Covering the alternating groups by products of cycle classes

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Abstract

Given integers $k, l \geq 2$, where either $l$ is odd or $k$ is even, we denote by $n = n(k, l)$ the largest integer such that each element of $A_n$ is a product of $k$ cycles of length $l$. For an odd $l$, $k$ is the diameter of the undirected Cayley graph $\text{Cay}(A_n, C_l)$, where $C_l$ is the set of all $l$-cycles in $A_n$. We prove that if $k \geq 2$ and $l \geq 9$ is odd and divisible by 3, then $2\frac{3}{2}kl \leq n(k, l) \leq 2\frac{3}{2}kl + 1$. This extends earlier results by Bertram [E. Bertram, Even permutations as a product of two conjugate cycles, J. Combin. Theory 12 (1972) 368–380] and Bertram and Herzog [E. Bertram, M. Herzog, Powers of cycle-classes in symmetric groups, J. Combin. Theory Ser. A 94 (2001) 87–99].© 2008 Elsevier Inc. All rights reserved.

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1. Introduction

Let $A_n$ be the group of all even permutations on $n$ letters. Given integers $k, l \geq 2$, we ask for the largest integer $n = n(k, l)$ such that every permutation in $A_n$ is a product of $k$ cycles of length $l$. By the definition of $A_n$, $n(k, l)$ exists only if either $l$ is odd or $k$ is even. E. Bertram solved the problem for $k = 2$ in 1972 (see also [6] for another proof of this result). He proved the following theorem:

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Theorem 1.1. (See [2, Corollary 2.1].) Each permutation in the alternating group $A_n$, $n \geq 2$, is a product of two $l$-cycles in $S_n$ if and only if either $\lfloor \frac{3n}{4} \rfloor \leq l \leq n$ or $n = 4$ and $l = 2$.

It follows from Theorem 1.1 that if $k = 2$ and $l > 2$, then $n(2, l)$ equals the largest integer $n$ satisfying $\lfloor \frac{3n}{4} \rfloor = l$. Suppose that $l = 3d + e$, where $e \in \{0, 1, 2\}$ and let $n = \lfloor \frac{4l}{3} \rfloor + 1 = 4d + e + 1$. Then $\lfloor \frac{3n}{4} \rfloor = \lfloor 3d + \frac{3}{4}(e + 1) \rfloor = 3d + e = l$, but $\lfloor \frac{3}{4}(n + 1) \rfloor = \lfloor 3d + \frac{3}{4}(e + 2) \rfloor = l + 1$. Hence

$$n(2, l) = \left\lfloor \frac{4l}{3} \right\rfloor + 1 = \left\lfloor \frac{2}{3} kl \right\rfloor + 1.$$

E. Bertram and M. Herzog proceeded by solving the problem for $k = 3$ and $k = 4$. They proved:

Theorem 1.2. (See [3, Theorem 2].) Each $\sigma \in A_n$, $n \geq 1$, is a product of three $l$-cycles in $S_n$ if and only if $l$ is odd and either $\lfloor \frac{7}{8} n \rfloor \leq l \leq n$ or $n = 7$ and $l = 3$.

Theorem 1.3. (See [3, Theorem 3].) Each $\sigma \in A_n$, $n \geq 2$, is a product of four $l$-cycles in $S_n$ if and only if:

1. $\lfloor \frac{3n}{8} \rfloor \leq l \leq n$ if $n \not\equiv 1 \pmod{8}$;
2. $\lfloor \frac{3n}{8} \rfloor \leq l \leq n$ if $n \equiv 1, 0 \pmod{8}$;
3. $n = 6$ and $l = 2$.

It follows from these results that if $2 \leq k \leq 4$ and $l$ is odd if $k = 3$, then $\lfloor \frac{3}{2} kl \rfloor \leq n(k, l) \leq \lfloor \frac{7}{8} kl \rfloor + 1$. Bertram and Herzog conjectured in [3] that $n(k, l) \approx \frac{2}{3} kl$ for every $k, l \geq 2$, provided that either $l$ is odd or $k$ is even. In the spirit of their conjecture, we conjecture the following:

Conjecture 1.1. Let $k, l \geq 2$ be integers and assume that either $l$ is odd or $k$ is even. Then $\lfloor \frac{3}{2} kl \rfloor \leq n(k, l) \leq \lfloor \frac{7}{8} kl \rfloor + 1$.

We note that by [2,3], $n(k, l) = \frac{2}{3} kl + 1$ when $k = 2, 4$ and $3 \mid l$. Hence we also conjecture the following:

Conjecture 1.2. Let $k, l$ be positive integers and assume that $k$ is even and $3 \mid l$. Then $n(k, l) = \frac{2}{3} kl + 1$.

In this paper we prove the validity of Conjecture 1.1 for every integer $k \geq 2$ and every odd integer $l \geq 9$ divisible by 3. Our main result is the following theorem:

Theorem 3.4. Let $k$ and $l$ be integers such that $k \geq 2$ and $l \geq 9$ is odd and divisible by $3$. Then $\frac{2}{3} kl \leq n(k, l) \leq \frac{7}{8} kl + 1$. Furthermore, if $k$ is odd, then $n(k, l) = \frac{5}{6} kl$.

The upper bound for $n(k, l)$ in Theorem 3.4 follows from the following more general result, which does not require $l$ being odd, $l \geq 9$ and $3 \mid l$, but only $l > 2$ and either $l$ is odd or $k$ is even.

Theorem 3.3. Let $k, l$ be natural numbers such that $k \geq 2$ and $l > 2$. Suppose that either $l$ is odd or $k$ is even. Denote $n_1 = \lfloor \frac{2kl}{3} \rfloor$ and $\delta = \frac{2kl}{3} - n_1$. Then:
The above results are closely related to problems on covering groups by products of conjugacy classes. We recall that for a group $G$ and an element $x$ in $G$, the conjugacy class of $x$ in $G$ is $C = x^G = \{g^{-1}xg \mid g \in G\}$. The covering number $cn(C)$ of a conjugacy class $C$ of $G$ is the least integer $m$ (if it exists) such that $C^m = G$, where $C^m = \{c_1c_2\cdots c_m \mid c_1, c_2, \ldots, c_m \in C\}$. The covering number of a group $G$, $cn(G)$, is the least integer $n$ (if it exists) such that $C^n = G$ for every non-trivial conjugacy class of $G$. We note that $cn(G)$ does not necessarily exist for an arbitrary group, but it exists whenever $G$ is a finite non-abelian simple group [1]. The covering numbers for the groups $A_n$ and $S_n$ were extensively studied by Brenner et al. (see [4] for a survey), Dvir [5], Vishne [11] and many others. In particular, since the set of all cycles of a given odd length $l$ ($2 \leq l < n - 1$) constitutes a conjugacy class of $A_n$ (see [10, 11, 15]), our results (as well as the results in [2, 3, 6]) deal with the covering numbers of these classes of $l$-cycles in $A_n$. The covering numbers for various groups other then $A_n$ (or $S_n$) were also extensively studied. See [9] for a recent survey.

