ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series A 115 (2008) 1235–1245
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Covering the alternating groups by products of cycle classes

Marcel Herzog [a], Gil Kaplan [b], Arieh Lev [b]

[a] *School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv, Israel*
[b] *School of Computer Sciences, The Academic College of Tel-Aviv-Yafo, 2 Rabenu Yeruham St., Tel-Aviv 61083, Israel*

Received 28 January 2007
Available online 18 March 2008

Abstract

Given integers $k, l \geqslant 2$, where either l is odd or k is even, we denote by $n = n(k, l)$ the largest integer such that each element of A_n is a product of k cycles of length l. For an odd l, k is the diameter of the undirected Cayley graph $Cay(A_n, \mathcal{C}_l)$, where $\mathcal{C}_l$ is the set of all l-cycles in A_n. We prove that if $k \geqslant 2$ and $l \geqslant 9$ is odd and divisible by 3, then $\frac{2}{3}kl \leqslant n(k, l) \leqslant \frac{2}{3}kl + 1$. This extends earlier results by Bertram [E. Bertram, Even permutations as a product of two conjugate cycles, J. Combin. Theory 12 (1972) 368–380] and Bertram and Herzog [E. Bertram, M. Herzog, Powers of cycle-classes in symmetric groups, J. Combin. Theory Ser. A 94 (2001) 87–99].

Keywords: Alternating groups; Products of cycles; Covering number

1. Introduction

Let A_n be the group of all even permutations on n letters. Given integers $k, l \geqslant 2$, we ask for the largest integer $n = n(k, l)$ such that every permutation in A_n is a product of k cycles of length l. By the definition of A_n, $n(k, l)$ exists only if either l is odd or k is even. E. Bertram solved the problem for $k = 2$ in 1972 (see also [6] for another proof of this result). He proved the following theorem:

E-mail address: herzogm@post.tau.ac.il (M. Herzog).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2008.01.010

Theorem 1.1. *(See [2, Corollary 2.1].) Each permutation in the alternating group A_n, $n \geqslant 2$, is a product of* two *l-cycles in S_n if and only if either $\lfloor \frac{3n}{4} \rfloor \leqslant l \leqslant n$ or $n = 4$ and $l = 2$.*

It follows from Theorem 1.1 that if $k = 2$ and $l > 2$, then $n(2, l)$ equals the largest integer n satisfying $\lfloor \frac{3n}{4} \rfloor = l$. Suppose that $l = 3d + e$, where $e \in \{0, 1, 2\}$ and let $n = \lfloor \frac{4l}{3} \rfloor + 1 = 4d + e + 1$. Then $\lfloor \frac{3n}{4} \rfloor = \lfloor 3d + \frac{3}{4}(e + 1) \rfloor = 3d + e = l$, but $\lfloor \frac{3}{4}(n + 1) \rfloor = \lfloor 3d + \frac{3}{4}(e + 2) \rfloor = l + 1$. Hence

$$n(2, l) = \left\lfloor \frac{4l}{3} \right\rfloor + 1 = \left\lfloor \frac{2}{3}kl \right\rfloor + 1.$$

E. Bertram and M. Herzog proceeded by solving the problem for $k = 3$ and $k = 4$. They proved:

Theorem 1.2. *(See [3, Theorem 2].) Each $\sigma \in A_n$, $n \geqslant 1$, is a product of* three *l-cycles in S_n if and only if l is odd and either $\lceil \frac{n}{2} \rceil \leqslant l \leqslant n$ or $n = 7$ and $l = 3$.*

Theorem 1.3. *(See [3, Theorem 3].) Each $\sigma \in A_n$, $n \geqslant 2$, is a product of* four *l-cycles in S_n if and only if:*

(1) $\lceil \frac{3n}{8} \rceil \leqslant l \leqslant n$ *if* $n \not\equiv 1 \pmod 8$;
(2) $\lfloor \frac{3n}{8} \rfloor \leqslant l \leqslant n$ *if* $n \equiv 1, 0 \pmod 8$;
(3) $n = 6$ *and* $l = 2$.

It follows from these results that if $2 \leqslant k \leqslant 4$ and l is odd if $k = 3$, then $\lfloor \frac{2}{3}kl \rfloor \leqslant n(k, l) \leqslant \lfloor \frac{2}{3}kl \rfloor + 1$. Bertram and Herzog conjectured in [3] that $n(k, l) \approx \frac{2}{3}kl$ for every $k, l \geqslant 2$, provided that either l is odd or k is even. In the spirit of their conjecture, we conjecture the following:

Conjecture 1.1. *Let $k, l \geqslant 2$ be integers and assume that either l is odd or k is even. Then $\lfloor \frac{2}{3}kl \rfloor \leqslant n(k, l) \leqslant \lfloor \frac{2}{3}kl \rfloor + 1$.*

We note that by [2,3], $n(k, l) = \frac{2}{3}kl + 1$ when $k = 2, 4$ and $3 \mid l$. Hence we also conjecture the following:

Conjecture 1.2. *Let k, l be positive integers and assume that k is even and $3 \mid l$. Then $n(k, l) = \frac{2}{3}kl + 1$.*

In this paper we prove the validity of Conjecture 1.1 for every integer $k \geqslant 2$ and every odd integer $l \geqslant 9$ divisible by 3. Our main result is the following theorem:

Theorem 3.4. *Let k and l be integers such that $k \geqslant 2$ and $l \geqslant 9$ is odd and divisible by 3. Then $\frac{2}{3}kl \leqslant n(k, l) \leqslant \frac{2}{3}kl + 1$. Furthermore, if k is odd, then $n(k, l) = \frac{2}{3}kl$.*

The upper bound for $n(k, l)$ in Theorem 3.4 follows from the following more general result, which does not require l being odd, $l \geqslant 9$ and $3 \mid l$, but only $l > 2$ and either l is odd or k is even.

Theorem 3.3. *Let k, l be natural numbers such that $k \geqslant 2$ and $l > 2$. Suppose that either l is odd or k is even. Denote $n_1 = \lfloor \frac{2kl}{3} \rfloor$ and $\delta = \frac{2kl}{3} - n_1$. Then:*

(1) *If* $n_1 \equiv 3 \pmod 4$*, then* $n(k,l) \leqslant n_1$;
(2) *If* $n_1 \equiv 1 \pmod 4$*, then* $n(k,l) \leqslant n_1 + 1$;
(3) *If* $n_1 \equiv 2 \pmod 4$*, then* $n(k,l) \leqslant n_1 + 1$; *if we further assume that* $l > 3$ *and* $\delta \in \{0, \frac{1}{3}\}$*, then* $n(k,l) \leqslant n_1$;
(4) *If* $n_1 \equiv 0 \pmod 4$*, then* $n(k,l) \leqslant n_1 + 1$.

