < -\alpha,
\quad \forall \text{Re}(s) \geq -\alpha, \quad (26)
\]
which implies that (22) holds. The proof is completed. \(\square\)
Because $A_0$ is a Hurwitz matrix, it is possible to choose an invertible matrix $P$ such that 
\[ \mu(P^{-1}A_0P) < 0. \]
If $A_0$ is diagonalisable, i.e., there exists an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that 
\[ P^{-1}A_0P = D \triangleq \text{diag}[\lambda_i(A_0)], \]
the following corollary follows directly from Theorem 1.

**Corollary 1.** If $A_0$ is diagonalisable, then the linear system (2) is $\alpha$-stable for some $\alpha > 0$ if the following inequality is satisfied:
\[ \xi_2 \triangleq \max_{i \in \Lambda_1^n} \{\text{Re}(\lambda_i(A_0))\} + \sum_{j=1}^{m} \phi(\tilde{A}_j) < 0, \]
where $\phi(\tilde{A}_j)$ is defined by (19). The convergence rate $\alpha$ is given by $\alpha = \beta - \epsilon$ where $\beta > 0$ is the unique positive solution of the following equation
\[ \max_{i \in \Lambda_1^n} \{\text{Re}(\lambda_i(A_0))\} + \beta + \sum_{j=1}^{m} \phi(\tilde{A}_j) \exp(\tau_j \beta) = 0 \]
and $\epsilon$ is any positive number such that $\epsilon < \beta$.

**Remark 1.** Using Lemma 4 it is easy to prove that if the linear system (2) is a two-dimensional real system, then
\[ \mu(P^{-1}A_0P) + \sum_{j=1}^{m} \phi(\tilde{A}_j) \leq \mu(P^{-1}A_0P) + \sum_{j=1}^{m} \|\tilde{A}_j\|. \]
The equality in (30) holds if and only if $P^{-1}A_jP$ ($j \in \Lambda_1^n$) are symmetric matrix or $P^{-1}A_jP = a_jI + U_j$ ($a_j \in R$ and $U_j^T = -U_j$, $j \in \Lambda_1^n$). Therefore, for a two-dimensional real system, the criterion (18) in Theorem 1 is less conservative than that in Theorem 1 derived by Sun and Hsieh [27]. Similar conclusion can be worked out between the criterion (28) in Corollary 1 and that in Corollary 1 of Sun and Hsieh [27].

**Remark 2.** From Lemma 5, for a linear $n$-dimensional real system (2), if $P^{-1}A_jP$ ($j \in \Lambda_1^n$) are symmetric matrix or $P^{-1}A_jP = a_jI + U_j$ ($a_j \in R$ and $U_j^T = -U_j$, $j \in \Lambda_1^n$), we have
\[ \mu(P^{-1}A_0P) + \sum_{j=1}^{m} \phi(\tilde{A}_j) = \mu(P^{-1}A_0P) + \sum_{j=1}^{m} \|\tilde{A}_j\|. \]
Thus, in this special case, our criterion is equivalent to that derived by Sun and Hsieh [27]. In general, it is not easy to compare our stability criterion and that in [27]. The numerical examples given in Section 4, however, show that our criterion provides a much less conservative result.
Now, we turn to investigate the following linear uncertain system with multiple time delays

\[
\begin{cases}
\dot{x}(t) = (A_0 + k_0 E_0)x(t) + \sum_{j=1}^{m}(A_j + k_j E_j)x(t - \tau_j), & t \geq 0, \\
x(t) = \theta(t), & t \in [-\tau, 0],
\end{cases}
\]  

(32)

where \( E_j \ (j \in A_0^n) \) are known constant matrices, \( k_j \ (j \in A_0^m) \) are uncertain parameters. Letting \( K = [k_0; k_1; \ldots; k_m]^T \), the problem is to find the criteria such that the system (32) is exponentially stable for any \( K \in \Omega \subset \mathbb{R}^{m+1} \) where

\[
\Omega = \{ |k_j| \leq \bar{k}_j, \ j \in A_0^m \}
\]

(33)

for given nonnegative constants \( \bar{k}_j \in \mathbb{R}^+ \ (j \in A_0^m) \).