We also note that for an odd $l$ and $n = n(k, l)$, $k$ is the diameter of the undirected Cayley graph $Cay(A_n, C_l)$, where $C_l$ is the set of all $l$-cycles in $A_n$. For related results, see [8].

Most of our notation is standard. The positive integers are denoted by $\mathbb{N}$. If $\Omega = \{1, 2, \ldots, n\}$, then $S_n$ denotes the symmetric group on $\Omega$, and $A_n$ denotes the alternating group on $\Omega$. Products of permutations will be executed from left to right. Suppose, first, that $\sigma \in S_n - \{1\}$. Then $\text{supp}(\sigma)$, the support of $\sigma$, is the set $\{i \in \Omega \mid \sigma(i) \neq i\}$ and $dcd * (\sigma)$, a non-trivial disjoint cycle decomposition of $\sigma$, denotes a representation of $\sigma$ as a product of disjoint cycles of length $> 1$.

It is well known that $dcd * (\sigma)$ is unique, except for a cyclic shift within the cycles and the order in which the cycles are written. We call $\sigma, \rho \in S_n$ disjoint permutations on $n_1$ and $n_2$ letters, respectively, if $\text{supp}(\sigma) \cap \text{supp}(\rho) = \emptyset$, $|\text{supp}(\sigma)| \leq n_1$ and $|\text{supp}(\rho)| \leq n_2$. We denote $m_\sigma = |\text{supp}(\sigma)|$ and the number of (non-trivial) cycles in $dcd * (\sigma)$ is denoted by $n_\sigma$. For $\sigma = 1$, we define $dcd * (1) = (1)$ and $m_1 = n_1 = 1$. If $G$ is a subgroup of $S_n$, we denote by $\text{supp}(G)$ the subset of letters in $\Omega$ which are moved by at least one element of $G$. If $q$ is a rational number, then $[q] = k$, where $k$ is the unique integer satisfying $k \leq q < k + 1$, and $[q] = k$, where $k$ is the unique integer satisfying $q \leq k < q + 1$.

The lower bound in Theorem 3.4 is proved in Section 2. Theorem 3.3 (which, in particular, provides the upper bound for Theorem 3.4) and Theorem 3.4 are proved in Section 3.

2. Lower bound for $n(k, l)$

**Definition 2.1.** Let $m, n$ be even integers satisfying $2 \leq m \leq n - 2$. A partition $A = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ of $n$ (i.e. $t, \alpha_i \in \mathbb{N}$ for $1 \leq i \leq t$ and $n = \sum_{i=1}^{t} \alpha_i$), with $\alpha_i \geq 2$ for all $i$, is called $(n, m)$-indecomposable if there does not exist a subset $B$ of $A$ such that $\sum_{\alpha_i \in B} \alpha_i = m$. Notice that $A$ is $(n, m)$-indecomposable if and only if it is $(n, n - m)$-indecomposable. If such $B$ does exist, then the partition $A$ will be called $(n, m)$-decomposable.
lary 2.2, together with the auxiliary Theorem 2.6, will play a key role in proving Theorem 2.7 at the end of this section.

**Lemma 2.1.** Let $m,n$ be even integers satisfying $2 \leq m \leq n-2$ and let $A = \{\alpha_1, \ldots, \alpha_t\}$ be a partition of $n$ which is $(n,m)$-indecomposable. Then $t \leq \frac{n}{3}$, unless $A = \{2, 3, \ldots, 3\}$, in which case $t = \frac{n+1}{3}$.

**Proof.** We may assume in this proof that the following ordering holds: $2 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$.

If $m = 2$ or $n - m = 2$, then $\alpha_i \geq 3$ for all $i$ and $t \leq \frac{n}{4}$, as claimed. If $m = 4$ or $n - m = 4$ and $t > \frac{n}{3}$, then $\alpha_1 = 2$ and $\alpha_j \geq 3$ for all $j > 1$. If $\alpha_j > 4$ for some $j$, then $t \leq \frac{n}{3}$, a contradiction. So $\alpha_j = 3$ for all $j > 1$ and $A = \{2, 3, \ldots, 3\}$. Moreover, $t = \frac{n-2}{3} + 1 = \frac{n+1}{3}$, so the theorem holds in this case too.

Suppose that $m \geq 6$ and $n - m \geq 6$, which implies that $n \geq 12$. We shall complete the proof by induction on $n$. Since $m$ is even, $\alpha_i > 2$ for some $i$. Since $n$ is even, one of the following holds: (i) $\alpha_r$ is even and $\alpha_r \geq 4$ for some $r$; (ii) condition (i) does not hold and there exist distinct integers $r$ and $s$ such that $\alpha_r, \alpha_s$ are odd and $\alpha_r + \alpha_s \geq 6$.

Denote $\beta = \alpha_r$ in case (i) and $\beta = \alpha_r + \alpha_s$ in case (ii). Clearly $\beta$ and $n - \beta$ are even numbers. Suppose, first, that $\beta \geq \frac{n}{2} + 2$. Then $t \leq \frac{n-\beta}{2} + 2 \leq \frac{n}{4} + 1 \leq \frac{n}{3}$, where the final inequality follows since $n \geq 12$.

Suppose, now, that either $\beta = \frac{n}{2}$ or $\beta = \frac{n}{2} + 1$. Since $\beta$ and $m$ are even, there exists $\alpha_i \geq 3$ for some $i \neq r$ in case (i) and for some $i \neq r, s$ in case (ii). But $n - \beta$ is even, so there exists $\alpha_j$ for some $j \neq i, r, s$ in case (i) and for some $j \neq i, r, s$ in case (ii), which satisfies $\alpha_i + \alpha_j \geq 6$. Hence

$$t \leq \frac{n-\beta}{2} + 2 \leq \frac{n}{4} + 1 \leq \frac{n}{3}$$

as $n \geq 12$. So we are done in this case too.

