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Filled function method for nonlinear equations

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ABSTRACT

Systems of nonlinear equations are ubiquitous in engineering, physics and mechanics, and have myriad applications. Generally, they are very difficult to solve. In this paper, we will present a filled function method to solve nonlinear systems. We will first convert the nonlinear systems into equivalent global optimization problems with the property: x^* is a global minimizer if and only if its function value is zero. A filled function method is proposed to solve the converted global optimization problem. Numerical examples are presented to illustrate our new techniques.

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1. Introduction

We consider the following box-constrained system of nonlinear equations:

$$\text{(SNE)} \quad F(x) = 0, \quad x \in X,$$

where the mapping $F : R^n \to R^m$ is continuously differentiable on Ω, which denotes an open set containing the box $X \subset R^n$.

Let $F(x) := (f_1(x), \ldots, f_n(x))^T$ and let

$$f(x) := \frac{1}{2}\sum_{i=1}^{n} f_i^2(x).$$

To solve (SNE), we can solve the following box-constrained global optimization problem:

$$\text{(GOP)} \quad \min f(x) \quad \text{s.t.} \quad x \in X.$$

Assume that (SNE) has at least one solution. Then, $\bar{x}$ is a solution of the system (SNE) if and only if it is a global minimizer of problem (GOP) and satisfies $f(\bar{x}) = 0$ (see Chapter 11 in [1]).

In this paper, a filled function $F(x, x^*)$ of problem (GOP) at x^* with $f(x^*) > 0$ will be proposed. We will introduce a filled function method to obtain a global minimizer of problem (GOP) by locally solving the following box-constrained optimization problem:

$$\text{(BOP)} \quad \min F(x, x^*) \quad \text{s.t.} \quad x \in X$$

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starting from the point near x^*. If x^* is not a global minimizer of $f(x)$, we are able to obtain a local minimizer $\bar{x}$ of $F(x, x^*)$ on $S_2 = \{x|f(x) < f(x^*), x \in X\}$ since $F(x, x^*)$ does have a minimizer in S_2. Then, we can manage to obtain a new local minimal point x_1^* of $f(x)$ satisfying $f(x_1^*) < f(x^*)$, by locally minimizing $f(x)$ starting from the point $\bar{x}$ (see e.g., [2–8]). Finally, we can obtain a global minimizer or an approximate global minimizer of problem (GOP) by solving a finite number of box-constrained problems (BOP).

The numerical results obtained show that our method is applicable and efficient. The rest of this paper is organized as follows. In Section 2, we propose a new filled function for the optimization problem. The corresponding algorithm is presented in Section 3. Several numerical examples are reported in Section 4. Finally, some conclusions are drawn in Section 5.

2. Properties of the filled function of problem (GOP)

Throughout this paper we make the following assumptions:

Assumption 1. $f_i(x)$ $(i = 1, \ldots, n)$ is continuously differentiable on Ω, which denotes an open set containing the box $X \subset R^n$.

Assumption 2. The value of $f(x)$ for x on the boundary of X is greater than the value of $f(x)$ for any x inside X.

Notice that Assumption 1 implies that $f(x)$ is Lipschitz continuous on $\operatorname{int} X$, i.e., there exists a constant $L > 0$ such that $|f(x) - f(y)| \le L\|x - y\|$ holds for all $x, y \in \operatorname{int} X$. Assumption 2 implies that the interior of X contains all minimizers of $f(x)$.

Definition 2.1. A continuous function $F(x, x^*)$ is said to be a filled function of problem (GOP) at a local minimal point x^* of $f(x)$ if it satisfies the following conditions:

1° x^* is a local maximizer of $F(x, x^*)$.