**Theorem 2.** The linear system (32) is \( \alpha \)-stable for any \( K \in \Omega \subset \mathbb{R}^{m+1} \) if there exists an invertible matrix \( P \in \mathbb{C}^{n \times n} \) such that the following inequality is satisfied:

\[
\xi_3 \triangleq \mu(P^{-1}A_0P) + \sum_{j=1}^{m} \phi(\tilde{A}_j) + \sum_{j=0}^{m} \bar{k}_j \phi(\tilde{E}_j) < 0,
\]

(34)

where \( \phi(\tilde{A}_j) \ (j \in A_0^n) \) are defined by (19) in Theorem 1, \( \phi(\tilde{E}_j) = \sqrt{\rho^2(\tilde{E}_{uj}) + \rho^2(\tilde{E}_{uj})} \) \( (j \in A_0^n) \), and

\[
\bar{E}_j = P^{-1}E_jP, \quad \bar{E}_{uj} = \frac{1}{2}(\bar{E}_j + \bar{E}_j^T), \quad \bar{E}_{uj} = \frac{1}{2}(\bar{E}_j - \bar{E}_j^T).
\]

(35)

The convergence rate \( \alpha \) is given by \( \alpha = \beta - \epsilon \) where \( \beta > 0 \) is the unique positive solution of the following equation

\[
\mu(P^{-1}A_0P) + \bar{k}_0 \phi(\bar{E}_0) + \beta + \sum_{j=1}^{m} \phi(\tilde{A}_j) + \bar{k}_j \phi(\tilde{E}_j)) \exp(\tau_j \beta) = 0
\]

(36)

and \( \epsilon \) is any positive number such that \( \epsilon < \beta \).

**Proof.** Taking notice of

\[
\phi(k_j P^{-1}E_j P) \leq |k_j| \phi(\bar{E}_j) \leq \bar{k}_j \phi(\tilde{E}_j), \quad j \in A_0^m,
\]

(37)

and following the same procedures in the proof of Theorem 1, it is easy to obtain the conclusion of the theorem. \( \square \)

**Corollary 2.** If \( A_0 \) is diagonalisable, i.e., there exists an invertible matrix \( P \in \mathbb{C}^{n \times n} \) such that (27) holds, then the linear system (32) is \( \alpha \)-stable for any \( K \in \Omega \subset \mathbb{R}^{m+1} \) if the following inequality is satisfied:

\[
\xi_4 \triangleq \max_{i \in A_1} \{ \text{Re}(\lambda_i(A_0)) \} + \sum_{j=1}^{m} \phi(\tilde{A}_j) + \sum_{j=0}^{m} \bar{k}_j \phi(\tilde{E}_j) < 0,
\]

(38)
where \( \phi(\tilde{A}_j) \) and \( \phi(\tilde{E}_j) \) are defined by (19) and (35), respectively. The convergence rate \( \alpha \) is given by \( \alpha = \beta - \epsilon \) where \( \beta > 0 \) is the unique positive solution of the following equation

\[
\max_{i \in \Lambda^k_1} \left\{ \text{Re}(\lambda_i(A_0)) \right\} + \bar{k}_0\phi(\tilde{E}_0) + \beta + \sum_{j=1}^m (\phi(\tilde{A}_j) + \bar{k}_j\phi(\tilde{E}_j)) \exp(\tau_j\beta) = 0 \tag{39}
\]

and \( \epsilon \) is any positive number such that \( \epsilon < \beta \).

**Proof.** It follows directly from Theorem 2. \( \Box \)

### 4. Illustrative examples

**Example 1.** Consider the linear system with multiple time delays worked out by Sun and Hsieh [27]

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau_1) + A_2 x(t - \sqrt{3}), \quad t \geq 0, \\
x(t) &= \theta(t), \quad t \in [-\sqrt{3}, 0],
\end{align*}
\tag{40}
\]

where \( \theta(t) \in \mathbb{R}^{2 \times 1} \) is a given continuous function, and

\[
A_0 = \begin{bmatrix} -7 & -1 \\ 0.5 & -5.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 10.1 & 14.2 \\ -6.6 & -10.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5.2 & -6.2 \\ 3.6 & 5.1 \end{bmatrix}.
\]

Comparing (40) with (2), we have \( m = 2, \tau_1 = 1, \tau_2 = \sqrt{3} \). Taking \( \mu(P^{-1}A_0P) = -6.0, \phi(\tilde{A}_1) = \phi(P^{-1}A_1P) = 3.1203 \) and \( \phi(\tilde{A}_2) = \phi(P^{-1}A_2P) = 2.1299 \). Therefore

\[
\xi_1 = \mu(P^{-1}A_0P) + \phi(\tilde{A}_1) + \phi(\tilde{A}_2) = -0.7498.
\tag{42}
\]

Moreover, the unique positive root of Eq. (20) is \( \beta = 0.0907 \). It follows from Theorem 1 that the system (40) is \( \alpha \)-stable with \( \alpha = 0.09 \) by selecting \( \epsilon = 0.0007 \). The stability criterion in [27] gives \( \alpha = 0.04 \). This implies that, for a two-dimensional real system, our result is less conservative than that in [27].

