The admissibility of sporadic simple groups

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ABSTRACT

In 1955 Hall and Paige conjectured that a finite group is admissible, i.e., admits complete mappings, if its Sylow 2-subgroup is trivial or noncyclic. In a recent paper, Wilcox proved that any minimal counterexample to this conjecture must be simple, and further, must be either the Tits group or a sporadic simple group. In this paper we improve on this result by proving that the fourth Janko group is the only possible minimal counterexample to this conjecture: John Bray reports having proved that this group is also not a counterexample, thus completing a proof of the Hall–Paige conjecture.

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1. Introduction

A complete mapping of a group $G$ is a bijection $\theta : G \rightarrow G$ for which the mapping $g \mapsto g\theta(g)$ is also a bijection: $G$ is admissible if $G$ admits complete mappings. The admissibility of a group is of combinatorial significance as if $G = \{g_1, \ldots, g_n\}$ is a finite group, then the Cayley table of $G$, i.e., the $n \times n$ matrix with $ij$th entry $g_i g_j$, is a Latin square, and this Latin square has an orthogonal mate if and only if $G$ is admissible.

A long-standing problem is that of determining which groups are admissible. In 1947 Paige [13] proved that a finite abelian group is admissible if and only if it does not contain a unique involution. In the same paper he proved that all groups of odd order were admissible: for odd-order groups the identity mapping is a complete mapping. Using transfinite induction, in 1950 Bateman [3] proved that all infinite groups are admissible.

Then, in 1955 Hall and Paige [12] proved that a finite group with a nontrivial, cyclic Sylow 2-subgroup is not admissible. They conjectured the converse, that all finite groups with trivial or noncyclic Sylow 2-subgroups are admissible: we will call such groups HP-groups as in [2]. Hall and Paige proved their conjecture true for alternating groups, symmetric groups, and solvable groups. In
their constructions they used bijections $\theta, \eta : T \to T$, $T$ a dual transversal of a subgroup $H$ of $G$, satisfying $t\theta(t)H = \eta(t)H$ for all $t \in T$, to extend complete mappings of $H$ to complete mappings of $G$: Aschbacher [2] calls $(H, T, \theta, \eta)$ an HP-system.

A number of classes of groups have since been proved to be admissible: $Sz(2^{2n+1})$, $SU(3, q^2)$ for $q$ even, and $PSU(3, q^2)$ for $q$ even by Di Vincenzo [9]; the Mathieu groups by Dalla Volta and Gavioli [7]; and most linear groups in papers by Saeli [15], Dalla Volta and Gavioli [7,8], and Evans [10,11].

Aschbacher [2] took a different approach by studying possible minimal counterexamples to the Hall–Paige conjecture. He proved that any minimal counterexample $G$ to the Hall–Paige conjecture must have a quasisimple normal subgroup $L$ for which $C_G(L)$ and $G/L$ are cyclic 2-groups. He showed that it was sufficient to construct HP-systems for almost simple groups to prove the Hall–Paige conjecture, and he constructed such systems for many almost simple groups with minimal normal subgroups of Lie type, and for almost simple groups whose minimal normal subgroups are Mathieu groups.

In a recent paper, Wilcox [16] improved on Aschbacher’s results by proving that any minimal counterexample to the Hall–Paige conjecture must be a simple group. Further, using double cosets, he was able to show that any minimal counterexample to the Hall–Paige conjecture must be either the Tits group or a sporadic simple group.

The Mathieu groups have already been dealt with: Aschbacher [2] proved that they could not be minimal counterexamples to the Hall–Paige conjecture, and Dalla Volta and Gavioli [7] proved them to be admissible. This leaves 22 possible minimal counterexamples to the Hall–Paige conjecture. In this paper we use Wilcox’s methods to further reduce the list of possible minimal counterexamples to the Hall–Paige conjecture to just one: $J_4$. Bray [4] has proved that $J_4$ is not a counterexample to the conjecture, and so the Hall–Paige conjecture has now been proved. In this paper we will use the notation for groups given in the Atlas of Finite Groups [6].