The above results are closely related to problems on covering groups by products of conjugacy classes. We recall that for a group G and an element x in G, the *conjugacy class* of x in G is $C = x^G = \{g^{-1}xg \mid g \in G\}$. The *covering number* $cn(C)$ of a conjugacy class C of G is the least integer m (if it exists) such that $C^m = G$, where $C^m = \{c_1c_2\cdots c_m \mid c_1, c_2, \ldots, c_m \in C\}$. The *covering number* of a group G, $cn(G)$, is the least integer n (if it exists) such that $C^n = G$ for every non-trivial conjugacy class of G. We note that $cn(G)$ does not necessarily exist for an arbitrary group, but it exists whenever G is a finite non-abelian simple group [1]. The covering numbers for the groups A_n and S_n were extensively studied by Brenner et al. (see [4] for a survey), Dvir [5], Vishne [11] and many others. In particular, since the set of all cycles of a given odd length l ($2 \leqslant l < n-1$) constitutes a conjugacy class of A_n (see [10, 11.1.5]), our results (as well as the results in [2,3,6]) deal with the covering numbers of these classes of l-cycles in A_n. The covering numbers for various groups other then A_n (or S_n) were also extensively studied. See [9] for a recent survey.

We also note that for an odd l and $n = n(k,l)$, k is the diameter of the undirected Cayley graph $Cay(A_n, \mathcal{C}_l)$, where $\mathcal{C}_l$ is the set of all l-cycles in A_n. For related results, see [8].

Most of our notation is standard. The positive integers are denoted by $\mathbb{N}$. If $\Omega = \{1, 2, \ldots, n\}$, then S_n denotes the symmetric group on Ω, and A_n denotes the alternating group on Ω. Products of permutations will be executed *from left to right*. Suppose, first, that $\sigma \in S_n - \{1\}$. Then $\mathrm{supp}(\sigma)$, *the support of* σ, is the set $\{i \in \Omega \mid \sigma(i) \neq i\}$ and $dcd*(\sigma)$, *a non-trivial disjoint cycle decomposition* of σ, denotes a representation of σ as a product of disjoint cycles of length > 1. It is well known that $dcd*(\sigma)$ is unique, except for a cyclic shift within the cycles and the order in which the cycles are written. We call $\sigma, \rho \in S_n$ *disjoint permutations on* n_1 *and* n_2 *letters*, respectively, if $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\rho) = \emptyset$, $|\mathrm{supp}(\sigma)| \leqslant n_1$ and $|\mathrm{supp}(\rho)| \leqslant n_2$. We denote $m_\sigma = |\mathrm{supp}(\sigma)|$ and the number of (non-trivial) cycles in $dcd*(\sigma)$ is denoted by n_σ. For $\sigma = 1$, we define $dcd*(1) = (1)$ and $m_1 = n_1 = 1$. If G is a subgroup of S_n, we denote by $\mathrm{supp}(G)$ the subset of letters in Ω which are moved by at least one element of G. If q is a rational number, then $\lfloor q \rfloor = k$, where k is the unique integer satisfying $k \leqslant q < k+1$, and $\lceil q \rceil = k$, where k is the unique integer satisfying $q \leqslant k < q+1$.

The lower bound in Theorem 3.4 is proved in Section 2. Theorem 3.3 (which, in particular, provides the upper bound for Theorem 3.4) and Theorem 3.4 are proved in Section 3.

2. Lower bound for $n(k,l)$

Definition 2.1. Let m, n be even integers satisfying $2 \leqslant m \leqslant n-2$. A partition $A = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ of n (i.e. $t, \alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant t$ and $n = \sum_{i=1}^{t} \alpha_i$), with $\alpha_i \geqslant 2$ for all i is called (n,m)*-indecomposable* if there does not exist a subset B of A such that $\sum_{\alpha_i \in B} \alpha_i = m$. Notice that A is (n,m)-indecomposable if and only if it is $(n, n-m)$-indecomposable. If such B does exist, then the partition A will be called (n,m)*-decomposable*.

The following Lemma 2.1 is a particular case of Theorem A in [7] (set $r \geqslant 2$ in that theorem). However, for completeness sake, a proof of the lemma is given below. Lemma 2.1 and Corollary 2.2, together with the auxiliary Theorem 2.6, will play a key role in proving Theorem 2.7 at the end of this section.

Lemma 2.1. *Let m, n be even integers satisfying $2 \leqslant m \leqslant n-2$ and let $A = \{\alpha_1, \ldots, \alpha_t\}$ be a partition of n which is (n,m)-indecomposable. Then $t \leqslant \frac{n}{3}$, unless $A = \{2, 3, \ldots, 3\}$, in which case $t = \frac{n+1}{3}$.*

Proof. We may assume in this proof that the following ordering holds: $2 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_t$. If $m = 2$ or $n - m = 2$, then $\alpha_i \geqslant 3$ for all i and $t \leqslant \frac{n}{3}$, as claimed. If $m = 4$ or $n - m = 4$ and $t > \frac{n}{3}$, then $\alpha_1 = 2$ and $\alpha_j \geqslant 3$ for all $j > 1$. If $\alpha_j \geqslant 4$ for some j, then $t \leqslant \frac{n}{3}$, a contradiction. So $\alpha_j = 3$ for all $j > 1$ and $A = \{2, 3, \ldots, 3\}$. Moreover, $t = \frac{n-2}{3} + 1 = \frac{n+1}{3}$, so the theorem holds in this case too.

Suppose that $m \geqslant 6$ and $n - m \geqslant 6$, which implies that $n \geqslant 12$. We shall complete the proof by induction on n. Since m is even, $\alpha_i > 2$ for some i. Since n is even, one of the following holds: (i) α_r is even and $\alpha_r \geqslant 4$ for some r; (ii) condition (i) does not hold and there exist distinct integers r and s such that α_r, α_s are odd and $\alpha_r + \alpha_s \geqslant 6$.

Denote $\beta = \alpha_r$ in case (i) and $\beta = \alpha_r + \alpha_s$ in case (ii). Clearly β and $n - \beta$ are even numbers. Suppose, first, that $\beta \geqslant \frac{n}{2} + 2$. Then $t \leqslant \frac{n-\beta}{2} + 2 \leqslant \frac{n}{4} + 1 \leqslant \frac{n}{3}$, where the final inequality follows since $n \geqslant 12$.