Suppose, finally, that $\beta < \frac{n}{2}$. Then either $m > \beta$ or $n - m > \beta$. So assume, without loss of generality, that $m > \beta$. Denote $m_1 = m - \beta$, $n_1 = n - \beta$, and let $A_1$ be the partition of $n_1$ obtained from $A$ by deleting the components of $\beta$. Then $A_1$ is a partition of $n_1$ which is $(n_1, m_1)$-indecomposable, $n_1, m_1$ are even integers and $2 \leq m_1 \leq n_1 - 2$. Let $t_1$ be the number of summands in $A_1$; clearly either $t_1 = t + 1$ or $t_1 = t + 2$.

Suppose that $A_1 = \{2, 3, \ldots, 3\}$. If $\beta = \alpha_r \geq 4$, then

$$t = t_1 + 1 = \frac{n-\beta-2}{3} + 2 \leq \frac{n-4+4}{3} = \frac{n}{3}$$

as required. If $\beta = \alpha_r + \alpha_s = 6$, then $\alpha_r = \alpha_s = 3$, $A = \{2, 3, \ldots, 3\}$ and $t = \frac{n-2}{3} + 1 = \frac{n+1}{3}$, as required. If $\beta = \alpha_r + \alpha_s > 6$, then $t = t_1 + 2 = \frac{n-\beta-2}{3} + 3 \leq \frac{n-8+2+9}{3} = \frac{n}{3}$, as required.

If $A_1 \neq \{2, 3, \ldots, 3\}$ and $\beta = \alpha_r \geq 4$, then by induction $t = t_1 + 1 \leq \frac{n-\beta}{3} + 1 \leq \frac{n-4+3}{3} < \frac{n}{3}$, as required. If $\beta = \alpha_r + \alpha_s > 6$, then by induction $t = t_1 + 2 \leq \frac{n-\beta}{3} + 2 \leq \frac{n-6+6}{3} = \frac{n}{3}$, again as required. The proof is complete. \qed

Using Lemma 2.1, we obtain

**Corollary 2.2.** Let $n$ be an even integer and let $\sigma \in S_n$, satisfying $n_\sigma > \frac{n+1}{3}$. Then for each even integer $m$ satisfying $2 \leq m \leq n-2$, there exist non-trivial permutations $\rho$ and $\phi$ in $S_n$ such that $\sigma = \rho \phi$, $\text{supp}(\rho) \cap \text{supp}(\phi) = \emptyset$, $|\text{supp}(\rho)| \leq m$ and $|\text{supp}(\phi)| \leq n-m$. 
Proof. Denote \( t = n_{\sigma} \). Since \( t > \frac{n + 1}{2} \geq 1 \), \( \sigma \neq 1 \). Let \( \sigma = (1)(2)(3)\ldots(f)C_{\alpha_1}C_{\alpha_2}\ldots C_{\alpha_t} \), where \( \{1, 2, 3, \ldots, f\} \) are the distinct fixed points of \( \sigma \) (this set can be empty) and the \( C_{\alpha_i} \) are disjoint cycles of length \( \alpha_i \) with \( \alpha_i \geq 2 \) for each \( i \), ordered in such a way that \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_t \geq 2 \). Then \( A = \{\alpha_1 + f, \alpha_2, \ldots, \alpha_t\} \) is a partition of the even integer \( n \), with all components larger than 1 and \( t > \frac{n + 1}{2} \). Fix an even \( m \) such that \( 2 \leq m \leq n - 2 \). It follows then by Lemma 2.1, that the partition \( A \) is \((n, m)\)-decomposable. Let \( A = A_1 \cup A_2 \) be a decomposition of \( A \) into two subpartitions of \( m \) and of \( n - m \), respectively. Define \( \rho = \prod_{\alpha_i \in A_1} C_i \) and \( \phi = \prod_{\alpha_i \in A_2} C_i \), where \( \alpha_1 \in A_j \) means \( \alpha_1 + f \in A_j \) for \( j = 1 \) or 2. Then \( \rho \) and \( \phi \) are non-trivial permutations in \( S_n \) satisfying \( \text{supp}(\rho) \cap \text{supp}(\phi) = \emptyset \), \( |\text{supp}(\rho)| \leq m \) and \( |\text{supp}(\phi)| \leq n - m \), as required. \( \square \)

The next lemma follows immediately from the following result in [6].

**Theorem 2.3.** (See [6, Theorem 7].) Let \( \sigma \in S_n \) and let \( l_1, l_2 \in \mathbb{N} \), \( n \geq l_1 \geq l_2 \geq 2 \). Then \( \sigma = C_1 C_2 \), where \( C_1, C_2 \) are cycles in \( S_n \) of lengths \( l_1, l_2 \), respectively, if and only if either \( n_{\sigma} = 2 \), \( l_1, l_2 \) are the lengths of the cycles in \( dcd \ast (\sigma) \) and \( l_1 + l_2 = m_{\sigma} \), or the following conditions hold:

1. \( l_1 + l_2 = m_{\sigma} + n_{\sigma} + 2s \) for some \( s \in \mathbb{N} \cup \{0\} \), and
2. \( l_1 - l_2 \leq m_{\sigma} - n_{\sigma} \).

**Lemma 2.4.** Let \( n, l_1, l_2, m \in \mathbb{N} \), satisfy \( n \geq l_1 \geq l_2 \geq 2 \) and let \( \sigma \) be an \( m \)-cycle in \( S_n \). Then there exist in \( S_n \) cycles \( C_1, C_2 \) of sizes \( l_1, l_2 \), respectively, such that \( \sigma = C_1 C_2 \), if and only if

\[
 m = l_1 + l_2 - (2s - 1) \leq n
\]

for some \( s \in \mathbb{N} \), \( s \leq l_2 \).

**Proof.** By Theorem 2.3, such \( C_1, C_2 \) exist if and only if

1. \( m \leq n \),
2. \( m = l_1 + l_2 - (2s - 1) \) for some \( s \in \mathbb{N} \), and
3. \( m \geq l_1 - l_2 + 1 \).