2. Double cosets and admissibility

Let $G$ be a finite group, $H$ a subgroup of $G$, $G/H$ the set of right cosets of $H$ in $G$, and $D = \{HgH \mid g \in G\}$ the set of double cosets of $H$ in $G$. The double cosets of $H$ in $G$ partition the element set of $G$. Each $D \in D$ is a union of elements of $G/H$ and $\sum_{D \in D} |D|/|H| = |G|/|H|$. We will make considerable use of the following results on double cosets and admissibility due to Wilcox [16].

Lemma 1. Suppose that $H$ is an admissible subgroup of a finite group $G$, and that $D$ is the set of double cosets of $H$ in $G$. If there exist bijections $\phi, \psi : D \to D$ satisfying $|D| = |\psi(D)| = |\phi(D)|$ and $\psi(D) \subseteq D\phi(D)$ for all $D \in D$, then $G$ is admissible.

Proof. See Corollary 15 in [16]. □

We will call $\{D, \phi, \psi\}$ in Lemma 1 a $W$-system, and we will say that a $W$-system $\{D, \phi, \psi\}$ is a simple $W$-system if $\phi$ and $\psi$ are equal to the identity mapping as in the following lemma.

Lemma 2. Suppose that $H$ is an admissible subgroup of a finite group $G$, and that $D$ is the set of double cosets of $H$ in $G$. If $D \subseteq D^2$ for all $D \in D$, then $G$ is admissible.

Proof. See Corollary 16 in [16]. □

We will say that $D$ is $W$-simple if $D \subseteq D^2$. Thus, by Lemma 2, if $H$ is admissible and every $D \in D$ is $W$-simple, then $G$ is admissible. For $D \in D$, the index of $H$ in $D$, denoted $|D : H|$, is the number of distinct right cosets of $H$ contained in $D$. If $D = HgH$, then $|D : H| = |D|/|H| = |H : H \cap H^g|$. We will call the double coset $HeH = H$ the trivial double coset. We will say that $D$ is solitary if $D' \in D$ and $|D' : H| = |D : H|$ implies that $D' = D$. Note that, if all the double cosets of $H$ in $G$ are solitary, then the double coset that an element $g \in G$ is contained in can be determined by computing $|H : H \cap H^g|$. Note also, that if $D$ contains a solitary double coset that is not $W$-simple, then a $W$-system cannot exist. The following lemma gives a simple test to determine whether a double coset is $W$-simple or not.
Lemma 3. Suppose that $H$ is a subgroup of a finite group $G$, and that $D$ is the set of double cosets of $H$ in $G$. Then $D \in \mathcal{D}$ is W-simple if and only if $g^2 \in D$ for some $g \in D$.

Proof. Let $D \in \mathcal{D}$. If $g^2 \in D$ for some $g \in D$, then $D^2 = HgHHgH \supseteq Hg^2H = D$, and so $D$ is W-simple.

If $D$ is W-simple and $h \in D$, then $h \in D^2$ and $h = m_1hm_2hm_3$ for some $m_1, m_2, m_3 \in H$. Setting $g = hm_2$, we see that $g \in D$ and $g^2 = m_1^{-1}hm_3^{-1}m_2$ and so $g^2 \in D$. □

As an example, we will show that $T = 2F_4(2)'$, the Tits group, and $HN$, the Harada–Norton group, are not minimal counterexamples to the Hall–Paige conjecture.

Theorem 1. $T$ is not a minimal counterexample to the Hall–Paige conjecture.

Proof. Using the permutation representation of degree 1,600 given in [17], we verified the following using magma. In the description given in [17], $T$ is generated by two elements, $x$ and $y$, that satisfy the relations $x^2 = y^3 = (xy)^6 = [x, y]^5 = [x, xy]^4 = ((xy)^4xy^4)^6 = 1$, where $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$. Let $H$ be the subgroup of $T$ generated by $y$ and $(xy)^3(xy^{-1})^4xy(xy^{-1})^2(xy)^2x$. Then $H$ is a maximal subgroup of $T$ of index 1,600, and is an HP-group. The set of double cosets of $H$ in $T$ is $\mathcal{D} = \{D_i = Hd_iH \mid i = 1, \ldots, 4\}$, where

\[
\begin{align*}
d_1 &= 1, \\
d_2 &= (xyy^2)^4, \\
d_3 &= xy^{-1}xy(xy^{-1})^2(xy)^2x, \text{ and} \\
d_4 &= xyx.
\end{align*}
\]

Now $|D_1 : H| = 1$,

$|D_2 : H| = 312$,

$|D_3 : H| = 351$, and

$|D_4 : H| = 936$.