**Example 2.** Consider the following three-dimensional system with multiple time delays

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + A_1 x(t - \tau_1) + A_2 x(t - \tau_2), \quad t \geq 0, \\
x(t) &= \theta(t), \quad t \in [-2, 0],
\end{align*}
\tag{43}
\]

where \( \theta(t) \in \mathbb{R}^{3 \times 1} \) is a given continuous function, \( \tau_1 = 1, \tau_2 = 2 \), and

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1 In Sun and Hsieh [27], the matrix \( P \) was chosen to be \( \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \). However, one cannot get \( \mu(P^{-1}A_0P) = -6.0 \) and other results stated in [27] by using this matrix. It could be a slip of the pen.
it can be computed that
\[ \mu(P^{-1}A_0 P) = -5.4856, \quad \phi(\tilde{A}_1) = 3.2215, \quad \phi(\tilde{A}_2) = 1.2254, \]
\[ \|\tilde{A}_1\| = \|P^{-1}A_1 P\| = 4.4622 \quad \text{and} \quad \|\tilde{A}_2\| = \|P^{-1}A_2 P\| = 1.6062. \]

Therefore
\[ \xi_1 = \mu(P^{-1}A_0 P) + \phi(\tilde{A}_1) + \phi(\tilde{A}_2) = -1.0387. \]

Moreover, the unique positive root of Eq. (20) is \( \beta = 0.1423. \) It follows from Theorem 1 that the system (43) is \( \alpha \)-stable with \( \alpha = 0.14 \) by selecting \( \epsilon = 0.0023. \) Since
\[ \mu(P^{-1}A_0 P) + \|P^{-1}A_1 P\| + \|P^{-1}A_2 P\| = 0.5828 > 0, \]
the stability criterion in Theorem 1 derived by Sun and Hsieh [27] cannot be satisfied. This shows that our criterion may complement Theorem 1 of Sun and Hsieh [27] in testing the exponential stability of high-dimensional (\( \geq 3 \)) system.

In order to compare our result with the criteria derived by Niculescu et al. [28], taking the same transformation matrix \( P \) and using the measure of matrix \( P^{-1}A_0 P \) by letting \( k_A = 1 \) and \( \eta_A = -\mu(A) = -\mu(P^{-1}A_0 P), \) we can obtain
\[ \frac{k_A}{\eta_A} (\|A_{d_1}\| + \|A_{d_2}\|) = \frac{\|P^{-1}A_1 P\| + \|P^{-1}A_2 P\|}{\mu(P^{-1}A_0 P)} = \frac{4.4622 + 1.6062}{5.4856} = 1.1062 > 1. \]

Thus, the delay-independent criterion (Theorem 3) in [28] is not available in this example. On the other hand, letting \( k = 1 \) and \( \eta = -\mu(P^{-1}(A_0 + A_1 + A_2) P), \) the delay-dependent criterion (Theorem 4) in [28] gives
\[ \tilde{\tau} < \frac{\eta}{k \sum_{i=1}^2 (\|A_{d_i} A\| + \sum_{j=1}^2 \|A_{d_i} A_{d_j}\|)} = 0.0852, \]
where \( A = P^{-1}A_0 P \) and \( A_{d_i} = P^{-1}A_i P \) (\( i = 1, 2 \)). Therefore, according to Theorem 4 in [28] the system (43) is exponentially stable if \( \tau_1, \tau_2 \leq \tilde{\tau} < 0.0852. \) This shows that, in this example, our criterion is less conservative than those proposed in [28]. It is worth to be noted that both delay-independent and delay-dependent criteria of [28] are for nonlinear systems with time-varying delays, whereas our criterion is for linear systems with time-invariant delays.