2° $F(x, x^*)$ has no stationary point in the region

$$S_1 = \{x|f(x) \ge f(x^*),\ x \in X \setminus \{x^*\}\}.$$

3° If x^* is not a global minimizer of $f(x)$, then $F(x, x^*)$ does have a minimizer in the region

$$S_2 = \{x|f(x) < f(x^*),\ x \in X\}.$$

In the following, we will introduce a function $F(x, x^*, q)$ which only has one parameter q and satisfies Definition 2.1. To begin with, we present a function

$$\phi_q(t) = \begin{cases} \exp\left(-\dfrac{q^2}{t^2}\right), & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases} \tag{2.1}$$

It is easy to prove that $\phi_q(t)$ is a continuously differentiable function (see Fig. 1).

The filled function given at x^* has the following form:

$$F(x, x^*, q) = \frac{1}{q + \|x - x^*\|}\phi_q\left(f(x) - \frac{f(x^*)}{2}\right), \quad x \in R^n,\ q > 0. \tag{2.2}$$

The following theorems show that $F(x, x^*, q)$ satisfies Definition 2.1.

Theorem 2.1. *Suppose that $f(x)$ is continuously differentiable on Ω and x^* is a local minimizer of $f(x)$ with $f(x^*) > 0$. Then, there exists a number $q' > 0$ such that, point x^* is a local maximizer of $F(x, x^*, q)$ for all $q \in (0, q')$.*

Proof. Since x^* is a local minimizer of $f(x)$, there exists a neighborhood $N(x^*, \delta)$ of x^* with $\delta > 0$ such that $f(x) \ge f(x^*)$ for all $x \in N(x^*, \delta)$, therefore $\frac{f(x)}{f(x) - \frac{f(x^*)}{2}} \le 2$. Then,

$$\begin{aligned} \frac{F(x, x^*, q)}{F(x^*, x^*, q)} &= \frac{q}{q + \|x - x^*\|} \exp\left(\frac{q^2}{(\frac{f(x^*)}{2})^2} - \frac{q^2}{(f(x) - \frac{f(x^*)}{2})^2}\right) \\ &= \frac{q}{q + \|x - x^*\|} \exp\left(\frac{q^2 f(x)(f(x) - f(x^*))}{(\frac{f(x^*)}{2})^2 (f(x) - \frac{f(x^*)}{2})^2}\right) \\ &\le \frac{q}{q + \|x - x^*\|} \exp\left(\frac{2q^2(f(x) - f(x^*))}{(\frac{f(x^*)}{2})^2 (f(x) - \frac{f(x^*)}{2})}\right). \end{aligned}$$

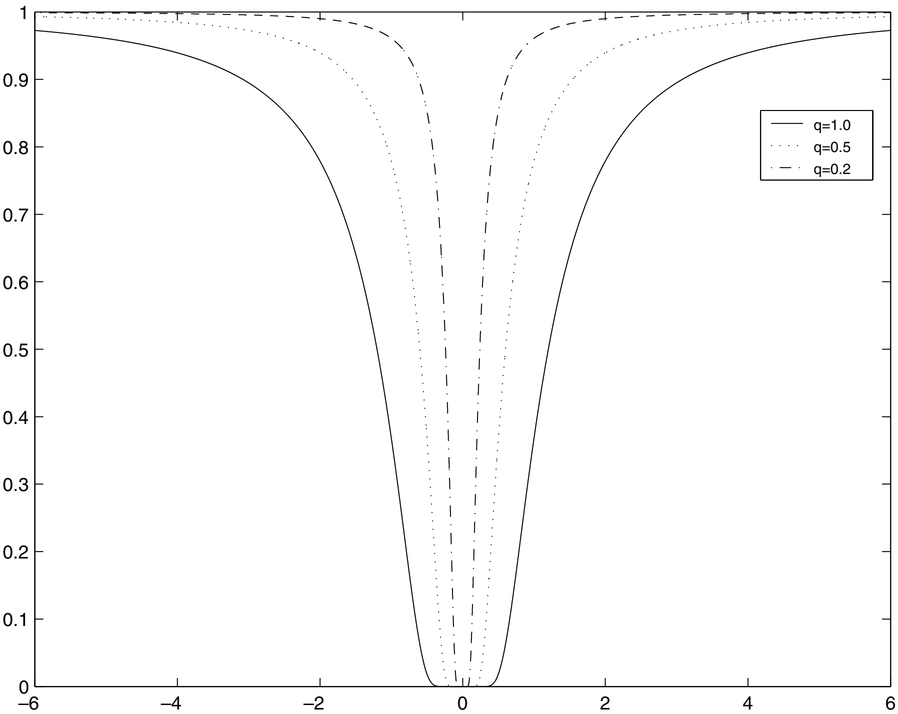

Fig. 1. The graph of $\phi_q(t)$ with $q = 1.0, 0.5, 0.2$, respectively.