Further $d_i^2 \in D_i$ for $i = 1, \ldots, 4$. Hence, by Lemma 3, each of $D_1, \ldots, D_4$ is W-simple, and thus, by Lemma 2, $T$ is not a minimal counterexample to the Hall–Paige conjecture. □

Theorem 2. $HN$ is not a minimal counterexample to the Hall–Paige conjecture.

Proof. Using the permutation representation of degree 1,140,000 given in [17] and built into magma, we verified the following using magma. In the description given in [17], $HN$ is generated by two elements, $x$ and $y$, and these satisfy the relations $x^2 = y^3 = (xy)^{22} = (xyxy)^5 = (x(xy^2xyxyxy^2)^{-1}(xy)^{11}(xy^2xyxyxy^2)) = 1$. Let $H$ be the subgroup of $HN$ generated by $a = xyxyxyxy^2xy^2x$ and $b = (xyxyxyxy^2xy)^{-2}xyxy(2)xyxyxy^2xy^2$. Then $H$ is a maximal subgroup of $HN$ of order 177,408,000 and index 1,539,000, and is an HP-group. The set of double cosets of $H$ in $HN$ is $\mathcal{D} = \{D_i = Hd_iH \mid i = 1, \ldots, 9\}$, where

\[
\begin{align*}
d_1 &= 1, \\
d_2 &= d_4abab^4ad_4, \\
d_3 &= xyx, \\
d_4 &= (xb^2ay)^3, \\
d_5 &= xyxy^2, \\
d_6 &= (xb^2ax)^4, \\
d_7 &= xy^2.
\end{align*}
\]
\[ d_8 = y, \quad \text{and} \]
\[ d_9 = xby^2. \]

Now \(|D_1 : H| = 1,\)
\[ |D_2 : H| = 1408, \]
\[ |D_3 : H| = 2200, \]
\[ |D_4 : H| = 5775, \]
\[ |D_5 : H| = 35,200, \]
\[ |D_6 : H| = 123,200, \]
\[ |D_7 : H| = 277,200, \]
\[ |D_8 : H| = 354,816, \quad \text{and} \]
\[ |D_9 : H| = 739,200. \]

Further \(d_i^2 \in D_i\) for \(i \in \{1, \ldots, 9\} \setminus \{2, 4\}\). Hence, by Lemma 3, each \(D_i, i \in \{1, \ldots, 9\} \setminus \{2, 4\}\), is W-simple. Now \(d_2(ab)^10d_2 \in D_2\), and so \(D_2^2 = H \cup Hd_2H \cup D_2\) is W-simple. Thus \(D_2\) is also W-simple. Now \(d_4ab^3(ab)^3d_4 \in D_4\), and so \(D_4^2 = H \cup H \cup H \cup D_4\) is W-simple. As each double coset of \(H\) in \(G\) is W-simple, \(HN\) is not a minimal counterexample to the Hall–Paige conjecture by Lemma 2. \(\Box\)

3. Doubly transitive sporadic simple groups

Doubly transitive groups can be dealt with using the fact that a group \(G\) acts doubly transitively on a set \(X\) if and only if the set of double cosets of \(G\) in \(G\) has order 2, where \(G_x\) is the stabilizer of a point \(x \in X\).

**Lemma 4.** If \(H\) is a point-stabilizer in a doubly transitive permutation representation of a finite simple group \(G\) and \(H\) is admissible, then \(G\) is admissible.