**Example 3.** Consider the following three-dimensional uncertain system with multiple time delays

\[ A_0 = \begin{bmatrix} -6.48 & 2.14 & -1.58 \\ -3.92 & -2.92 & -3.35 \\ -0.48 & 2.14 & -7.58 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.0 & 1.4 & 0.1 \\ -0.6 & -1.0 & 0.22 \\ 0.35 & 0.26 & 0.22 \end{bmatrix}, \]
\[ A_2 = \begin{bmatrix} -0.5 & -0.6 & 0.2 \\ 0.36 & 0.52 & -0.32 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}. \]

Taking an invertible matrix
\[ P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & -2 \\ 2 & 1 & -3 \end{bmatrix}, \]

it can be computed that \( \mu(P^{-1}A_0 P) = -5.4856, \) \( \phi(\tilde{A}_1) = 3.2215, \) \( \phi(\tilde{A}_2) = 1.2254, \)
\[ \|\tilde{A}_1\| = \|P^{-1}A_1 P\| = 4.4622 \quad \text{and} \quad \|\tilde{A}_2\| = \|P^{-1}A_2 P\| = 1.6062. \]

Therefore
\[ \xi_1 = \mu(P^{-1}A_0 P) + \phi(\tilde{A}_1) + \phi(\tilde{A}_2) = -1.0387. \]
\[
\begin{align*}
    \dot{x}(t) &= (A_0 + k_0 E_0) x(t) + (A_1 + k_1 E_1) x(t - 1) + k_2 E_2 x(t - 2), \quad t \geq 0, \\
    x(t) &= \theta(t), \quad t \in [-2, 0], 
\end{align*}
\]

where \( \theta(t) \in C^{3 \times 1} \) is a given continuous function.

\[
A_0 = \begin{bmatrix}
    -6.48 & 2.14 & -1.58 \\
    -3.92 & -2.92 & -3.35 \\
    -0.48 & 2.14 & -7.58
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
    1.0 & 0.0 & 0.1 \\
    -0.6 & -1.0 & 0.22 \\
    0.0 & 0.26 & 0.22
\end{bmatrix}
\]

and

\[
E_0 = \begin{bmatrix}
    0 & 0.2 & 0 \\
    -0.3 & -0.2 & 0 \\
    0 & 0.3 & 1.0
\end{bmatrix}, \quad E_1 = \begin{bmatrix}
    0.3 & 0.5 & 0.1 \\
    -0.1 & 0.0 & -0.2 \\
    0.3 & 0.0 & 0.2
\end{bmatrix},
\]

\[
E_2 = \begin{bmatrix}
    -0.3 & -0.2 & 0.0 \\
    -0.4 & 0.1 & 0.0 \\
    0.0 & 0.0 & 0.2
\end{bmatrix}.
\]

The bounds of uncertain parameters are given as \( |k_0| \leq \bar{k}_0 = 0.6, \quad |k_1| \leq \bar{k}_1 = 0.8, \quad |k_2| \leq \bar{k}_2 = 0.4 \).

Comparing (47) with (32), we have \( m = 2, \tau_1 = 1, \tau_2 = 2 \). Using the same matrix \( P \) introduced in Example 2 it can be computed that \( \mu(P^{-1} A_0 P) = -5.4856, \phi(A_1) = 1.7967, \phi(E_0) = 1.7462, \phi(E_1) = 1.1588 \) and \( \phi(E_2) = 0.5392 \). Therefore

\[
    \xi_3 = \mu(P^{-1} A_0 P) + \phi(A_1) + 2 \sum_{j=0}^{2} \bar{k}_j \phi(E_j) = -1.4984.
\]

Moreover, the unique positive root of Eq. (36) is \( \beta = 0.3124 \). It follows from Theorem 2 that the system (47) is \( \alpha \)-stable with \( \alpha = 0.31 \) by selecting \( \epsilon = 0.0024 \).

5. Conclusion

In this article, we have studied the exponential stability of uncertain linear systems with multiple time delays. Using the characteristic function, \( \alpha \)-stability criteria have been derived in terms of scalar inequalities involving spectral radius and matrix measure. The estimation of the convergence rate \( \alpha \) can be easily calculated using the results obtained. For two-dimensional time-invariant linear systems, the proposed stability criteria have been shown to be less conservative than those in [27]. For linear systems with multiple time-invariant delays, numerical examples are given to further show that the new stability criteria are more powerful comparing to those in [27] and [28].

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