Suppose, now, that either $\beta = \frac{n}{2}$ or $\beta = \frac{n}{2} + 1$. Since β and m are even, there exists $\alpha_i \geqslant 3$ for some $i \neq r$ in case (i) and for some $i \neq r, s$ in case (ii). But $n - \beta$ is even, so there exists α_j for some $j \neq i, r$ in case (i) and for some $j \neq i, r, s$ in case (ii), which satisfies $\alpha_i + \alpha_j \geqslant 6$. Hence

$$t \leqslant \frac{\frac{n}{2} - 6}{2} + 4 = \frac{n}{4} + 1 \leqslant \frac{n}{3}$$

as $n \geqslant 12$. So we are done in this case too.

Suppose, finally, that $\beta < \frac{n}{2}$. Then either $m > \beta$ or $n - m > \beta$. So assume, without loss of generality, that $m > \beta$. Denote $m_1 = m - \beta$, $n_1 = n - \beta$ and let A_1 be the partition of n_1 obtained from A by deleting the components of β. Then A_1 is a partition of n_1 which is (n_1, m_1)-indecomposable, n_1, m_1 are even integers and $2 \leqslant m_1 \leqslant n_1 - 2$. Let t_1 be the number of summands in A_1; clearly either $t = t_1 + 1$ or $t = t_1 + 2$.

Suppose that $A_1 = \{2, 3, \ldots, 3\}$. If $\beta = a_r \geqslant 4$, then

$$t = t_1 + 1 = \frac{n - \beta - 2}{3} + 2 \leqslant \frac{n - 4 + 4}{3} = \frac{n}{3}$$

as required. If $\beta = \alpha_r + \alpha_s = 6$, then $\alpha_r = \alpha_s = 3$, $A = \{2, 3, \ldots, 3\}$ and $t = \frac{n-2}{3} + 1 = \frac{n+1}{3}$, as required. If $\beta = \alpha_r + \alpha_s > 6$, then $t = t_1 + 2 = \frac{n-\beta-2}{3} + 3 \leqslant \frac{n-8-2+9}{3} < \frac{n}{3}$, as required.

If $A_1 \neq \{2, 3, \ldots, 3\}$ and $\beta = a_r \geqslant 4$, then by induction $t = t_1 + 1 \leqslant \frac{n-\beta}{3} + 1 \leqslant \frac{n-4+3}{3} < \frac{n}{3}$, as required. If $\beta = \alpha_r + \alpha_s \geqslant 6$, then by induction $t = t_1 + 2 \leqslant \frac{n-\beta}{3} + 2 \leqslant \frac{n-6+6}{3} = \frac{n}{3}$, again as required. The proof is complete. □

Using Lemma 2.1, we obtain

Corollary 2.2. *Let n be an even integer and let $\sigma \in S_n$, satisfying $n_\sigma > \frac{n+1}{3}$. Then for each even integer m satisfying $2 \leqslant m \leqslant n-2$, there exist non-trivial permutations ρ and ϕ in S_n such that $\sigma = \rho\phi$, $\mathrm{supp}(\rho) \cap \mathrm{supp}(\phi) = \emptyset$, $|\mathrm{supp}(\rho)| \leqslant m$ and $|\mathrm{supp}(\phi)| \leqslant n - m$.*

Proof. Denote $t = n_\sigma$. Since $t > \frac{n+1}{3} \geqslant 1$, $\sigma \neq 1$. Let $\sigma = (1)(2)(3)\ldots(f)C_{\alpha_1}C_{\alpha_2}\ldots C_{\alpha_t}$, where $\{1, 2, 3, \ldots, f\}$ are the distinct fixed points of σ (this set can be empty) and the C_{α_i} are disjoint cycles of length α_i with $\alpha_i \geqslant 2$ for each i, ordered in such a way that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_t \geqslant 2$. Then $A = \{\alpha_1 + f, \alpha_2, \ldots, \alpha_t\}$ is a partition of the even integer n, with all components larger than 1 and $t > \frac{n+1}{3}$. Fix an even m such that $2 \leqslant m \leqslant n-2$. It follows then by Lemma 2.1, that the partition A is (n,m)-decomposable. Let $A = A_1 \cup A_2$ be a decomposition of A into two subpartitions of m and of $n-m$, respectively. Define $\rho = \prod_{\alpha_i \in A_1} C_i$ and $\phi = \prod_{\alpha_i \in A_2} C_i$, where $\alpha_1 \in A_j$ means $\alpha_1 + f \in A_j$ for $j = 1$ or 2. Then ρ and ϕ are non-trivial permutations in S_n satisfying $\operatorname{supp}(\rho) \cap \operatorname{supp}(\phi) = \emptyset$, $|\operatorname{supp}(\rho)| \leqslant m$ and $|\operatorname{supp}(\phi)| \leqslant n-m$, as required. □

The next lemma follows immediately from the following result in [6].

Theorem 2.3. *(See [6, Theorem 7].) Let $\sigma \in S_n$ and let $l_1, l_2 \in \mathbb{N}$, $n \geqslant l_1 \geqslant l_2 \geqslant 2$. Then $\sigma = C_1C_2$, where C_1, C_2 are cycles in S_n of lengths l_1, l_2, respectively, if and only if either $n_\sigma = 2$, l_1, l_2 are the lengths of the cycles in $dcd*(\sigma)$ and $l_1 + l_2 = m_\sigma$, or the following conditions hold*:

(1) $l_1 + l_2 = m_\sigma + n_\sigma + 2s$ *for some* $s \in \mathbb{N} \cup \{0\}$, *and*
(2) $l_1 - l_2 \leqslant m_\sigma - n_\sigma$.

Lemma 2.4. *Let $n, l_1, l_2, m \in \mathbb{N}$, satisfy $n \geqslant l_1 \geqslant l_2 \geqslant 2$ and let σ be an m-cycle in S_n. Then there exist in S_n cycles C_1, C_2 of sizes l_1, l_2, respectively, such that $\sigma = C_1C_2$, if and only if*

$$m = l_1 + l_2 - (2s-1) \leqslant n$$

for some $s \in \mathbb{N}$, $s \leqslant l_2$.

Proof. By Theorem 2.3, such C_1, C_2 exist if and only if

(1) $m \leqslant n$,
(2) $m = l_1 + l_2 - (2s-1)$ for some $s \in \mathbb{N}$, and
(3) $m \geqslant l_1 - l_2 + 1$.

Conditions (2) and (3) are clearly equivalent to conditions (2) and

(3′) $s \leqslant l_2$

and the lemma follows. □

We prove now a generalization of Lemma 2.4 for odd l_i's and m.