If $q^2 < \frac{(\frac{f(x^*)}{2})^2}{2}$ and $f(x^*) > 0$ is known, then

$$\frac{2q^2(f(x) - f(x^*))}{(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2})} \le \frac{2q^2}{(\frac{f(x^*)}{2})^2} < 1.$$

Hence, by $e^y < \frac{1}{1-y}$ for all $y < 1$ and $y \neq 0$, we have

$$\begin{aligned}\frac{F(x, x^*, q)}{F(x^*, x^*, q)} &< \frac{q}{q + \|x - x^*\|} \exp\left(\frac{2q^2(f(x) - f(x^*))}{(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2})}\right) \\ &< \frac{q}{q + \|x - x^*\|} \Bigg/ \left(1 - \frac{2q^2(f(x) - f(x^*))}{(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2})}\right) \\ &= \frac{q}{q + \|x - x^*\|} \times \frac{(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2})}{(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2}) - 2q^2(f(x) - f(x^*))} \\ &= \frac{q(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2})}{q(\frac{f(x^*)}{2})^2(f(x) - \frac{f(x^*)}{2}) + A},\end{aligned}$$

where

$$A = \left(\frac{f(x^*)}{2}\right)^2 \|x - x^*\| \left(f(x) - \frac{f(x^*)}{2}\right) - 2q^3\left(f(x) - f(x^*)\right) - 2q^2\|x - x^*\|\left(f(x) - f(x^*)\right).$$

Denote $q' = \min\{\frac{(\frac{f(x^*)}{2})}{\sqrt{2}}, \frac{(\frac{f(x^*)}{2})}{\sqrt[3]{2L}}\}$ and let $q \in (0, q')$, we have

$$A \ge \left(\left(\frac{f(x^*)}{2}\right)^2 - 2q^2\right) \|x - x^*\| \left(f(x) - f(x^*)\right) + \left(\left(\frac{f(x^*)}{2}\right)^3 - 2q^3 L\right) \|x - x^*\| > 0.$$

All of above inequalities hold for $0 < q < q'$, thus,

$$\frac{F(x, x^*, q)}{F(x^*, x^*, q)} < 1,$$

x^* is a local maximizer of $F(x, x^*, q)$. □

Theorem 2.2. *Suppose that $f(x)$ is continuously differentiable on Ω and x^* is a local minimizer of $f(x)$ with $f(x^*) > 0$. Then, there exists a number $q'' > 0$ such that, function $F(x, x^*, q)$ has no stationary points on the set $S_1 = \{x | f(x) \geq f(x^*), x \in X/\{x^*\}\}$ for all $q \in (0, q'')$.*

Proof. Let $x \in S_1$, i.e., $f(x) \geq f(x^*)$ and $x \neq x^*$, we have

$$\begin{aligned} \nabla F(x, x^*, q)^T \frac{x - x^*}{\|x - x^*\|} &= -\frac{1}{(q + \|x - x^*\|)^2} \exp\left(-\frac{q^2}{(f(x) - \frac{f(x^*)}{2})^2}\right) \\ &\quad + \frac{2}{q + \|x - x^*\|} \exp\left(-\frac{q^2}{(f(x) - \frac{f(x^*)}{2})^2}\right) \frac{\nabla f(x)^T (x - x^*)}{\|x - x^*\|} \frac{q^2}{(f(x) - \frac{f(x^*)}{2})^3} \\ &\leq -\frac{1}{(q + \|x - x^*\|)^2} \exp\left(-\frac{q^2}{(\frac{f(x^*)}{2})^2}\right) + \frac{2L}{q + \|x - x^*\|} \frac{q^2}{(\frac{f(x^*)}{2})^3} \\ &\leq -\frac{1}{(q + \max_{x \in \Omega} \|x - x^*\|)^2} \exp\left(-\frac{q^2}{(\frac{f(x^*)}{2})^2}\right) + 2L \frac{q}{(\frac{f(x^*)}{2})^3}. \end{aligned}$$