Conditions (2) and (3) are clearly equivalent to conditions (2) and

\[
 s \leq l_2
\]

and the lemma follows. \( \square \)

We prove now a generalization of Lemma 2.4 for odd \( l_i \)'s and \( m \).

**Lemma 2.5.** Let \( n, t \in \mathbb{N} \) and let \( l_1, l_2, \ldots, l_t \) be odd integers satisfying \( n \geq l_1 \geq l_2 \geq \cdots \geq l_t \geq 3 \) and

\[
l_1 \leq m \leq \min \left( l_1 + \sum_{i=2}^{t} (d_i - 1), n \right).
\]

Then for each \( m \)-cycle \( \sigma \in S_n \) there exist in \( S_n \) cycles \( C_i \), \( 1 \leq i \leq t \), of sizes \( l_i \), respectively, such that \( \sigma = C_1 C_2 \cdots C_t \).
Definition 2.2. Let $t$ transpositions and if $\sigma$.

Proof. Let $m \geq l_1 + \sum_{i=2}^{t-1}(l_i - 1)$, which implies that $m' = l_1 + \sum_{i=2}^{t-1}(l_i - 1)$. Then $m' < m \leq l_1 + \sum_{i=2}^{t}(l_i - 1) = m' + l_t - 1$

and since $m, m'$ and $l_t$ are odd, it follows that $m = m' + l_t - (2s - 1)$ for some $s \in \mathbb{N}, s \leq l_t$.

Thus, by Lemma 2.4, there exist in $S_n$ cycles $\tau$ and $C_t$ of lengths $m'$ and $l_t$, respectively, such that $\sigma = \tau C_t$. Since $l_1 \leq m' \leq \min(l_1 + \sum_{i=2}^{t-1}(l_i - 1), m)$ and $m \leq n$, it follows by the induction hypothesis that there exist in $S_n$ cycles $C_1, \ldots, C_{t-1}$ such that $\tau = C_1 \cdots C_{t-1}$ and the result follows.

It remains to prove the lemma in the case that $l_1 \leq m \leq l_1 + \sum_{i=2}^{t-1}(l_i - 1)$. Then, by the induction hypothesis, there exist in $S_n$ cycles $C_1, C_2, \ldots, C_t, C_{t-1}$ of lengths $l_1, l_2, \ldots, l_t$, respectively, such that $\sigma = C_1 C_2 \cdots C_{t-1} C_t$. Since the $l_t$ are odd integers, it follows again by Lemma 2.4 that there exist in $S_n$ cycles $C_{t-1}, C_t$ of lengths $l_{t-1}, l_t$, respectively, such that $C_{t-1} = C_{t-1} C_t$, and the lemma follows. \square

For our auxiliary Theorem 2.6 we need the following definition.

Definition 2.2. Let $k, l, n \in \mathbb{N}$ such that $k, l \geq 2$ and $n \geq l$. Denote by $C(l)$ the set of all cycles of length $l$ in $S_n$ and by $P(k,l;n)$ the set of all permutations in $S_n$ which may be written as a product of $k$ cycles of length $l$.

Theorem 2.6. Let $k, l, n \in \mathbb{N}$ be such that $k \geq 2$ and $l$ is odd and suppose that $9 \leq l \leq \frac{3}{4} \cdot l + 1$. Moreover, let $\sigma \in A_n$ and suppose that $n_\sigma \leq \frac{n+2}{3}$ if $k \geq 3$. Then $\sigma \in P(k,l;n)$.

Proof. Since $n$ is fixed, we shall denote $P(i, j; n)$ by $P(i,j)$. If $k = 2$, then $n \leq \frac{3}{4}l + 1$, which implies that $l \geq \lceil \frac{3}{4}(n-1) \rceil \geq \lceil \frac{3}{4}n \rceil$ and, by Theorem 1.1, $\sigma \in P(2,l)$. So assume that $k \geq 3$ and $n_\sigma \leq \frac{n+2}{3}$.

For the continuation of the proof, we need the following three observations:

$O_1$. $P(r, l) \subseteq P(t, l)$ if $r \leq t$.

$O_2$. $P(k, l) \subseteq P(k, l + 2)$ if $l + 2 \leq n$.

$O_3$. $m_\sigma + n_\sigma$ is even for all $\sigma \in A_n$.

Concerning $O_1$, we notice that if $r \in \mathbb{N}$ and $C_1, C_2, \ldots, C_r \in C(l)$, then, since $l$ is odd, also $C_r^{-1}, C_2 \in C(l)$ and $C_1 C_2 \cdots C_r = C_1 C_2 \cdots C_{r-1} C_2 C_r^{-1}$. Observation $O_2$ follows from Proposition 15 in [3]. Finally, observation $O_3$ follows from the fact that if $l > 1$, then each $l$-cycle can be decomposed into $l - 1$ transpositions. Thus any $\sigma \in S_n - \{1\}$ can be decomposed into $m_\sigma - n_\sigma$ transpositions and if $\sigma \in A_n$, then this number, and hence also $m_\sigma + n_\sigma$, must be even.

We continue now with our proof. If $\sigma = 1$, then clearly $\sigma \in P(2, l)$ and, by $O_1$, $\sigma \in P(k, l)$. So assume that $\sigma \in A_n - \{1\}$, which implies that $m_\sigma \geq 3$ and $n_\sigma \geq 1$.

Suppose, first, that $l \leq m_\sigma \leq 2l - 1$. Then $m_\sigma \geq l \geq \frac{m_\sigma + 1}{2} \geq \lceil \frac{m_\sigma}{2} \rceil$ and as $l$ is odd and $l \geq 9$, it follows by Theorem 1.2 that $\sigma \in P(3, l)$. As $k \geq 3$, $O_1$ implies that $\sigma \in P(k, l)$, as required.
Suppose, next, that $3 \leq m_\sigma < l$. Then either $m_\sigma$ or $m_\sigma - 1$ is an odd integer, say $l_1$, and since $m_\sigma \geq l_1 \geq m_\sigma - 1 \geq \lceil \frac{m_\sigma}{2} \rceil$, it follows by Theorem 1.2 that $\sigma \in P(3, l_1)$. Hence, by $O_1$, $\sigma \in P(k, l_1)$ and since both $l$ and $l_1$ are odd and $l_1 < l$, it follows by $O_2$ that $\sigma \in P(k, l)$.