Lemma 2.5. *Let $n, t \in \mathbb{N}$ and let $l_1, l_2, \ldots, l_t, m$ be odd integers satisfying $n \geqslant l_1 \geqslant l_2 \geqslant \cdots \geqslant l_t \geqslant 3$ and*

$$l_1 \leqslant m \leqslant \min\left(l_1 + \sum_{l=2}^{t}(l_i - 1), n\right).$$

Then for each m-cycle $\sigma \in S_n$ there exist in S_n cycles C_i, $1 \leqslant i \leqslant t$, of sizes l_i, respectively, such that $\sigma = C_1C_2\cdots C_t$.

Proof. We shall prove Lemma 2.5 by induction on t. If $t = 1$, then the result is trivial. So assume that $t \geqslant 2$ and the result holds for $t - 1$. Set $m' = \min(l_1 + \sum_{i=2}^{t-1}(l_i - 1), m)$ and note that m' is odd and $m' \geqslant l_1$ since $m \geqslant l_1$.

Assume first that $m > l_1 + \sum_{i=2}^{t-1}(l_i - 1)$, which implies that $m' = l_1 + \sum_{i=2}^{t-1}(l_i - 1)$. Then

$$m' < m \leqslant l_1 + \sum_{l=2}^{t}(l_i - 1) = m' + l_t - 1$$

and since m, m' and l_t are odd, it follows that $m = m' + l_t - (2s - 1)$ for some $s \in \mathbb{N}$, $s \leqslant l_t$. Thus, by Lemma 2.4, there exist in S_n cycles τ and C_t of lengths m' and l_t, respectively, such that $\sigma = \tau C_t$. Since $l_1 \leqslant m' \leqslant \min(l_1 + \sum_{i=2}^{t-1}(l_i - 1), m)$ and $m \leqslant n$, it follows by the induction hypothesis that there exist in S_n cycles $C_1, \ldots, C_{t-1}$ such that $\tau = C_1 \cdots C_{t-1}$ and the result follows.

It remains to prove the lemma in the case that $l_1 \leqslant m \leqslant l_1 + \sum_{i=2}^{t-1}(l_i - 1)$. Then, by the induction hypothesis, there exist in S_n cycles $C_1, C_2, \ldots, C_{t-2}, C'_{t-1}$ of lengths $l_1, l_2, \ldots, l_{t-1}$, respectively, such that $\sigma = C_1 C_2 \cdots C_{t-2} C'_{t-1}$. Since the l_i are odd integers, it follows again by Lemma 2.4 that there exist in S_n cycles C_{t-1}, C_t of lengths l_{t-1}, l_t, respectively, such that $C'_{t-1} = C_{t-1} C_t$, and the lemma follows. $\square$

For our auxiliary Theorem 2.6 we need the following definition.

Definition 2.2. Let $k, l, n \in \mathbb{N}$ such that $k, l \geqslant 2$ and $n \geqslant l$. Denote by $C(l)$ the set of all cycles of length l in S_n and by $P(k, l; n)$ the set of all permutations in S_n which may be written as a product of k cycles of length l.

Theorem 2.6. *Let $k, l, n \in \mathbb{N}$ be such that $k \geqslant 2$ and l is odd and suppose that $9 \leqslant l \leqslant n \leqslant \frac{2}{3}kl + 1$. Moreover, let $\sigma \in A_n$ and suppose that $n_\sigma \leqslant \frac{n+2}{3}$ if $k \geqslant 3$. Then $\sigma \in P(k, l; n)$.*

Proof. Since n is fixed, we shall denote $P(i, j; n)$ by $P(i, j)$. If $k = 2$, then $n \leqslant \frac{4}{3}l + 1$, which implies that $l \geqslant \lceil \frac{3}{4}(n - 1) \rceil \geqslant \lfloor \frac{3}{4}n \rfloor$ and, by Theorem 1.1, $\sigma \in P(2, l)$. So assume that $k \geqslant 3$ and $n_\sigma \leqslant \frac{n+2}{3}$.

For the continuation of the proof, we need the following three observations:

O_1. $P(r, l) \subseteq P(t, l)$ if $r \leqslant t$.
O_2. $P(k, l) \subseteq P(k, l + 2)$ if $l + 2 \leqslant n$.
O_3. $m_\sigma + n_\sigma$ is even for all $\sigma \in A_n$.

Concerning O_1, we notice that if $r \in \mathbb{N}$ and $C_1, C_2, \ldots, C_r \in C(l)$, then, since l is odd, also $C_r^{-1}, C_r^2 \in C(l)$ and $C_1 C_2 \cdots C_r = C_1 C_2 \cdots C_{r-1} C_r^2 C_r^{-1}$. Observation O_2 follows from Proposition 15 in [3]. Finally, observation O_3 follows from the fact that if $l > 1$, then each l-cycle can be decomposed into $l - 1$ transpositions. Thus any $\sigma \in S_n - \{1\}$ can be decomposed into $m_\sigma - n_\sigma$ transpositions and if $\sigma \in A_n$, then this number, and hence also $m_\sigma + n_\sigma$, must be even.

We continue now with our proof. If $\sigma = 1$, then clearly $\sigma \in P(2, l)$ and, by O_1, $\sigma \in P(k, l)$. So assume that $\sigma \in A_n - \{1\}$, which implies that $m_\sigma \geqslant 3$ and $n_\sigma \geqslant 1$.

Suppose, first, that $l \leqslant m_\sigma \leqslant 2l - 1$. Then $m_\sigma \geqslant l \geqslant \frac{m_\sigma + 1}{2} \geqslant \lceil \frac{m_\sigma}{2} \rceil$ and as l is odd and $l \geqslant 9$, it follows by Theorem 1.2 that $\sigma \in P(3, l)$. As $k \geqslant 3$, O_1 implies that $\sigma \in P(k, l)$, as required.

Suppose, next, that $3 \leqslant m_\sigma < l$. Then either m_σ or $m_\sigma - 1$ is an odd integer, say l_1, and since $m_\sigma \geqslant l_1 \geqslant m_\sigma - 1 \geqslant \lceil \frac{m_\sigma}{2} \rceil$, it follows by Theorem 1.2 that $\sigma \in P(3, l_1)$. Hence, by O_1, $\sigma \in P(k, l_1)$ and since both l and l_1 are odd and $l_1 < l$, it follows by O_2 that $\sigma \in P(k, l)$.