**Proof.** If \(G\) acts doubly transitively on a set \(X\) and \(H = G_x\), for some \(x \in X\), then the set of double cosets of \(H\) in \(G\) is \(D = \{H, D\}\). Clearly \(H\) is W-simple. To prove that \(D\) is W-simple we will assume the contrary, and thus \(g^2 \in H\) for all \(g \in G\). In particular, if \(|g|\) is odd, then \(g \in H\). Let \(K\) be the subgroup of \(G\) generated by the set of odd-order elements of \(G\). \(K\) is a nontrivial characteristic subgroup of \(G\) and is contained in \(H\), contradicting the simplicity of \(G\). Thus \(D\) is W-simple and the result then follows from Lemma 2. \(\Box\)

In Lemma 4, if \(G\) is not assumed to be simple, then an alternative proof that \(D\) is W-simple can be given in the case that \(|G : H| > 2\). \(D^2\) is a union of double cosets of \(H\) in \(G\) and \(D^2 \neq H\) as \(|D^2|/|H| > |D : H| > 1 = |H|/|H|\). Thus \(D^2\) is either \(D\) or \(D \cup H\).

**Theorem 3.** The groups \(HS\) and \(Co_3\) are not minimal counterexamples to the Hall–Paige conjecture.

**Proof.** The group \(HS\) has a doubly transitive permutation representation of degree 176 with point-stabilizer isomorphic to \(U_3(5) : 2\), and the group \(Co_3\) has a doubly transitive permutation representation of degree 276 with point-stabilizer isomorphic to \(McL : 2\). In each case, the point-stabilizer is an HP-group. The result then follows from Lemma 4. \(\Box\)

It should be noted that from Lemma 4 we can obtain yet another proof that none of the Mathieu groups are minimal counterexamples to the Hall–Paige conjecture. We can also obtain from Lemma 4 an alternative, inductive, proof that the alternating groups are admissible: \(A_4\) is admissible, and, for \(n \geq 5\), \(A_n\) has a natural doubly transitive permutation representation with point-stabilizer isomorphic to \(A_{n-1}\).
4. Rank-3 sporadic simple groups

If \( G \) acts transitively on a set \( X \), then this action extends naturally to an action on \( X \times X \). The orbits of this action are called orbitals, and their number is the rank of this permutation representation. The orbital \( \{(x, x) \mid x \in X\} \) is the trivial orbital. If \( \{E_1, \ldots, E_r\} \) is the set of orbitals and \( H = G_x \) for some \( x \in X \), then the sets \( O_i \), \( i = 1, \ldots, r \), defined by \( O_i = \{y \mid (x, y) \in E_i\} \) are the orbitals of \( H \) on \( X \). For each \( O_i \), the set \( D_i = \{g \in G \mid x^g = y \text{ for some } y \in O_i\} \) is a double coset of \( H \) in \( G \). This establishes one–one correspondences between the set of orbitals, the set of orbits of \( H \) on \( X \), and the set of double cosets of \( H \) in \( G \): the trivial orbital corresponds to the trivial double coset of \( H \) in \( G \). To each orbital \( E \) we associate a paired orbital \( E' = \{(y, x) \mid (x, y) \in E\} \): \( E \) is self-paired if \( E' = E \). If \( D \) is a double coset of \( H \) in \( G \), then the set \( D^{(1)} = \{g \in D \} \) is also a double coset of \( H \) in \( G \), which we will call the inverse of \( D \), and we will say that \( D \) is self-inverse if \( D^{(1)} = D \). Note that, as \( |D : H| = |D^{(1)} : H| \), if \( D \) is solitary, then \( D \) is self-inverse. If \( D \) is the double coset corresponding to the orbital \( E \), then \( D^{(1)} \) is the double coset corresponding to the orbital \( E' \), as \( (x, x^g) \in E \) if and only if \( (x^g, x) \in E' \) if and only if \((x, x^{g^{-1}}) \in E' \). Hence \( D \) is self-inverse if and only if \( E \) is self-paired. For each orbital \( E \), \( G_E \) will denote the digraph with vertices the elements of \( X \) and directed edges the elements of \( E \): \( G_E \) is an undirected graph if \( E \) is self-paired. \( G_E \) is called an orbital (di)graph. If \( \{E_1, \ldots, E_r\} \) are the orbitals of \( G \) ordered so that \( |E_1| \leq |E_2| \leq \cdots \leq |E_r| \), \( E_1 \) the trivial orbital, \( O_1, \ldots, O_r \) are the corresponding orbits of \( H \) on \( X \), \( D_1, \ldots, D_r \) are the corresponding double cosets of \( H \) in \( G \), and \( x = a_1, \ldots, a_n \) are representatives of \( O_1, \ldots, O_r \), respectively, then the collapsed adjacency matrix for \( G_{E_k} \) is the \( r \times r \) matrix \( A_{kk}^i \) with \( i \)th entry \( A_{ij}^k = |\{y \in O_j \mid (a_i, y) \in E_k\}| \).