Let $M = \max_{x \in X} \|x - x^*\|$, and $0 < q < \min\{1, \frac{(\frac{f(x^*)}{2})^3}{4L(1+M)^2}\}$, we have that

$$\nabla F(x, x^*, q)^T \frac{x - x^*}{\|x - x^*\|} < -\frac{1}{(1 + M)^2} \exp\left(-\frac{q}{(\frac{f(x^*)}{2})^2}\right) + \frac{1}{2(1 + M)^2}.$$

Thus, let $q'' = \min\{\frac{(\frac{f(x^*)}{2})^3}{4L(1+M)^2}, (\frac{f(x^*)}{2})^2 \ln 2, 1\}$, when $q \in (0, q'')$, we have

$$\nabla F(x, x^*, q)^T \frac{x - x^*}{\|x - x^*\|} < 0.$$

It implies that the function $F(x, x^*, q)$ has no stationary points in the region $S_1 = \{x | f(x) \geq f(x^*), x \in X \setminus \{x^*\}\}$ when $0 < q < q''$. □

Theorem 2.3. *If x^* is a local minimizer and it is not a global minimizer of $f(x)$ in X, then for arbitrary $q > 0$, there exists a minimizer $\overline{x^*}$ of $F(x, x^*, q)$ in the region $S_2 = \{x | f(x) < f(x^*), x \in X\}$.*

Proof. Since $f(x)$ is continuous and x^* is not its global minimizer, and the global minimum of $f(x)$ is zero, there exists a point $\overline{x^*}$, such that

$$f(\overline{x^*}) - \frac{f(x^*)}{2} = 0,$$

therefore,

$$F(\overline{x^*}, x^*, q) = \frac{1}{q + \|x - x^*\|} \phi_q\left(f(\overline{x^*}) - \frac{f(x^*)}{2}\right) = 0.$$

On the other hand $F(x, x^*, q) \geq 0$ from the form of the filled function. Therefore,

$$F(x, x^*, q) \geq F(\overline{x^*}, x^*, q).$$

Thus $\overline{x^*}$ is a minimizer of $F(x, x^*, q)$ for any $q > 0$. □

Theorems 2.1–2.3 show that, for all numbers $q \in (0, \min\{q', q''\})$, function $F(x, x^*, q)$ satisfies all the conditions of filled function in Definition 2.1. In other words, $F(x, x^*, q)$ is a filled function for sufficiently small number $q > 0$.

We note the following two important issues. Firstly, in the phase of minimizing the filled function, Theorems 2.1–2.3 guarantee that the present local minimizer x^* of the objective function is avoided. Moreover the minimum of the filled function will always be achieved at a point where the objective function value is not higher than the current minimum of the objective function.

Secondly, the parameters q are easier to be appropriately chosen than those of the original filled function [2].

In Section 3, we describe an optimization algorithm that employs the filled function $F(x, x^*, q)$ presented above.

3. Algorithm

In Section 2, we discussed some properties of the filled function. Thus, a global minimizer or an approximate global minimizer of (GOP) can be obtained in finite steps. The corresponding algorithm is denoted by Algorithm MSNE (method of solving nonlinear equations) and detailed as follows:

MSNE

1. Initial Step Choose $0 < r_0 < 1$ as the tolerance parameters for terminating the minimization process of problem (GOP).

Choose $q = \ln 2$, $0 < q_0 < 1$ and $M > 0$.