So assume that $m_\sigma \geqslant \max(2l, 2n_\sigma)$. It follows that

$$m_\sigma - n_\sigma \geqslant \max(2l - n_\sigma, n_\sigma) \geqslant l > l - 1.$$

Clearly $m_\sigma \leqslant n$. Hence, by our assumptions, we have

$$\begin{aligned} m_\sigma + n_\sigma &\leqslant n + \frac{n+2}{3} = \frac{4}{3}n + \frac{2}{3} \leqslant \frac{4}{3}\left(\frac{2}{3}kl + 1\right) + \frac{2}{3} = \frac{8}{9}kl + 2 \\ &= 2l + \frac{l}{l-1}\left(\frac{8}{9}k - 2\right)(l-1) + 2 \leqslant 2l + \frac{9}{8}\left(\frac{8}{9}k - 2\right)(l-1) + 2 \\ &= 2l + (k-2)(l-1) - \frac{l-1}{4} + 2 \\ &\leqslant 2l + (k-2)(l-1). \end{aligned}$$

Let $s \in \mathbb{N}$ be minimal such that

$$m_\sigma + n_\sigma \leqslant 2l + (s-2)(l-1). \tag{3}$$

Clearly $s \leqslant k$. Since $m_\sigma \geqslant 2l$ and $n_\sigma \geqslant 1$, it follows that $3 \leqslant s \leqslant k$.

If $s = 3$, then $m_\sigma + n_\sigma \leqslant 3l - 1 = l_1 + l$, where $l_1 = 2l - 1$. Since $\sigma \in A_n$ and $l \geqslant 9$ is odd, it follows, in view of O_3, that both sides of the above inequality are even and $l < l_1 < m_\sigma \leqslant n$. As shown above, $m_\sigma - n_\sigma > l - 1 = l_1 - l$, and it follows by Theorem 2.3 that $\sigma = C_1C_2$, with $C_1 \in C(l_1)$ and $C_2 \in C(l)$. By Lemma 2.4, $C_1 \in P(2, l)$ and we may conclude, in view of O_1, that $\sigma \in P(3, l) \subseteq P(k, l)$, as required.

So suppose that $s \geqslant 4$. It follows from inequality (3) that

$$m_\sigma + n_\sigma \leqslant \left[l + \left(\left\lceil \frac{s}{2} \right\rceil - 1\right)(l-1)\right] + \left[l + \left(\left\lfloor \frac{s}{2} \right\rfloor - 1\right)(l-1)\right] = l_1 + l_2 \tag{4}$$

with the obvious notation. By our assumptions and the minimality of s, we must have $\frac{4}{3}n + \frac{2}{3} \geqslant m_\sigma + n_\sigma > 2l + (s-3)(l-1)$ and since l is odd, it follows that $n > \frac{3}{2}l + \frac{3}{4}(s-3)(l-1) - \frac{1}{2} = \frac{m-1}{2}$ for some integer m. Hence $n \geqslant \frac{3}{2}l + \frac{3}{4}(s-3)(l-1)$ and if $l_1 > n$, then

$$l + \left(\left\lceil \frac{s}{2} \right\rceil - 1\right)(l-1) > \frac{3}{2}l + \frac{3}{4}(s-3)(l-1). \tag{5}$$

This implies that $\frac{s-1}{2} > \frac{3}{4}(s-3) + \frac{1}{2}$, whence $2s - 2 > 3s - 9 + 2$ and $s < 5$, leaving us with the case $s = 4$. But then inequality (5) implies that $l + (l-1) > \frac{3}{2}l + \frac{3}{4}(l-1)$, or $-1 > l$, a contradiction. Hence $n \geqslant l_1 \geqslant l_2$ and $l_1 - l_2 \leqslant l - 1 < m_\sigma - n_\sigma$. As, in view of O_3, $m_\sigma + n_\sigma$ and $l_1 + l_2$ are both even, it follows by Theorem 2.3 that $\sigma = C_1C_2$, with $C_1 \in C(l_1)$ and $C_2 \in C(l_2)$. In view of the definition of l_1 and l_2, it follows by Lemma 2.5 that $C_1 \in P(\lceil \frac{s}{2} \rceil, l)$ and $C_2 \in P(\lfloor \frac{s}{2} \rfloor, l)$. Thus $\sigma \in P(s, l)$ and since $s \leqslant k$, it follows by O_1 that $\sigma \in P(k, l)$, as required. The proof is complete. $\square$

Theorem 2.7. *Let $k, l \in \mathbb{N}$ be such that $k \geqslant 2$ and $l \geqslant 9$ is odd and divisible by 3. Moreover, let $n = \frac{2kl}{3}$ and $\sigma \in A_n$. Then $\sigma \in P(k, l; n)$.*

Proof. If $k=2$, then $l=\frac{3}{4}n$ and $\sigma \in P(2,l;n)$ by Theorem 1.1.

If $k=3$, then $l=\frac{n}{2}=\lceil\frac{n}{2}\rceil$ and $\sigma \in P(3,l;n)$ by Theorem 1.2.

If $k=4$, then $l=\frac{3}{8}n=\lceil\frac{3}{8}n\rceil$ and $\sigma \in P(4,l;n)$ by Theorem 1.3.

So suppose that $k \geqslant 5$. If $n_\sigma \leqslant \frac{n+2}{3}$, then $\sigma \in P(k,l;n)$ by Theorem 2.6. So we may also assume that $n_\sigma > \frac{n+2}{3}$. Denote $t=n_\sigma$.

Let m be an even integer satisfying $2 \leqslant m \leqslant n-2$ and let A be the partition of n obtained from the disjoint cycle decomposition of σ as in the proof of Corollary 2.2. Since $t > \frac{n+2}{3}$, it follows by Lemma 2.1 that A is (n,m)-decomposable. In particular, A is $(\frac{2}{3}kl, \frac{4}{3}l)$-decomposable, and as shown in the proof of Corollary 2.2, there exist non-trivial disjoint permutations ρ and ϕ on $\frac{2}{3}(k-2)l$ and $\frac{4}{3}l$ letters, respectively, such that $\sigma=\rho\phi$.

We proceed by induction on k. If ρ and ϕ are both even, then, by induction, $\rho \in P(k-2,l;\frac{2}{3}(k-2)l)$ and $\phi \in P(2,l;\frac{4}{3}l)$, which implies that $\sigma \in P(k,l;n)$, as required. Since $\sigma \in A_n$, it remains to deal with the case when both ρ and ϕ are odd.