So assume that $m_\sigma \geq \max(2l, 2n_\sigma)$. It follows that

$$m_\sigma - n_\sigma \geq \max(2l - n_\sigma, n_\sigma) \geq l > l - 1.$$  

Clearly $m_\sigma \leq n$. Hence, by our assumptions, we have

$$m_\sigma + n_\sigma \leq n + \frac{n + 2}{3} = \frac{4}{3} n + \frac{2}{3} \leq \frac{4}{3} \left( \frac{2}{3} kl + 1 \right) + \frac{2}{3} = \frac{8}{9} kl + 2$$

$$= 2l + \frac{l}{l - 1} \left( \frac{8}{9} k - 2 \right) (l - 1) + 2 \leq 2l + \frac{9}{8} \left( \frac{8}{9} k - 2 \right) (l - 1) + 2$$

$$= 2l + (k - 2)(l - 1) \frac{l - 1}{4} + 2$$

$$\leq 2l + (k - 2)(l - 1).$$

Let $s \in \mathbb{N}$ be minimal such that

$$m_\sigma + n_\sigma \leq 2l + (s - 2)(l - 1). \tag{3}$$

Clearly $s \leq k$. Since $m_\sigma \geq 2l$ and $n_\sigma \geq 1$, it follows that $3 \leq s \leq k$.

If $s = 3$, then $m_\sigma + n_\sigma \leq 3l - 1 = l_1 + l$, where $l_1 = 2l - 1$. Since $\sigma \in A_n$ and $l \geq 9$ is odd, it follows, in view of $O_3$, that both sides of the above inequality are even and $l < l_1 < m_\sigma \leq n$. As shown above, $m_\sigma - n_\sigma > l - 1 = l_1 - l$, and it follows by Theorem 2.3 that $\sigma = C_1 C_2$, with $C_1 \in C(l_1)$ and $C_2 \in C(l)$. By Lemma 2.4, $C_1 \in P(2, l)$ and we may conclude, in view of $O_1$, that $\sigma \in P(3, l) \subseteq P(k, l)$, as required.

So suppose that $s \geq 4$. It follows from inequality (3) that

$$m_\sigma + n_\sigma \leq \left[ l + \left( \left\lceil \frac{s}{2} \right\rceil - 1 \right) (l - 1) \right] + \left[ l + \left( \left\lfloor \frac{s}{2} \right\rfloor - 1 \right) (l - 1) \right] = l_1 + l_2 \tag{4}$$

with the obvious notation. By our assumptions and the minimality of $s$, we must have $\frac{4}{3} n + \frac{2}{3} \geq m_\sigma + n_\sigma \geq 2l + (s - 3)(l - 1)$ and since $l$ is odd, it follows that $n \geq \frac{3}{2} l + \frac{3}{4} (s - 3)(l - 1) - \frac{1}{2} = \frac{m - 1}{2}$ for some integer $m$. Hence $n \geq \frac{3}{2} l + \frac{3}{4} (s - 3)(l - 1) + m_\sigma - n_\sigma = \frac{3}{2} l + \frac{3}{4} (s - 3)(l - 1) - \frac{1}{2}$. Hence, by Theorem 2.3 that $\sigma = C_1 C_2$, with $C_1 \in C(l_1)$ and $C_2 \in C(l_2)$. In view of the definition of $l_1$ and $l_2$, it follows by Lemma 2.5 that $C_1 \in P(\lceil \frac{s}{2} \rceil, l)$ and $C_2 \in P(\lfloor \frac{s}{2} \rfloor, l)$. Thus $\sigma \in P(s, l)$ and since $s \leq k$, it follows by $O_1$ that $\sigma \in P(k, l)$, as required.

The proof is complete. \qed

**Theorem 2.7.** Let $k, l \in \mathbb{N}$ be such that $k \geq 2$ and $l \geq 9$ is odd and divisible by 3. Moreover, let $n = \frac{2k}{3}$ and $\sigma \in A_n$. Then $\sigma \in P(k, l; n)$. 
Proof. If \( k = 2 \), then \( l = \frac{3}{4}n \) and \( \sigma \in P(2, l; n) \) by Theorem 1.1.

If \( k = 3 \), then \( l = \frac{2}{3} \left\lceil \frac{n}{3} \right\rceil \) and \( \sigma \in P(3, l; n) \) by Theorem 1.2.

If \( k = 4 \), then \( l = \frac{3}{8}n = \left\lceil \frac{3}{8}n \right\rceil \) and \( \sigma \in P(4, l; n) \) by Theorem 1.3.

So suppose that \( k \geq 5 \). If \( n_\sigma \leq \frac{n+2}{3} \), then \( \sigma \in P(k, l; n) \) by Theorem 2.6. So we may also assume that \( n_\sigma > \frac{n+2}{3} \). Denote \( l = n_\sigma \).

Let \( m \) be an even integer satisfying \( 2 \leq m \leq n - 2 \) and let \( A \) be the partition of \( n \) obtained from the disjoint cycle decomposition of \( \sigma \) as in the proof of Corollary 2.2. Since \( t > \frac{n+2}{3} \), it follows by Lemma 2.1 that \( A \) is \((n, m)\)-decomposable. In particular, \( A \) is \( (\frac{2}{3}k, \frac{2}{3}l)\)-decomposable, and as shown in the proof of Corollary 2.2, there exist non-trivial disjoint permutations \( \rho \) and \( \phi \) on \( \frac{2}{3}(k - 2)l \) and \( \frac{4}{3}l \) letters, respectively, such that \( \sigma = \rho \phi \).

We proceed by induction on \( k \). If \( \rho \) and \( \phi \) are both even, then, by induction, \( \rho \in P(k - 2, l; \frac{2}{3}(k - 2)l) \) and \( \phi \in P(2, l; \frac{4}{3}l) \), which implies that \( \sigma \in P(k, l; n) \), as required. Since \( \sigma \in A_n \), it remains to deal with the case when both \( \rho \) and \( \phi \) are odd.