Choose direction e_i, $i = 1, 2, \ldots, k_0$ with integer $k_0 > 2n$, where n is the number of variable.

Choose an initial point $x_1^0 \in \Omega$.

Let $k = 1$.

2. Main Step

$\mathbf{1^0}$. Obtain a local minimizer of prime problem (GOP) by implementing a local downhill search procedure starting from the x_k^0. Let x_k^* be the local minimizer obtained. Let $i = 1$, $q = \ln 2$.

$\mathbf{2^0}$. If $i \le k_0$, then goto 5^0, otherwise goto 3^0.

$\mathbf{3^0}$. If $\frac{f(x_k^*)}{2} \le r_0$, then terminate the iteration, the x_k^* is the global minimizer of problem(GOP), otherwise, goto 4^0.

$\mathbf{4^0}$. If $q \le q_0$, then let $q = \frac{f(x_k^*)}{2} \ln 2$ and $i = 1$, goto 5^0, otherwise, let $q = q/10$, $i = 1$, goto 5^0.

$\mathbf{5^0}$. $\bar{x}_k^* = x_k^* + \sigma e_i$ (where σ is a very small positive number), if $f(\bar{x}_k^*) < f(x_k^*)$ then let $k = k + 1$, $x_k^0 = \bar{x}_k^*$ and goto 1^0; otherwise, goto 6^0.

$\mathbf{6^0}$. Let

$$F(x, x_k^*, q) = \frac{1}{q + \|x - x_k^*\|}\phi_q\left(f(x) - \frac{f(x_k^*)}{2}\right),$$

and $y_0 = \bar{x}_k^*$. Turn to inner loop.

3. Inner Loop

$\mathbf{1^0}$. Let $m = 0$.

$\mathbf{2^0}$. $y_{m+1} = \varphi(y_m)$, where φ is an iteration function. It denotes a local downhill search method for the following problem:

$$\min \quad F(x, x_k^*, q) \quad \text{s.t.} \quad x \in X.$$

Such as F-R method, BFGS method, etc.

$\mathbf{3^0}$. If $\|y_{m+1} - x_1^0\| \ge M$, then let $i = i + 1$, goto main step 2^0, otherwise goto 4^0.

$\mathbf{4^0}$. If $f(y_{m+1}) \le f(x_k^*)$ then let $k = k + 1$, $x_k^0 = y_{m+1}$ and goto main step 1^0, otherwise let $m = m + 1$ and goto 2^0.

The idea and mechanism of algorithm are explained as follows:

There are two phrases in the algorithm. One is that of minimizing the original function f, and the other is that of minimizing the new filled function $F(x, x^*, q)$ in the inner loop. We let $q = \ln 2$ in the initialization, afterwards, $\frac{f(x_k^*)}{2}$ and q are gradually reduced via the two-phase cycle until they are less than sufficiently small positive scales. If $\frac{f(x_k^*)}{2}$ is less than r_0, then we think that x_k^* is the global minimizer of $f(x)$. The algorithm is terminated.

4. Numerical examples

In this section, we apply Algorithm MSNS to several test examples. The proposed algorithm is programmed in Fortran 95 for working on the windows XP system with Intel cl.7G CPU and 256M RAM.

In this section, as a local optimization method in both Main Step and Inner loop, we use BFGS Method to get the search direction and the *Armijo* line search to get the step size. We use $\varepsilon = 10^{-3}$ as a termination condition in $\|\nabla F\| \le \varepsilon$.

In the numerical experiments below, we take $k_0 \le 1000$ and define the set $\{e_i \in R^n : i = 1, \ldots, k_0\}$ as a subset of points $e_i = (e_i(1), \ldots, e_i(n))$ defined by

$$\begin{aligned}
&e_i(1) = \sin(\theta_{1i})\sin(\theta_{2i})\cdots\sin(\theta_{n-2,i})\sin(\theta_{n-1,i}),\\
&e_i(2) = \sin(\theta_{1i})\sin(\theta_{2i})\cdots\sin(\theta_{n-2,i})\cos(\theta_{n-1,i}),\\
&e_i(3) = \sin(\theta_{1i})\cdots\sin(\theta_{n-3,i})\cos(\theta_{n-2,i}),\\
&\ldots,\\
&e_i(n-1) = \sin(\theta_{1i})\cos(\theta_{2i}),\\
&e_i(n) = \cos(\theta_{1i}),
\end{aligned}$$

where n is the number of variables and $\theta_{ji} \in \{k\frac{\pi}{16} : k = 1, \ldots, 32\}$, $j = 1, \ldots, n-1$, $i = 1, \ldots, k_0$.