If $n_\rho \leqslant \frac{\frac{2}{3}(k-2)l+2}{3}$, let τ be the transposition $\tau=(u,v)$, where $u \in \operatorname{supp}(\rho)$ and $v \in \operatorname{supp}(\phi)$, and let $\rho^*=\rho\tau$ and $\phi^*=\tau\phi$. Then ρ^* and ϕ^* are even permutations on $\frac{2}{3}(k-2)l+1$ and $\frac{4}{3}l+1$ letters, respectively, $\sigma=\rho^*\phi^*$ and $n_{\rho^*}=n_\rho \leqslant \frac{\frac{2}{3}(k-2)l+2}{3}$. By Theorem 2.6, $\rho^* \in P(k-2,l;\frac{2}{3}(k-2)l+1)$. Moreover, since $\lfloor\frac{3}{4}(\frac{4}{3}l+1)\rfloor=l$, it follows by Theorem 1.1 that $\phi^* \in P(2,l;\frac{4}{3}l+1)$. Consequently, $\sigma \in P(k,l;n)$, as required. So we may assume that $n_\rho > \frac{\frac{2}{3}(k-2)l+2}{3}$. It follows then, by Corollary 2.2, that we can write $\rho=\rho_1\rho_2$, where ρ_1,ρ_2 are non-trivial disjoint permutations of opposite parity on $\frac{2}{3}(k-4)l$ and $\frac{4}{3}l$ letters, respectively. Now, if ρ_1 is odd, then $\rho_1\phi$ and ρ_2 are even permutations on $\frac{2}{3}(k-2)l$ and $\frac{4}{3}l$ letters, respectively, and hence, by induction, $\rho_1\phi \in P(k-2,l;\frac{2}{3}(k-2)l)$ and $\rho_2 \in P(2,l;\frac{4}{3}l)$. Since $\sigma=(\rho_1\phi)\rho_2$, $\sigma \in P(k,l;n)$ as required (notice that the permutations ρ_1,ρ_2 and ϕ commute). So we may assume that ρ_1 is even and ρ_2 is odd.

Suppose, first, that $k=5$. Then $n=\frac{10l}{3}$, ρ is an odd permutation on $2l$ letters and ϕ is an odd permutation on $\frac{4}{3}l$ letters. Moreover, $\rho=\rho_1\rho_2$, where ρ_1 is an even permutation on $\frac{2}{3}l$ letters and ρ_2 is an odd permutation on $\frac{4}{3}l$ letters. If either $n_\phi > \frac{\frac{4}{3}l+1}{3}$ or $n_{\rho_2} > \frac{\frac{4}{3}l+1}{3}$, assume, without loss of generality, that $n_\phi > \frac{\frac{4}{3}l+1}{3}$. By Corollary 2.2 we can write $\phi=\phi_1\phi_2$, where ϕ_1 is a non-trivial even permutation on $\frac{2}{3}l$ letters and ϕ_2 is a non-trivial odd permutation on the remaining $\frac{2}{3}l$ letters. Define $\beta=\phi_1\rho_1$ and $\gamma=\phi_2\rho_2$. Then β and γ are even permutations on $\frac{4}{3}l$ and $2l$ letters, respectively, and $\sigma=\beta\gamma$. By induction we get $\beta \in P(2,l;\frac{4}{3}l)$, $\gamma \in P(3,l;2l)$ and hence $\sigma \in P(5,l;n)$, as required. Thus we assume that $n_\phi \leqslant \frac{\frac{4}{3}l+1}{3}$ and $n_{\rho_2} \leqslant \frac{\frac{4}{3}l+1}{3}$. Choose $u \in \operatorname{supp}(\rho_1)$ and $v \in \operatorname{supp}(\rho_2)$ and let $\tau=(u,v)$. Denote $\rho^*=\rho\tau$, an even permutation on $2l$ letters, and denote $\phi^*=\tau\phi$, an even permutation on $\frac{4}{3}l+2$ letters. Then, by induction, $\rho^* \in P(3,l;2l)$ and, by Theorem 2 in [2], $\phi^* \in P(2,l;\frac{4}{3}l+2)$ provided that

$$\frac{m_{\phi^*}+n_{\phi^*}}{2} \leqslant \frac{\frac{4}{3}l+2+n_\phi+1}{2} \leqslant \frac{\frac{4}{3}l+2+\frac{\frac{4}{3}l+1}{3}+1}{2}=\frac{16l+30}{18} \leqslant l$$

which holds for $l \geqslant 15$. So it remains only to deal with the case $l=9$. In this case, $n=\frac{2}{3}\cdot 5\cdot 9=30$, ϕ and ρ_2 each acts on 12 letters and ρ_1 acts on 6 letters. Moreover, $n_\phi, n_{\rho_2} \leqslant \frac{12+1}{3}$, which implies $n_\phi, n_{\rho_2} \leqslant 4$. Furthermore, since ρ_1 is even, we obtain $n_{\rho_1} \leqslant 2$. Thus $n_\sigma = n_{\rho_1} + n_{\rho_2} + n_\phi \leqslant 2+4+4 = 10 \leqslant \frac{30}{3}$ and $\sigma \in P(5, l; n)$ by Theorem 2.6.

So assume that $k \geqslant 6$. As ρ_1 is an even permutation on $\frac{2}{3}(k-4)l$ letters, with $k-4 \geqslant 2$, and $\phi\rho_2$ is an even permutation on $\frac{8}{3}l$ letters, it follows by induction that $\rho_1 \in P(k-4, l; \frac{2}{3}(k-4)l)$ and $\phi\rho_2 \in P(4, l; \frac{8}{3}l)$. But $\sigma = \rho_1(\phi\rho_2)$, so it follows that $\sigma \in P(k, l; n)$, as required. The proof is complete. $\square$

3. Upper bound for $n(k, l)$

Our aim in this section is to prove Theorem 3.3 below, which provides bounds from above on $n(k, l)$, and use it for the proof of Theorem 3.4. We define and discuss first the important notion of *movements* of a permutation.

Let $\sigma \in S_n$ be a permutation on the set $\Omega = \{1, 2, \dots, n\}$. For distinct $u, v \in \Omega$ we shall say that (u, v) is a *movement* of σ if $u^\sigma = v$. Clearly σ is completely determined by the set of all its movements, which will be denoted by R_σ. Let $\sigma = C_1 C_2 \cdots C_r$, where $C_1, C_2, \dots, C_r$ are arbitrary cycles in S_n, and let $(u, v) \in R_\sigma$ be a movement of σ. Let i_1 be the minimal index such that $u \in \operatorname{supp}(C_{i_1})$ and let i_s be the maximal index such that $v \in \operatorname{supp}(C_{i_s})$. Then we have a unique series $1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant r$ (allowing $i_s = i_1$) with (not necessarily distinct) elements $u_{i_j} \in \Omega$ such that (u, u_{i_1}) is a movement of C_{i_1}, (u_{i_1}, u_{i_2}) is a movement of C_{i_2} and so on, finishing with the movement $(u_{i_{s-1}}, u_{i_s}) = (u_{i_{s-1}}, v)$ of C_{i_s}. For $(u, v) \in R_\sigma$ we define the set $T_{u,v}(C_1 C_2 \cdots C_r)$ as follows: $T_{u,v}(C_1 C_2 \cdots C_r) = \{t^{i_1}_{u,u_{i_1}}, t^{i_2}_{u_{i_1},u_{i_2}}, \dots, t^{i_s}_{u_{i_{s-1}},u_{i_s}}\}$, where $t^i_{p,q}$ is the notation for the movement (p, q) of the cycle C_i. Moreover, we define $T(C_1 C_2 \cdots C_r)$ as follows:

$$T(C_1 C_2 \cdots C_r) = \bigcup_{i=1}^{r} \bigl\{ t^i_{p,q} \bigm| (p, q) \in R_{C_i} \bigr\}.$$