If \( n_\rho \leq \frac{2}{3}(k-2)+2 \), let \( \tau \) be the transposition \( (u, v) \), where \( u \in \text{supp}(\rho) \) and \( v \in \text{supp}(\phi) \), and let \( \rho^* = \rho \tau \) and \( \phi^* = \tau \phi \). Then \( \rho^* \) and \( \phi^* \) are even permutations on \( \frac{2}{3}(k - 2)l + 1 \) and \( \frac{4}{3}l + 1 \) letters, respectively, \( \sigma = \rho^* \phi^* \) and \( n_{\rho^*} = n_\rho \leq \frac{2}{3}(k-2)+2 \). By Theorem 2.6, \( \rho^* \in P(k - 2, l; \frac{2}{3}(k - 2)l + 1) \). Moreover, since \( \frac{1}{3}(\frac{2}{3}l + 1)j = l \), it follows by Theorem 1.1 that \( \phi^* \in P(2, l; \frac{4}{3}l + 1) \). Consequently, \( \sigma \in P(k, l; n) \), as required. So we may assume that \( n_\rho > \frac{2}{3}(k-2)+2 \). It follows then, by Corollary 2.2, that we can write \( \rho = \rho_1 \rho_2 \), where \( \rho_1, \rho_2 \) are non-trivial disjoint permutations of opposite parity on \( \frac{2}{3}(k - 4)l \) and \( \frac{4}{3}l \) letters, respectively. Now, if \( \rho_1 \) is odd, then \( \rho_1 \phi \) and \( \rho_2 \) are even permutations on \( \frac{2}{3}(k - 2)l \) and \( \frac{4}{3}l \) letters, respectively, and hence, by induction, \( \rho_1 \phi \in P(k - 2, l; \frac{2}{3}(k - 2)l) \) and \( \rho_2 \in P(2, l; \frac{4}{3}l) \). Since \( \sigma = (\rho_1 \phi) \rho_2 \), \( \sigma \in P(k, l; n) \) as required (notice that the permutations \( \rho_1, \rho_2 \) and \( \phi \) commute). So we may assume that \( \rho_1 \) is even and \( \rho_2 \) is odd.

Suppose, first, that \( k = 5 \). Then \( n = \frac{10l}{3} \), \( \rho \) is an odd permutation on \( 2l \) letters and \( \phi \) is an odd permutation on \( \frac{4}{3}l \) letters. Moreover, \( \rho = \rho_1 \rho_2 \), where \( \rho_1 \) is an even permutation on \( \frac{2}{3}l \) letters and \( \rho_2 \) is an odd permutation on \( \frac{4}{3}l \) letters. If either \( n_\phi > \frac{4}{3}l + 1 \) or \( n_{\rho_2} > \frac{4}{3}l + 1 \), assume, without loss of generality, that \( n_\phi > \frac{4}{3}l + 1 \). By Corollary 2.2 we can write \( \phi = \phi_1 \phi_2 \), where \( \phi_1 \) is a non-trivial even permutation on \( \frac{2}{3}l \) letters and \( \phi_2 \) is a non-trivial odd permutation on the remaining \( \frac{4}{3}l \) letters. Define \( \beta = \phi_1 \rho_1 \) and \( \gamma = \phi_2 \rho_2 \). Then \( \beta \) and \( \gamma \) are even permutations on \( \frac{2}{3}l \) and \( 2l \) letters, respectively, and \( \sigma = \beta \gamma \). By induction we get \( \beta \in P(2, l; \frac{4}{3}l) \), \( \gamma \in P(3, l; 2l) \) and hence \( \sigma \in P(5, l; n) \), as required. Thus we assume that \( n_\phi \leq \frac{4}{3}l + 1 \) and \( n_{\rho_2} \leq \frac{4}{3}l + 1 \). Choose \( u \in \text{supp}(\rho_1) \) and \( v \in \text{supp}(\rho_2) \) and let \( \tau = (u, v) \). Denote \( \rho^* = \rho \tau \), an even permutation on \( 2l \) letters, and denote \( \phi^* = \tau \phi \), an even permutation on \( \frac{4}{3}l + 2 \) letters. Then, by induction, \( \rho^* \in P(3, l; 2l) \) and, by Theorem 2 in [2], \( \phi^* \in P(2, l; \frac{4}{3}l + 2) \) provided that

\[
\frac{m_{\phi^*} + n_{\phi^*}}{2} \leq \frac{4}{3}l + 2 + n_{\phi^*} + 1 \leq \frac{4}{3}l + 2 + \frac{4}{3}l + 1 = \frac{16l + 30}{18} \leq l
\]

which holds for \( l \geq 15 \). So it remains only to deal with the case \( l = 9 \). In this case, \( n = \frac{2}{3} \cdot 5 \cdot 9 = 30 \), \( \phi \) and \( \rho_2 \) each acts on 12 letters and \( \rho_1 \) acts on 6 letters. Moreover, \( n_\phi, n_{\rho_2} \leq \frac{12}{3} + 1 \),
which implies \( n_\phi, n_{\rho_2} \leq 4 \). Furthermore, since \( \rho_1 \) is even, we obtain \( n_{\rho_1} \leq 2 \). Thus \( n_\sigma = n_{\rho_1} + n_{\rho_2} + n_\phi \leq 2 + 4 + 4 = 10 \leq \frac{30}{3} \) and \( \sigma \in P(k, l; n) \) by Theorem 2.6.

So assume that \( k \geq 6 \). As \( \rho_1 \) is an even permutation on \( \frac{2}{3}(k-4) \) letters, with \( k-4 \geq 2 \), and \( \phi \rho_2 \) is an even permutation on \( \frac{8}{3} l \) letters, it follows by induction that \( \rho_1 \in P(k-4, l; \frac{2}{3}(k-4)l) \) and \( \phi \rho_2 \in P(4, l; \frac{8}{3} l) \). But \( \sigma = \rho_1(\phi \rho_2) \), so it follows that \( \sigma \in P(k, l; n) \), as required. The proof is complete. \( \square \)

3. Upper bound for \( n(k, l) \)

Our aim in this section is to prove Theorem 3.3 below, which provides bounds from above on \( n(k, l) \), and use it for the proof of Theorem 3.4. We define and discuss first the important notion of movements of a permutation.