Table 4.1
Numerical results for Example 4.1.

k	x_k	Local minimizer x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix} 6.0000 \\ -2.0000 \end{pmatrix}$	$\begin{pmatrix} 5.7220 \\ -1.8806 \end{pmatrix}$	2.5070	$\begin{pmatrix} -0.7614 \\ -1.3883 \end{pmatrix}$
1	$\begin{pmatrix} 4.6221 \\ -1.8801 \end{pmatrix}$	$\begin{pmatrix} 4.7387 \\ -1.7417 \end{pmatrix}$	1.6212	$\begin{pmatrix} -0.2762 \\ -1.2429 \end{pmatrix}$
2	$\begin{pmatrix} 3.7595 \\ -1.4863 \end{pmatrix}$	$\begin{pmatrix} 3.7387 \\ -1.2649 \end{pmatrix}$	0.61647	$\begin{pmatrix} -0.1714 \\ -0.7662 \end{pmatrix}$
3	$\begin{pmatrix} 2.5976 \\ -0.3300 \end{pmatrix}$	$\begin{pmatrix} 2.5762 \\ -0.4316 \end{pmatrix}$	0.35576	$\begin{pmatrix} -0.5615 \\ -0.2013 \end{pmatrix}$
4	$\begin{pmatrix} 1.7395 \\ -0.4335 \end{pmatrix}$	$\begin{pmatrix} 1.8784 \\ -0.3459 \end{pmatrix}$	2.9166×10^{-12}	$\begin{pmatrix} -3.1896 \times 10^{-7} \\ -1.6778 \times 10^{-6} \end{pmatrix}$
0	$\begin{pmatrix} -5.0000 \\ -3.0000 \end{pmatrix}$	$\begin{pmatrix} -2.1001 \\ -1.0313 \end{pmatrix}$	9.9739	$\begin{pmatrix} 3.0069 \\ -0.9655 \end{pmatrix}$
1	$\begin{pmatrix} 0.9887 \\ -1.0153 \end{pmatrix}$	$\begin{pmatrix} 0.9140 \\ -1.0268 \end{pmatrix}$	4.8916	$\begin{pmatrix} 2.0734 \\ -0.7696 \end{pmatrix}$
2	$\begin{pmatrix} 1.9211 \\ -0.9692 \end{pmatrix}$	$\begin{pmatrix} 1.8784 \\ -0.3458 \end{pmatrix}$	3.5917×10^{-12}	$\begin{pmatrix} 2.5847 \times 10^{-7} \\ 1.8775 \times 10^{-6} \end{pmatrix}$

Table 4.2
Numerical results for Example 4.2.

k	x_k	Local minimizer x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix} 3.0000 \\ 8.0000 \end{pmatrix}$	$\begin{pmatrix} 0.0000 \\ 6.8750 \end{pmatrix}$	1.0000	$\begin{pmatrix} -1.0000 \\ 3.3251 \end{pmatrix}$
1	$\begin{pmatrix} 9.9999 \times 10^{-6} \\ 6.8750 \end{pmatrix}$	$\begin{pmatrix} 1.4545 \times 10^{-5} \\ 6.8750 \end{pmatrix}$	3.49897×10^{-10}	$\begin{pmatrix} 2.8777 \times 10^{-10} \\ 1.8706 \times 10^{-5} \end{pmatrix}$
0	$\begin{pmatrix} 2.0000 \\ 7.5000 \end{pmatrix}$	$\begin{pmatrix} 0.0000 \\ 6.9667 \end{pmatrix}$	1.0000	$\begin{pmatrix} -1.0000 \\ 5.7256 \times 10^{-5} \end{pmatrix}$
1	$\begin{pmatrix} 9.9999 \times 10^{-6} \\ 9667 \end{pmatrix}$	$\begin{pmatrix} 1.4354 \times 10^{-5} \\ 6.9667 \end{pmatrix}$	5.1280×10^{-9}	$\begin{pmatrix} -9.8870 \times 10^{-10} \\ -7.1610 \times 10^{-5} \end{pmatrix}$

