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Note

Two Remarks on Affine Designs with Classical Parameters

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Simple proofs are given for Dembowski's theorem characterizing the classical affine designs and for the existence of affine designs with classical parameters but not isomorphic to any affine space. © 1985 Academic Press, Inc.

In this note, we present simple alternative proofs for two well-known and fundamental results on affine designs: Dembowski's [3, 4] characterization of the affine spaces among the affine designs and the existence of non-classical affine designs with classical parameters, which is due to Kantor, cf. [5]. We assume the reader to be familiar with the standard notions of design theory. For background, we refer to [1, 5]. Let us begin with recalling Dembowski's theorem.

THEOREM 1 (Dembowski). *Let $\mathscr{D}=(V,\mathscr{B})$ be a non-trivial resolvable $S_\lambda(2,k;v)$ with $\lambda>1$. Furthermore assume that $s=v/k>2$ or that each plane of $\mathscr{D}$ contains at least four points. Then the following conditions are equivalent.*

(i) *$\mathscr{D}$ is isomorphic to the design $AG_{n-1}(n,s)$ of points and hyperplanes of some n-dimensional affine space $AG(n,s)$, where $n>2$.*

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(ii) *Each line of* $\mathscr{D}$ *meets each non-parallel block.*

(iii) *Each line has exactly s points.*

(iv) *Each plane is contained in exactly* $\rho = \lambda - (r - \lambda)/s$ *blocks.*

(v) $\mathscr{D}$ *is smooth and affine.*

Proof. By linear algebra (i) implies each of (ii), (iii), and (v). Moreover, (ii), (iii), and (iv) are equivalent and (v) implies (iv). For these parts of the proof—which are not too difficult—the reader is referred to Dembowski's original proof [3] or to [1] where details are given. The main part of the proof consists of showing that (ii), (iii), and (iv) together imply (i). We will do this in five steps.

Step 1. Two distinct lines of $\mathscr{D}$ are parallel iff they are contained in a plane of $\mathscr{D}$ and have no point in common.

Proof: Let L and L' be disjoint lines in a common plane P, and assume $L \nparallel L'$. Then there is a block B containing L but not parallel to L'. Then $P \not\subseteq B$ and thus $P \cap B = L$, as $\mathscr{D}$ is smooth by (iv). By (ii), L' meets B in a point p; then $L' \subseteq P$ implies $p \in P \cap B = L$, a contradiction. Thus indeed $L \parallel L'$. The converse may be left to the reader as a simple exercise.

Step 2. If (p, L) is a non-incident point–line pair, then there is exactly one line G with $p \in G \parallel L$ (Euclid's parallel axiom).

Proof: Clearly there is at most one such line G. To see the existence of G, choose a block B containing L but not p; thus $P \cap B = L$, where P is the plane determined by p and L. Furthermore, let C be the unique block through p parallel to B and let q be a point on L. We claim that q is on a line $H \subseteq P$ with $H \neq L$ and $p \notin H$. To see this, let $t \in L \setminus \{q\}$ be a point. If the line $\overline{tp}$ contains a third point $x \neq p$, t, we may choose $H = \overline{qx}$. Otherwise $\mathscr{D}$ has constant line size $s = 2$ by (iii). Then P contains a point $y \neq p, q, t$ by hypothesis, and we may choose $H = \overline{qy}$. By (ii), H intersects C in a unique point $z \neq p$. Then $\overline{pz} = P \cap C$. Since $B \cap C = \emptyset$, we have $\overline{pz} \cap L = \emptyset$ and thus $\overline{pz} \parallel L$ by Step 1.

Step 3. The points and lines of $\mathscr{D}$ form an affine space $AG(n, s)$ $\mathscr{A}$ with $n > 2$.

Proof: We shall verify the axioms of Lenz [8] for affine spaces; cf. also [2]. By definition, any two points of $\mathscr{D}$ are on a unique line, which is the first axiom. Also any line has at least two points. Since parallelism of lines is an equivalence relation and since Euclid's parallel axiom holds by Step 2, the second axiom is satisfied. By Steps 1 and 2, each plane of $\mathscr{D}$ is an affine plane. This implies the validity of the next two axioms: if A, B, C are lines with $A \parallel B \neq A$, if $a \in A \cap C$, $b \in B \cap C$, and $c \in C$ are distinct points, and if $x \in A$ is a point, then there is a point $d = B \cap \overline{cx}$. Moreover, if a, b, c are non-collinear points, then there is a point d such that $\overline{ab} \parallel \overline{cd}$ and $\overline{ac} \parallel \overline{bd}$. Finally, there are two non-parallel disjoint lines, since V is neither a line nor a plane.

Step 4. Each block of $\mathscr{D}$ is a hyperplane of $\mathscr{A}$.

Proof: If B is a block of $\mathscr{D}$ and $L \subseteq B$ a line, then B contains each line G parallel to L and containing a point of B. Hence B is an affine subspace of $\mathscr{A}$ and thus (by (ii)) a hyperplane.