**Lemma 5.** If \( A_{kk}^i \neq 0 \) then the double coset \( D_k \) of \( H \) in \( G \) is \( W \)-simple.

**Proof.** If \( A_{kk}^i \neq 0 \), then there exist edges \((x, x^a), (x, x^b)\), \((x^a, x^b) \in E_k \) for some \( a, b \in D_k \). Action by \( a^{-1} \) shows that \((x, x^{ba^{-1}}) \in E_k \). Thus \( ba^{-1} \in D_k \) and \( D_k^2 = Hba^{-1}HHaH \supseteq Hbh \) if \( D_k \) is \( W \)-simple. \( \Box \)

Let \( E_1, E_2, E_3 \) be the orbitals in a rank-3 permutation group \( G \) on \( X \), \( E_1 \) the trivial orbital. Let \( H = G_x \) for some \( x \in X \), let \( O_1, O_2, O_3 \) be the orbitals of \( H \) on \( X \) corresponding to the orbitals \( E_1, E_2, E_3 \), respectively, and let \( w \in O_2 \) and \( z \in O_3 \). If \( |G| \) is even, then \( E_2 \) and \( E_3 \) are self-paired. As in Aschbacher [1], we set \( n = |X|, k = |O_2|, l = |O_3|, \lambda = |\{y \in O_2 \mid (w, y) \in E_2\}|, \) and \( \mu = |\{y \in O_2 \mid (z, y) \in E_2\}| \). \( (n, k, l, \lambda, \mu) \) are the parameters of the rank-3 graph \( G_{E_2} \), and the corresponding parameters for \( G_{E_3} \) are \( (n, l, k, l-k+\mu - 1, l-k+\lambda + 1) \).

**Lemma 6.** Let \( G \) be an even-order, rank-3 permutation group, with point-stabilizer \( H \), and parameters \((n, k, l, \lambda, \mu) \). If \( H \) is admissible, \( \lambda > 0 \), and \( l-k+\mu-1 > 0 \), then \( G \) is admissible.

**Proof.** Let \( E_1, E_2, E_3 \) be the orbitals of an even-order, rank-3 permutation group \( G \), where \( E_1 \) is the trivial orbital, let the parameters of \( G_{E_2} \) be \((n, k, l, \lambda, \mu) \), and let \( H \) be a point-stabilizer. Direct computation shows that \( A_{22}^3 = \lambda \) and \( A_{33}^3 = l-k+\mu - 1 \). Thus if \( \lambda > 0 \), then, by Lemma 5, the double coset of \( H \) in \( G \) corresponding to the orbital \( E_2 \) is \( W \)-simple, and if \( l-k+\mu-1 > 0 \), then, by Lemma 5, the double coset of \( H \) in \( G \) corresponding to the orbital \( E_3 \) is \( W \)-simple. As \( H \) is also \( W \)-simple the result follows from Lemma 2. \( \Box \)

**Theorem 4.** \( J_2, \) McI, \( Ru, \) Suz, \( Co_2, \) \( Fi_{22}, \) \( Fi_{23}, \) and \( Fi_{24}' \) are not minimal counterexamples to the Hall–Paige conjecture.

**Proof.** From the collapsed adjacency matrices in [14] we can read off the parameters for rank-3 permutation representations of these groups: \( k = A_{12}^2, \lambda = A_{13}^2, \lambda = A_{22}^2, \) and \( \mu = A_{32}^2 \). These parameters are displayed in Table 1.