The computational results are summarized in tables for each example. The symbols used in the tables are given as follows:

- k is the number of iterations in finding the k-th local minimizer.
- x_k is the starting point in the k-th iteration in finding the k-th local minimizer.
- x_k^* is the k-th local minimizer.
- $f(x_k^*)$ is the function value of $f(x)$ in finding the k-th local minimizer.
- $F(x_k^*)$ is the function value of $F(x)$ in finding the k-th local minimizer.

Example 4.1 (*Test Problem 1 in [8]*).

$$f_1(x) = 1 - 2x_2 + 0.2\sin(4\pi x_2) - x_1 = 0,$$
$$f_2(x) = x_2 - 0.5\sin(2\pi x_1) = 0$$

where $-10 \le x_1, x_2 \le 10$. The solution is $(0.1025250, 0.3005036)$. The computational results are summarized in Table 4.1.

Example 4.2 (*Test Problem 14.1.3 in [9]*).

$$f_1(x) = 10^4 x_1 x_2 - 1 = 0,$$
$$f_2(x) = \mathrm{e}^{-x_1} + \mathrm{e}^{-x_2} - 1.001 = 0$$

where $5.49\times 10^{-6} \le x_1 \le 4.553, 2.196\times 10^{-3} \le x_2 \le 18.21$. The solution is $(1.450\times 10^{-5}, 6.8933353)$. The computational results are summarized in Table 4.2.

Example 4.3 (*Test Problem 14.1.4 in [9]*).

$$f_1(x) = 0.5\sin(x_1x_2) - 0.25x_2/\pi - 0.5x_1 = 0,$$
$$f_2(x) = (1 - 0.25/\pi)(\mathrm{e}^{2x_1} - e) + ex_2/\pi - 2ex_1 = 0$$

where $0.25 \le x_1 \le 1$, $1.5 \le x_2 \le 2\pi$. The solution is $(0.29945, 2.83693)$ and $(0.5, 3.14159)$. The computational results are summarized in Table 4.3.

Table 4.3
Numerical results for Example 4.3.

k	x_k	Local minimizer x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix}1.0000\\6.0000\end{pmatrix}$	$\begin{pmatrix}0.9717\\1.5610\end{pmatrix}$	6.1640×10^{-3}	$\begin{pmatrix}-0.1108\\6.8414 \times 10^{-3}\end{pmatrix}$
1	$\begin{pmatrix}0.30293\\3.83692\end{pmatrix}$	$\begin{pmatrix}0.50000\\3.14159\end{pmatrix}$	1.27661×10^{-15}	$\begin{pmatrix}3.39038 \times 10^{-8}\\-3.74668 \times 10^{-8}\end{pmatrix}$
0	$\begin{pmatrix}0.8000\\1.5000\end{pmatrix}$	$\begin{pmatrix}0.9717\\1.5610\end{pmatrix}$	1.2328×10^{-2}	$\begin{pmatrix}-0.1108\\-6.8459 \times 10^{-3}\end{pmatrix}$
1	$\begin{pmatrix}0.1505\\2.3834\end{pmatrix}$	$\begin{pmatrix}0.2994\\2.8369\end{pmatrix}$	8.3734×10^{-13}	$\begin{pmatrix}-8.4537 \times 10^{-9}\\-9.1502 \times 10^{-7}\end{pmatrix}$

Table 4.4
Numerical results for Example 4.4.