Step 5. Each hyperplane of $\mathscr{A}$ is a block of $\mathscr{D}$.

Proof: By Step 4, $\mathscr{D}$ is affine with $\mu = k/s = q^{n-2}$ by Step 3 (where μ is the intersection number of non-parallel blocks). Thus $b = s(s^2\mu - 1)/(s-1) = s(s^n - 1)/(s-1)$ which is the number of hyperplanes of $\mathscr{A}$. ∎

It may be remarked that this proof is not only considerably shorter than the original proof of Dembowski; it also avoids a somewhat circular argument: Dembowski first shows that $\mathscr{D}$ is affine and in fact residual. Then he uses the Dembowski–Wagner theorem [6] to show that $\mathscr{D}$ is the residual of a projective space. But the Dembowski–Wagner theorem is proved by using the Veblen–Young axioms [10] for projective spaces. We have seen that projective spaces are not needed in this problem.

Next we construct non-classical designs with the parameters of some affine space by using symmetric nets. Recall that a symmetric (s, μ) net is an affine $S_{s\mu}(1, s\mu; s^2\mu)$ whose dual is likewise affine. The classical examples of symmetric nets are obtained from a design $AG_{n-1}(n, q)$ by discarding all blocks parallel to a given line; here $s = q$ and $\mu = q^{n-2}$. For background, we refer to [1, 7], or [9]. The following result is due to Kantor (cf. Dembowski [5, 2.4.36]). We remark that the proof given in [5] seems to us incomplete (if correct). Moreover, the possibility of using symmetric nets in constructing non-isomorphic affine designs is interesting in itself; e.g., Mavron [9] has recently used this method to obtain asymptotic results on the number of non-isomorphic affine designs with the parameters of some affine space, whenever the order q is the order of a proper near-field.

THEOREM 2 (Kantor). *Let q be any prime power and n an integer with $n > 2$ and $(q, n) \neq (2, 3)$. Then there exists an affine design with the parameters of but not isomorphic to $AG_{n-1}(n, q)$.*

Proof. Let $\mathscr{S}$ be a classical symmetric (q, q^{n-2})-net constructed from $AG_{n-1}(n, q)$ by removing all blocks parallel to a line L, and let $\mathscr{D}$ be $AG_{n-2}(n-1, q)$. As the dual of $\mathscr{S}$ is affine, too, we may consider parallel classes of points which we will simply call *groops*. Note that the number of groops of $\mathscr{S}$ and the number of points of $\mathscr{D}$ both equal q^{n-1}. Now let α be any bijection between the groops of $\mathscr{S}$ and the points of $\mathscr{D}$. For each block

B of $\mathscr{S}$, form a set B' as the union of the groops corresponding to the points of B under α. Then adjoin all B' as new blocks to $\mathscr{S}$ to obtain a larger structure $\mathscr{S}'$. It is easily seen that $\mathscr{S}'$ is an affine 1-design with parameters q and q^{n-1} and with $r=(q^n-1)/(q-1)$ parallel classes of blocks. (This is a special case of a much more general construction; cf. [7, 9].) But it is well known that an affine 1-design is in fact a 2-design iff $r=(s^2\mu-1)/(s-1)$. Thus $\mathscr{S}'$ is an affine 2-design with the same parameters as $AG_{n-1}(n, q)$, i.e., with $s=q$ and $\mu=q^{n-2}$. It remains to choose α in such a way that $\mathscr{S}'$ is not isomorphic to $AG_{n-1}(n, q)$.

To this end, choose a block B of $\mathscr{D}$ and any two joined points p and r of $\mathscr{S}$. The line $\overline{pr}$ of $\mathscr{S}$ (i.e., the intersection of all blocks containing p and r) is easily seen to consist of q points of $\mathscr{S}$ (in fact, it is just a line not parallel to L of $AG_{n-1}(n, q)$). Choose α in such a way that two points of B are mapped on the groops of p and r, respectively, whereas all other points of B are mapped onto groops not intersecting the line $\overline{pr}$. By construction of B', it is obvious that $\overline{pr} \cap B' = \{p, r\}$. Therefore the line $\overline{pr}$ of $\mathscr{S}'$ consists of p and r only. Thus $\mathscr{S}'$ is not isomorphic to $AG_{n-1}(n, q)$ except possibly for $q=2$. In this case, choose a point $p' \neq p, r$ which is joined to both p and r, and consider the plane $\overline{prp'}$ of $\mathscr{S}$. This plane contains a unique fourth point r' (it is in fact just a plane not parallel to L of $AG_{n-1}(n, 2)$). This time choose α in such a way that the groops of p, r, and p' correspond to points of B, whereas the groop of r' does not, which is possible as $n \neq 3$ in this case. Then the plane $\overline{prp'}$ of $\mathscr{S}'$ has only 3 points, and hence $\mathscr{S}'$ is not classical.

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Note added in proof. Using symmetric nets, one may in fact show that the number of affine designs with classical parameters grows exponentially. Similar results hold for symmetric designs, symmetric nets, and biaffine designs. See [11].

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