We note that in each case the point-stabilizer, \( \lambda > 0 \), and \( l-k+\mu-1 > 0 \). The result then follows from Lemma 6. \( \Box \)
Table 1

Some rank-3 permutation representations.

<table>
<thead>
<tr>
<th>G</th>
<th>H</th>
<th>n</th>
<th>k</th>
<th>l</th>
<th>λ</th>
<th>μ</th>
</tr>
</thead>
<tbody>
<tr>
<td>J2</td>
<td>U₃(3)</td>
<td>100</td>
<td>36</td>
<td>63</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>McL</td>
<td>U₄(3)</td>
<td>275</td>
<td>112</td>
<td>162</td>
<td>30</td>
<td>56</td>
</tr>
<tr>
<td>Ru</td>
<td>2F₄(2)</td>
<td>4060</td>
<td>1755</td>
<td>2304</td>
<td>730</td>
<td>780</td>
</tr>
<tr>
<td>Suz</td>
<td>G₂(4) : 2</td>
<td>1782</td>
<td>416</td>
<td>1365</td>
<td>100</td>
<td>96</td>
</tr>
<tr>
<td>Co₂</td>
<td>U₆(2) : 2</td>
<td>2300</td>
<td>891</td>
<td>1408</td>
<td>378</td>
<td>324</td>
</tr>
<tr>
<td>Fi₂₂</td>
<td>2 U₄(2)</td>
<td>3510</td>
<td>693</td>
<td>2816</td>
<td>180</td>
<td>126</td>
</tr>
<tr>
<td>Fi₂₃</td>
<td>O₇(3)</td>
<td>14,080</td>
<td>3159</td>
<td>10,920</td>
<td>918</td>
<td>648</td>
</tr>
<tr>
<td>Fi₂₄</td>
<td>2 Fi₂₂</td>
<td>31,671</td>
<td>3510</td>
<td>28,160</td>
<td>693</td>
<td>351</td>
</tr>
<tr>
<td></td>
<td>O₇⁺(3) : S₃</td>
<td>137,632</td>
<td>28,431</td>
<td>109,200</td>
<td>6030</td>
<td>5832</td>
</tr>
</tbody>
</table>

5. Rank-4 and rank-5 sporadic simple groups

Praeger and Soicher [14] give collapsed adjacency matrices for permutation representations of sporadic simple groups of rank 5 or less: this is a particularly useful resource as a number of proofs that a given sporadic simple group is not a minimal counterexample to the Hall–Paige conjecture can be deduced from these matrices.

Theorem 5. O’N, Ly, and Co₁ are not minimal counterexamples to the Hall–Paige conjecture.

Proof. O’N has a rank-5 permutation representation of degree 122,760 with point-stabilizer H isomorphic to L₃(7) : 2. The indices are

\[ |D₁ : H| = 1, \]
\[ |D₂ : H| = 5586, \]
\[ |D₃ : H| = 6384, \]
\[ |D₄ : H| = 52,136, \]
\[ |D₅ : H| = 58,653. \]

From the collapsed adjacency matrices in [14] we see that

\[ A₁₁ = 1, \]
\[ A₂₂ = 364, \]
\[ A₃₃ = 349, \]
\[ A₄₄ = 22,057, \]
\[ A₅₅ = 27,972. \]

Thus, by Lemma 5, O’N is not a minimal counterexample to the Hall–Paige conjecture.

Ly has a rank-5 permutation representation of degree 8,835,156 with point-stabilizer H isomorphic to G₂(5). The indices are

\[ |D₁ : H| = 1, \]
\[ |D₂ : H| = 19,530, \]
\[ |D₃ : H| = 968,750, \]
\[ |D₄ : H| = 2,034,375, \]
\[ |D₅ : H| = 5,812,500. \]

From the collapsed adjacency matrices in [14] we see that
$A_{11}^1 = 1,$
$A_{22}^2 = 154,$
$A_{33}^3 = 114,013,$
$A_{34}^4 = 468,670,$ and
$A_{35}^5 = 3,821,006.$

Thus, by Lemma 5, $L_Y$ is not a minimal counterexample to the Hall–Paige conjecture.