Let \( \sigma \in S_n \) be a permutation on the set \( \Omega = \{1, 2, \ldots, n\} \). For distinct \( u, v \in \Omega \) we shall say that \( (u, v) \) is a movement of \( \sigma \) if \( u^{\sigma} = v \). Clearly \( \sigma \) is completely determined by the set of all its movements, which will be denoted by \( R_\sigma \). Let \( \sigma = C_1 C_2 \cdots C_r \), where \( C_1, C_2, \ldots, C_r \) are arbitrary cycles in \( S_n \), and let \( (u, v) \in R_\sigma \) be a movement of \( \sigma \). Let \( i_1 \) be the minimal index such that \( u \in \text{supp}(C_{i_1}) \) and let \( i_s \) be the maximal index such that \( v \in \text{supp}(C_{i_s}) \). Then we have a unique series \( 1 \leq i_1 < i_2 < \cdots < i_s \leq r \) (allowing \( i_s = i_1 \) with (not necessarily distinct) elements \( u_{ij} \in \Omega \) such that \( (u_{ij}, u_{i_{j+1}}) \) is a movement of \( C_{i_j} \), \( (u_{ij}, u_{i_{j+1}}) \) is a movement of \( C_{i_j} \) and so on, finishing with the movement \( u_{i_{i_s-1}}, u_{i_s} \) of \( C_{i_s} \). For \( (u, v) \in R_\sigma \) we define the set \( T_{u,v}(C_1 C_2 \cdots C_r) \) as follows: \( T_{u,v}(C_1 C_2 \cdots C_r) = \{ t_{p,q}^{i_1}, t_{u_{i_1},u_{i_2}}, \ldots, t_{u_{i_{s-1}},u_{i_s}} \} \), where \( t_{p,q}^{i} \) is the notation for the movement \( (p, q) \) of the cycle \( C_{i_s} \). Moreover, we define \( T(C_1 C_2 \cdots C_r) \) as follows:

\[
T(C_1 C_2 \cdots C_r) = \bigcup_{i=1}^{r} \{ t_{p,q}^{i} \mid (p, q) \in R_{C_i} \}.
\]

Thus \( T(C_1 C_2 \cdots C_r) \) is the set of all the \( t_{p,q}^{i} \)'s occurring in the cycles \( C_i \) and

\[
T(C_1 C_2 \cdots C_r) \supseteq \bigcup_{(u, v) \in R_\sigma} T_{u,v}(C_1 C_2 \cdots C_r),
\]

where \( \bigcup \) denotes a disjoint union.

Example. Let \( \sigma = (23) = (12)(23)(31) \in S_3 \). Then

\[
T((12)(23)(31)) = \{ t_{1,2}^{1}, t_{2,1}^{2}, t_{2,3}^{2}, t_{3,2}^{3}, t_{3,1}^{3}, t_{1,3}^{3} \}
\supseteq \bigcup_{(u, v) \in R_{(23)}} T_{u,v}((12)(23)(31)) = \{ t_{2,1}^{1}, t_{1,3}^{3} \} \cup \{ t_{3,2}^{2} \}.
\]

Lemma 3.1. Let \( \sigma \in S_n \) be a permutation on \( \Omega = \{1, 2, \ldots, n\} \) with the non-trivial disjoint cycle decomposition \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_r \). Let \( l_i \) denote the length of \( \sigma_i \) for \( i = 1, \ldots, r \). Suppose that \( \sigma = C_1 C_2 \cdots C_k \), where each \( C_i \) is a cycle of size \( l_i \). Fix \( 1 \leq j \leq r \), and let \( B_j = \bigcup_{(p, q) \in R_{\sigma_j}} T_{p,q}(C_1 C_2 \cdots C_k) \). Then:

1. \( |B_j| \geq l_j \);
2. If \( \sigma_j \) is not equal to one of the \( C_i \)'s, then \( |B_j| \geq l_j + 1 \).
Proof. Part (1) is obvious, since \(|R_{\sigma_j}| = l_j\) and \(|T_{p,q}(C_1C_2 \cdots C_k)| \geq 1\) for each \((p, q) \in R_{\sigma_j}\).

We continue with a proof of (2). Suppose that \(|B_j| = l_j\). It suffices to prove that \(C_{m_1} = \sigma_j\) for some \(m_1 \in \{1, \ldots, k\}\). Without loss of generality we may assume that \(\sigma_j = (1, 2, \ldots, l_j)\). For each \((p, q) \in R_{\sigma_j}\), we have \(|T_{p,q}(C_1 \cdots C_k)| = 1\) and so \(T_{p,q}(C_1 \cdots C_k) = \{t_{p,q}^m\}\) for some \(m \in \{1, \ldots, k\}\). Let \(m_1\) denote the smallest such \(m\). We may assume, without loss of generality, that \(T_{1,2}(C_1 \cdots C_k) = \{t_{1,2}^{m_1}\}\) and hence \((1, 2) \in R_{C_{m_1}}\). Now \(T_{2,3}(C_1 \cdots C_k) = \{t_{2,3}^m\}\) for some \(m_2\) satisfying \(m_2 \geq m_1\). But if \(m_2 > m_1\), then \(m_1\) is not the maximal index such that \(2 \in C_{m_1}\), contradicting \(|T_{1,2}(C_1 \cdots C_k)| = 1\). Hence \(m_2 = m_1\) and also \((2, 3) \in R_{C_{m_1}}\). Similar arguments imply that each movement of \(\sigma_j\) is a movement of \(C_{m_1}\). Since the movements of \(\sigma_j\) are circular, we must have \(C_{m_1} = \sigma_j\), as required. □

Let \(\sigma \in S_n\). We denote by \(\alpha_i = \alpha_i(\sigma)\) the number of cycles of size \(i\) in the disjoint cycle decomposition of \(\sigma\). We denote further \(d_\sigma = \sum_{i=2}^{n}(i+1)\alpha_i\).

Corollary 3.2. Under the assumptions of Lemma 3.1, \(d_\sigma - \alpha_1 \leq kl\).