Thus $T(C_1 C_2 \cdots C_r)$ is the set of all the $t^i_{p,q}$'s occurring in the cycles C_i and

$$T(C_1 C_2 \cdots C_r) \supseteq \dot{\bigcup_{(u,v)\in R_\sigma}} T_{u,v}(C_1 C_2 \cdots C_r),$$

where $\dot{\bigcup}$ denotes a disjoint union.

Example. Let $\sigma = (23) = (12)(23)(31) \in S_3$. Then

$$\begin{aligned} T\bigl((12)(23)(31)\bigr) &= \bigl\{t^1_{1,2}, t^1_{2,1}, t^2_{2,3}, t^2_{3,2}, t^3_{3,1}, t^3_{1,3}\bigr\} \\ &\supset \dot{\bigcup_{(u,v)\in R_{(23)}}} T_{u,v}\bigl((12)(23)(31)\bigr) = \bigl\{t^1_{2,1}, t^3_{1,3}\bigr\} \dot\cup \bigl\{t^2_{3,2}\bigr\}. \end{aligned}$$

Lemma 3.1. *Let $\sigma \in S_n$ be a permutation on $\Omega = \{1, 2, \dots, n\}$ with the non-trivial disjoint cycle decomposition $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$. Let l_i denote the length of σ_i for $i = 1, \dots, r$. Suppose that $\sigma = C_1 C_2 \cdots C_k$, where each C_i is a cycle of size l. Fix $1 \leqslant j \leqslant r$, and let $B_j = \dot{\bigcup}_{(p,q)\in R_{\sigma_j}} T_{p,q}(C_1 C_2 \cdots C_k)$. Then*:

(1) $|B_j| \geqslant l_j$;
(2) *If σ_j is not equal to one of the C_i's, then $|B_j| \geqslant l_j + 1$.*

Proof. Part (1) is obvious, since $|R_{\sigma_j}| = l_j$ and $|T_{p,q}(C_1C_2\cdots C_k)| \geqslant 1$ for each $(p,q) \in R_{\sigma_j}$.

We continue with a proof of (2). Suppose that $|B_j| = l_j$. It suffices to prove that $C_{m_1} = \sigma_j$ for some $m_1 \in \{1,\dots,k\}$. Without loss of generality we may assume that $\sigma_j = (1,2,\dots,l_j)$. For each $(p,q) \in R_{\sigma_j}$ we have $|T_{p,q}(C_1\cdots C_k)| = 1$ and so $T_{p,q}(C_1\cdots C_k) = \{t^m_{p,q}\}$ for some $m \in \{1,\dots,k\}$. Let m_1 denote the smallest such m. We may assume, without loss of generality, that $T_{1,2}(C_1\cdots C_k) = \{t^{m_1}_{1,2}\}$ and hence $(1,2) \in R_{C_{m_1}}$. Now $T_{2,3}(C_1\cdots C_k) = \{t^{m_2}_{2,3}\}$ for some m_2 satisfying $m_2 \geqslant m_1$. But if $m_2 > m_1$, then m_1 is not the maximal index such that $2 \in C_{m_1}$, contradicting $|T_{1,2}(C_1\cdots C_k)| = 1$. Hence $m_2 = m_1$ and also $(2,3) \in R_{C_{m_1}}$. Similar arguments imply that each movement of σ_j is a movement of C_{m_1}. Since the movements of σ_j are circular, we must have $C_{m_1} = \sigma_j$, as required. □

Let $\sigma \in S_n$. We denote by $\alpha_i = \alpha_i(\sigma)$ the number of cycles of size i in the disjoint cycle decomposition of σ. We denote further $d_\sigma = \sum_{i=2}^n (i+1)\alpha_i$.

Corollary 3.2. *Under the assumptions of Lemma* 3.1, $d_\sigma - \alpha_l \leqslant kl$.

Proof. We use the notation of Lemma 3.1. Let $T(C_1C_2\cdots C_r) = \bigcup_{i=1}^r \{t^i_{p,q} \mid (p,q) \in R_{C_i}\}$. As remarked above,

$$T(C_1C_2\cdots C_r) \supseteq \dot{\bigcup_{(u,v)\in R_\sigma}} T_{u,v}(C_1C_2\cdots C_r) = \dot{\bigcup_{1\leqslant j\leqslant r}} B_j.$$

Clearly $|T(C_1C_2\cdots C_r)| = kl$ and, by Lemma 3.1, we have $|B_j| \geqslant l_j + 1$ if $l_j \neq l$ and $|B_j| \geqslant l_j$ if $l_j = l$. Therefore $kl \geqslant d_\sigma - \alpha_l$, and the proof is complete. □

Theorem 3.3. *Let k, l be natural numbers such that $k \geqslant 2$ and $l > 2$. Suppose that either l is odd or k is even. Denote $n_1 = \lfloor \frac{2kl}{3} \rfloor$ and $\delta = \frac{2kl}{3} - n_1$. Then*:

(1) *If* $n_1 \equiv 3 \pmod 4$, *then* $n(k,l) \leqslant n_1$;
(2) *If* $n_1 \equiv 1 \pmod 4$, *then* $n(k,l) \leqslant n_1 + 1$;
(3) *If* $n_1 \equiv 2 \pmod 4$, *then* $n(k,l) \leqslant n_1 + 1$; *if we further assume that* $l > 3$ *and* $\delta \in \{0, \frac{1}{3}\}$, *then* $n(k,l) \leqslant n_1$;
(4) *If* $n_1 \equiv 0 \pmod 4$, *then* $n(k,l) \leqslant n_1 + 1$.

Proof. We recall first that either l is odd or k is even, and

$$n(k,l) = \max\{m \geqslant l \mid \text{each permutation in } A_m \text{ is a product of } k \text{ cycles of size } l\}.$$

Moreover, $k \geqslant 2$ and $l \geqslant 3$. We shall prove each item separately.