$C_0_1$ has a rank-4 permutation representation of degree 98,280 with point-stabilizer $H$ isomorphic to $C_0_2$. The indices are

$|D_1 : H| = 1,$
$|D_2 : H| = 4600,$
$|D_3 : H| = 46,575,$ and
$|D_4 : H| = 47,104.$

From the collapsed adjacency matrices in [14] we see that

$A_{11}^1 = 1,$
$A_{22}^2 = 892,$
$A_{33}^3 = 21,582,$ and
$A_{34}^4 = 22,528.$

Thus, by Lemma 5, $C_0_1$ is not a minimal counterexample to the Hall–Paige conjecture. □

So far every W-system, used in our proofs, has been a simple W-system. The following lemma allows us to construct other W-systems.

**Lemma 7.** If $A_{kj}^k > 0$, then $D_k^2 \supseteq D_j$.

**Proof.** If $A_{kj}^k > 0$, then there exist edges $(x, x^g), (x^h, x^h g) \in E_k$ for some $g, h \in D_k$, where $h g \in D_j$. Then $D_k^2 = H h H h g H \supseteq H h g H = D_j$. □

**Theorem 6.** He is not a minimal counterexample to the Hall–Paige conjecture.

**Proof.** He has a rank-5 permutation representation of degree 2058 with point-stabilizer $H$ isomorphic to $S_4(4) : 2$. The indices are

$|D_1 : H| = 1,$
$|D_2 : H| = 136,$
$|D_3 : H| = 136,$
$|D_4 : H| = 425,$ and
$|D_5 : H| = 1360.$

From the collapsed adjacency matrices in [14] we see that

$A_{11}^1 = 1,$
$A_{23}^2 = 36,$
$A_{32}^3 = 36,$
\[ A_{44}^4 = 136, \text{ and } A_{55}^5 = 894. \]

Thus, by Lemma 5, \( D_1, D_4, \) and \( D_5 \) are W-simple; and, by Lemma 7, \( D_2^2 \cong D_3 \) and \( D_3^2 \cong D_2 \). If we define \( \phi, \psi : \mathcal{D} \to \mathcal{D} \) by \( \phi(D_i) = D_i \) for all \( i \), and \( \psi(D_i) = D_i \) if \( i \neq 2, 3, \) \( \psi(D_2) = D_3 \), and \( \psi(D_3) = D_2 \), then \( \{ \mathcal{D}, \phi, \psi \} \) is a W-system. As \( H \) is an HP-group, by Lemma 1, \( He \) is not a minimal counterexample to the Hall–Paige conjecture.

6. The groups \( Th, B, \) and \( M \)

The groups \( Th, B, \) and \( M \) can be handled using the GAP-database in [5]. In this database a different definition of collapsed adjacency matrices is used. The \( ij \)th entry of a \( k \)th collapsed adjacency matrix in [5] is \( A_{ij}^{kp} \) by the definition used in this paper and in [14]; here, \( kp \) is defined by \( E_{kp} = E_k^p \). If the \( k \)th orbital is self-paired, then the \( kk \)th entry of the \( k \)th collapsed adjacency matrix will be the same in either definition.

**Theorem 7.** \( Th, B, \) and \( M \) are not minimal counterexamples to the Hall–Paige conjecture.

**Proof.** The GAP-database in [5] contains two permutation representations of \( Th \). The second representation is a rank-11 permutation representation of degree 283,599,225 with point-stabilizer \( H \) isomorphic to \( 2^5.L_5(2) \), an HP-group of order 319,979,520. The indices are

\[
|D_1 : H| = 1, \\
|D_2 : H| = 248, \\
|D_3 : H| = 59,520, \\
|D_4 : H| = 2,064,384, \\
|D_5 : H| = 2,064,384, \\
|D_6 : H| = 2,539,520, \\
|D_7 : H| = 6,666,240, \\
|D_8 : H| = 35,553,280, \\
|D_9 : H| = 63,995,904, \\
|D_{10} : H| = 63,995,904, \text{ and } \\
|D_{11} : H| = 106,659,840.
\]