Proof. We use the notation of Lemma 3.1. Let \(T(C_1C_2 \cdots C_r) = \bigcup_{i=1}^{r}T_{p,q}(C_1C_2 \cdots C_r)\). As remarked above,

\[T(C_1C_2 \cdots C_r) \supseteq \bigcup_{(u,v) \in R_{\sigma}}T_{u,v}(C_1C_2 \cdots C_r) = \bigcup_{1 \leq j \leq r}B_j.\]

Clearly \(|T(C_1C_2 \cdots C_r)| = kl\) and, by Lemma 3.1, we have \(|B_j| \geq l_j + 1\) if \(l_j \neq l\) and \(|B_j| \geq l_j\) if \(l_j = l\). Therefore \(kl \geq d_\sigma - \alpha_1\), and the proof is complete. □

Theorem 3.3. Let \(k, l\) be natural numbers such that \(k \geq 2\) and \(l > 2\). Suppose that either \(l\) is odd or \(k\) is even. Denote \(n_1 = \lfloor \frac{2kl}{3} \rfloor\) and \(\delta = \frac{2kl}{3} - n_1\). Then:

1. If \(n_1 \equiv 3 \pmod{4}\), then \(n(k, l) \leq n_1\);
2. If \(n_1 \equiv 1 \pmod{4}\), then \(n(k, l) \leq n_1 + 1\);
3. If \(n_1 \equiv 2 \pmod{4}\), then \(n(k, l) \leq n_1 + 1\); if we further assume that \(l > 3\) and \(\delta \in \{0, \frac{1}{3}\}\), then \(n(k, l) \leq n_1\);
4. If \(n_1 \equiv 0 \pmod{4}\), then \(n(k, l) \leq n_1 + 1\).

Proof. We recall first that either \(l\) is odd or \(k\) is even, and

\[n(k, l) = \max \{m \geq l \mid \text{each permutation in } A_m \text{ is a product of } k \text{ cycles of size } l\}.\]

Moreover, \(k \geq 2\) and \(l \geq 3\). We shall prove each item separately.

1. Suppose that \(n_1 \equiv 3 \pmod{4}\). Then \((n_1 + 1)/2\) is even, and we can choose a permutation \(\sigma \in A_{n_1+1}\) such that the disjoint cycle decomposition of \(\sigma\) is a product of \((n_1 + 1)/2\) transpositions. Since \(l \geq 3\), we have \(d_\sigma - \alpha_1(\sigma) = 3 \cdot \frac{n_1+1}{2} - 0 = \frac{3}{2}(n_1 + 1) > \frac{3}{2} \cdot \frac{2}{3} kl = kl\). Thus, by Corollary 3.2, \(\sigma\) is not a product of \(k\) \(l\)-cycles, and so \(n(k, l) \leq n_1\).

2. Suppose that \(n_1 \equiv 1 \pmod{4}\). We choose a permutation \(\sigma \in A_{n_1+2}\), such that the disjoint cycle decomposition of \(\sigma\) is a product of \((n_1 - 1)/2\) transpositions and one 3-cycle. If \(l = 3\), then \(d_\sigma - \alpha_3(\sigma) = 3 \cdot \frac{n_1-1}{2} + 3 = \frac{3}{2}(n_1 + 1) > \frac{3}{2} \cdot \frac{2}{3} kl = kl\). If \(l > 3\), then \(d_\sigma - \alpha_1(\sigma) = 3 \cdot \frac{n_1-1}{2} + 4 - 0 > kl\). Thus, in both cases, we obtain \(n(k, l) \leq n_1 + 1\) by Corollary 3.2.
(3) Suppose that \( n_1 \equiv 2 \pmod{4} \). We choose a permutation \( \sigma \in A_{n_1+1} \) such that the disjoint cycle decomposition of \( \sigma \) is a product of \( (n_1 - 2)/2 \) transpositions and one 3-cycle. Assume first that \( l > 3 \). Then
\[
d_\sigma - \alpha_l(\sigma) = 3 \cdot \frac{n_1 - 2}{2} + 4 - 0 = 3 \left( \frac{2}{3} kl - \delta - 2 \right) + 4 = kl + 1 - \frac{3}{2} \delta \cdot
\]
If \( \delta = 0 \) or \( \frac{1}{3} \), then we obtain \( d_\sigma - \alpha_l(\sigma) > kl \) and so \( n(k,l) \leq n_1 \) by Corollary 3.2. For the case \( \delta = \frac{2}{3} \), choose \( \rho \in A_{n_1+2} \) with the disjoint cycle decomposition consisting of a product of \( (n_1 + 2)/2 \) transpositions. Then \( d_\rho - \alpha_{l}(\rho) = 3 \cdot \frac{n_1 + 2}{2} > kl \), yielding \( n(k,l) \leq n_1 + 1 \) as required.

Assume now that \( l = 3 \). Then the above permutation \( \rho \) satisfies \( d_\rho - \alpha_{3}(\rho) > kl \), and so \( n(k,l) \leq n_1 + 1 \) also in this case.

(4) Suppose that \( n_1 \equiv 0 \pmod{4} \). If \( l > 3 \), then choose \( \sigma \in A_{n_1+2} \) with the disjoint cycle decomposition consisting of a product of \( (n_1 - 2)/2 \) transpositions and two 3-cycles. Then \( d_\sigma - \alpha_l(\sigma) = 3 \cdot \frac{n_1 - 4}{2} + 4 - 0 > \frac{3}{2} (\frac{2}{3} kl - 1 - 4) + 8 = kl + \frac{1}{2} > kl \). If \( l = 3 \), then choose \( \rho \in A_{n_1+2} \) with the disjoint cycle decomposition consisting of a product of \( (n_1 - 2)/2 \) transpositions and one 4-cycle. Then \( d_\rho - \alpha_{3}(\rho) = 3 \cdot \frac{n_1 - 2}{2} + 5 - 0 > \frac{3}{2} (\frac{2}{3} kl - 1 - 2) + 5 = kl + \frac{1}{2} > kl \). Thus in both cases we deduce by Corollary 3.2 that \( n(k,l) \leq n_1 + 1 \), as required. \( \square \)

The main result of this paper now follows.

**Theorem 3.4.** Let \( k \) and \( l \) be integers such that \( k \geq 2 \) and \( l \geq 3 \) is odd and divisible by 3. Then \( \frac{2}{3} kl \leq n(k,l) \leq \frac{2}{3} kl + 1 \). Furthermore, \( n(k,l) = \frac{2}{3} kl \) whenever \( k \) is odd.

**Proof.** By Theorem 2.7, \( \frac{2}{3} kl \leq n(k,l) \) and, by Theorem 3.3, \( n(k,l) \leq \frac{2}{3} kl + 1 \). If \( k \) is odd, then \( \frac{2}{3} kl = \lfloor \frac{2}{3} kl \rfloor \equiv 2 \pmod{4} \) and, by Theorem 3.3(3), \( n(k,l) \leq \frac{2}{3} kl \). Hence in this case \( n(k,l) = \frac{2}{3} kl \). \( \square \)

**References**