(1) Suppose that $n_1 \equiv 3 \pmod 4$. Then $(n_1+1)/2$ is even, and we can choose a permutation $\sigma \in A_{n_1+1}$ such that the disjoint cycle decomposition of σ is a product of $(n_1+1)/2$ transpositions. Since $l \geqslant 3$, we have $d_\sigma - \alpha_l(\sigma) = 3 \cdot \frac{n_1+1}{2} - 0 = \frac{3}{2}(n_1+1) > \frac{3}{2} \cdot \frac{2}{3}kl = kl$. Thus, by Corollary 3.2, σ is not a product of k l-cycles, and so $n(k,l) \leqslant n_1$.

(2) Suppose that $n_1 \equiv 1 \pmod 4$. We choose a permutation $\sigma \in A_{n_1+2}$, such that the disjoint cycle decomposition of σ is a product of $(n_1-1)/2$ transpositions and one 3-cycle. If $l = 3$, then $d_\sigma - \alpha_3(\sigma) = 3 \cdot \frac{n_1-1}{2} + 3 = \frac{3}{2}(n_1+1) > \frac{3}{2} \cdot \frac{2}{3}kl = kl$. If $l > 3$, then $d_\sigma - \alpha_l(\sigma) = 3 \cdot \frac{n_1-1}{2} + 4 - 0 > kl$. Thus, in both cases, we obtain $n(k,l) \leqslant n_1 + 1$ by Corollary 3.2.

(3) Suppose that $n_1 \equiv 2 \pmod 4$. We choose a permutation $\sigma \in A_{n_1+1}$ such that the disjoint cycle decomposition of σ is a product of $(n_1-2)/2$ transpositions and one 3-cycle. Assume first that $l > 3$. Then

$$d_\sigma - \alpha_l(\sigma) = 3 \cdot \frac{n_1-2}{2} + 4 - 0 = \frac{3}{2}\left(\frac{2}{3}kl - \delta - 2\right) + 4 = kl + \left(1 - \frac{3}{2}\delta\right).$$

If $\delta = 0$ or $\frac{1}{3}$, then we obtain $d_\sigma - \alpha_l(\sigma) > kl$ and so $n(k,l) \leqslant n_1$ by Corollary 3.2. For the case $\delta = \frac{2}{3}$, choose $\rho \in A_{n_1+2}$ with the disjoint cycle decomposition consisting of a product of $(n_1+2)/2$ transpositions. Then $d_\rho - \alpha_l(\rho) = 3 \cdot \frac{n_1+2}{2} > kl$, yielding $n(k,l) \leqslant n_1 + 1$ as required.

Assume now that $l = 3$. Then the above permutation ρ satisfies $d_\rho - \alpha_3(\rho) > kl$, and so $n(k,l) \leqslant n_1 + 1$ also in this case.

(4) Suppose that $n_1 \equiv 0 \pmod 4$. If $l > 3$, then choose $\sigma \in A_{n_1+2}$ with the disjoint cycle decomposition consisting of a product of $(n_1-4)/2$ transpositions and two 3-cycles. Then $d_\sigma - \alpha_l(\sigma) = 3 \cdot \frac{n_1-4}{2} + 4 \cdot 2 - 0 > \frac{3}{2}(\frac{2}{3}kl - 1 - 4) + 8 = kl + \frac{1}{2} > kl$. If $l = 3$, then choose $\rho \in A_{n_1+2}$ with the disjoint cycle decomposition consisting of a product of $(n_1-2)/2$ transpositions and one 4-cycle. Then $d_\rho - \alpha_3(\rho) = 3 \cdot \frac{n_1-2}{2} + 5 - 0 > \frac{3}{2}(\frac{2}{3}kl - 1 - 2) + 5 = kl + \frac{1}{2} > kl$. Thus in both cases we deduce by Corollary 3.2 that $n(k,l) \leqslant n_1 + 1$, as required. □

The main result of this paper now follows.

Theorem 3.4. *Let k and l be integers such that $k \geqslant 2$ and $l \geqslant 9$ is odd and divisible by 3. Then $\frac{2}{3}kl \leqslant n(k,l) \leqslant \frac{2}{3}kl + 1$. Furthermore, $n(k,l) = \frac{2}{3}kl$ whenever k is odd.*

Proof. By Theorem 2.7, $\frac{2}{3}kl \leqslant n(k,l)$ and, by Theorem 3.3, $n(k,l) \leqslant \frac{2}{3}kl + 1$. If k is odd, then $\frac{2}{3}kl = \lfloor \frac{2}{3}kl \rfloor \equiv 2 \pmod 4$ and, by Theorem 3.3(3), $n(k,l) \leqslant \frac{2}{3}kl$. Hence in this case $n(k,l) = \frac{2}{3}kl$. □

References

[1] Z. Arad, M. Herzog, J. Stavi, Powers and products of conjugacy classes in groups, in: Z. Arad, M. Herzog (Eds.), Products of Conjugacy Classes in Groups, in: Lecture Notes in Math., vol. 1112, Springer-Verlag, New York/Berlin, 1985, Chapter 1.

[2] E. Bertram, Even permutations as a product of two conjugate cycles, J. Combin. Theory 12 (1972) 368–380.

[3] E. Bertram, M. Herzog, Powers of cycle-classes in symmetric groups, J. Combin. Theory Ser. A 94 (2001) 87–99.

[4] J.L. Brenner, R.J. List, Applications of partition theory to groups: Covering the alternating groups by products of conjugacy classes, in: J.M. De Konninc, C. Levesque (Eds.), Number Theory, Walter de Gruyter, 1989, pp. 65–71.

[5] Y. Dvir, Covering properties of permutation groups, in: Z. Arad, M. Herzog (Eds.), Products of Conjugacy Classes in Groups, in: Lecture Notes in Math., vol. 1112, Springer-Verlag, New York/Berlin, 1985, Chapter 3.

[6] M. Herzog, G. Kaplan, A. Lev, Representation of permutations as products of two cycles, Discrete Math. 285 (2004) 323–327.

[7] M. Herzog, G. Kaplan, A. Lev, Universal partitions, submitted for publication.

[8] R. Lawther, M.W. Liebeck, On the diameter of a Cayley graph of a simple group of Lie type based on conjugacy classes, J. Combin. Theory Ser. A 83 (1998) 118–137.

[9] A. Lev, On the covering numbers of finite groups: Some old and new results, in: Contemp. Math., vol. 402, 2006, pp. 201–209.

[10] W.R. Scott, Group Theory, Prentice Hall, Inc./Dover Publications, Inc., London, 1964.

[11] U. Vishne, Mixing and covering in the symmetric group, J. Algebra 205 (1998) 119–140.