k	x_k	Local minimizer x_k^*	$f(x_k^*)$	$F(x_k^*)$
0	$\begin{pmatrix}30.0000\\30.0000\\30.0000\\30.0000\\30.0000\end{pmatrix}$	$\begin{pmatrix}0.0000\\15.50422\\0.0000\\0.0000\\0.0000\end{pmatrix}$	0.49976	$\begin{pmatrix}0.0000\\7.0035\\0.0000\\0.0000\\-0.99976\end{pmatrix}$
1	$\begin{pmatrix}4.0169 \times 10^{-9}\\15.5042\\2.52286 \times 10^{-6}\\1.59228 \times 10^{-3}\\1.0000\end{pmatrix}$	$\begin{pmatrix}3.4305 \times 10^{-3}\\31.3238\\6.8353 \times 10^{-2}\\0.8592\\3.6962 \times 10^{-2}\end{pmatrix}$	9.0569×10^{-14}	$\begin{pmatrix}3.6892 \times 10^{-7}\\-1.8673 \times 10^{-7}\\1.0065 \times 10^{-7}\\-4.8858 \times 10^{-9}\\-3.5825 \times 10^{-9}\end{pmatrix}$
0	$\begin{pmatrix}40.0000\\40.0000\\40.0000\\40.0000\\40.0000\end{pmatrix}$	$\begin{pmatrix}0.0000\\20.5481\\0.0000\\0.0000\\0.0000\end{pmatrix}$	0.4996	$\begin{pmatrix}0.0000\\1.2272 \times 10^{-3}\\0.000\\0.0000\\-0.9996\end{pmatrix}$
1	$\begin{pmatrix}4.0168 \times 10^{-9}\\20.5482\\2.5226 \times 10^{-6}\\1.5923 \times 10^{-3}\\1.0000\end{pmatrix}$	$\begin{pmatrix}3.4302 \times 10^{-3}\\31.3264\\6.8350 \times 10^{-2}\\0.8595\\3.6962 \times 10^{-2}\end{pmatrix}$	9.1852×10^{-17}	$\begin{pmatrix}5.2640 \times 10^{-9}\\-1.0104 \times 10^{-8}\\5.8779 \times 10^{-9}\\2.7078 \times 10^{-9}\\-3.4383 \times 10^{-9}\end{pmatrix}$

Example 4.4 (*Test Problem 14.1.2 in [9]*).

$$\begin{aligned}
f_1(x) &= x_1x_2 + x_1 - 3x_5 = 0,\\
f_2(x) &= 2x_1x_2 + x_1 + 3R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 + R_9x_2x_4 + R_8x_2 - Rx_5 = 0,\\
f_3(x) &= 2x_2x_3^2 + R_7x_2x_3 + 2R_5x_3^2 + R_6x_3 - 8x_5 = 0,\\
f_4(x) &= R_9x_2x_4 + 2x_4^2 - 4Rx_5 = 0,\\
f_5(x) &= x_1x_2 + x_1 + R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 + R_9x_2x_4 + R_8x_2 + R_5x_3^2 + R_6x_3 + x_4^2 - 1 = 0
\end{aligned}$$

where $0.0001 \le x_i \le 100, i = 1, \ldots, 5$, and $R = 10, R_5 = 0.193, R_6 = 4.10622 \times 10^{-4}, R_7 = 5.45177 \times 10^{-4}, R_8 = 4.4975 \times 10^{-7}, R_9 = 3.40735 \times 10^{-5}, R_{10} = 9.615 \times 10^{-7}$. The solution is (0.003431, 31.325636, 0.068352, 0.859530, 0.036963). The computational results are summarized in Table 4.4.

5. Conclusions

In this paper, a new global optimization approach based on the filled function method is proposed for finding solutions to nonlinear systems of equations. We transform given system into an optimization problem and construct a new filled function by employing some special properties of the transformed optimization problem. This approach allows us to reduce a currently found best value of objective function in each iteration. The computational results show the efficiency of the algorithm developed.

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