We used the GAP-database in [5] and the GAP-program in this database to compute the collapsed adjacency matrices for this representation from the character table in GAP for \( Th \). Translating from the definition of collapsed adjacency matrices used in the database to the definition of collapsed adjacency matrices used in this paper and in [14] we found the following.

\[
A_{11}^1 = 1, \\
A_{22}^2 = 7, \\
A_{33}^3 = 488, \\
A_{44}^4 = 13,517, \\
A_{54}^5 = 13,517, \\
A_{66}^6 = 30,856, \\
A_{77}^7 = 149,968, \\
A_{88}^8 = 4,511,008, \\
A_{99}^9 = 14,435,003,
\]
Thus, by Lemma 5, $D_i$ is W-simple for $i \in \{1, \ldots, 11\} \setminus \{4, 5\}$, and, by Lemma 7, $D_4^2 \supseteq D_5$ and $D_2^2 \supseteq D_4$. If we define $\phi, \psi : D \to D$ by $\phi(D_i) = D_i$ for all $i$, and $\psi(D_i) = D_i$ for all $i \neq 4, 5$, $\psi(D_4) = D_5$, and $\psi(D_5) = D_4$, then $\{D, \phi, \psi\}$ is a W-system. Hence, by Lemma 1, $Th$ is not a minimal counterexample to the Hall–Paige conjecture.

The GAP-database in [5] contains exactly one permutation representation of $B$. The third representation is a rank-10 permutation representation of degree $11,707,448,673,375$ with point-stabilizer $H$ isomorphic to $2^{1+22}.CO_2$, an HP-group of order $354,883,595,661,213,696,000$. The indices are

$$|D_1 : H| = 1,$$
$$|D_2 : H| = 93,150,$$
$$|D_3 : H| = 7,286,400,$$
$$|D_4 : H| = 262,310,400,$$
$$|D_5 : H| = 4,196,966,400,$$
$$|D_6 : H| = 9,646,899,200,$$
$$|D_7 : H| = 470,060,236,800,$$
$$|D_8 : H| = 537,211,699,200,$$
$$|D_9 : H| = 4,000,762,036,224,$$ and
$$|D_{10} : H| = 6,685,301,145,600.$$
which it follows, by Lemma 5, that each double coset of \( H \).

7. The Janko groups, \( J_1 \), and \( J_3 \)

In this section we will see that the distribution of elements of order 3 can play an important role in determining that a given group is not a minimal counterexample to the Hall–Paige conjecture.

**Lemma 8.** Suppose that \( H \) is a subgroup of a finite group \( G \), and that \( D \) is the set of double cosets of \( H \) in \( G \). If \( D \subseteq D \subseteq D \) contains an element of order 3, then \( D \subseteq (D^{-1})^2 \) and \( D^{-1} \subseteq D^2 \).

**Proof.** Let \( g \in D \) be of order 3. Then \( D^2 = HgHHg \supseteq Hg^2H = Hg^{-1}H = D^{-1} \). Similarly \( (D^{-1})^2 \supseteq D^2 \). □

**Corollary 1.** Suppose that \( H \) is a subgroup of a finite group \( G \), and that \( D \) is the set of double cosets of \( H \) in \( G \). If \( D \in D \) is solitary and contains an element of order 3, then \( D \) is \( W \)-simple.

**Proof.** If \( D \) is solitary, then \( D^{-1} = D \), and so \( D \) is \( W \)-simple by Lemma 8. □
Lemma 9. If $H$ is an admissible subgroup of a finite group $G$ and every nontrivial double coset of $H$ in $G$ contains an element of order 3, then $G$ is admissible.

Proof. Let $D$ be the set of double cosets of $H$ in $G$. Define $\phi, \psi : D \rightarrow D$ by $\phi(D) = D$ and $\psi(D) = D^{(-1)}$. Then $(D, \phi, \psi)$ is a W-system by Lemma 8, and so $G$ is admissible by Lemma 1.

To determine that a double coset $D$ of $H$ in a finite group $G$ contains an element of order 3 or not it is sufficient to check any right coset of $H$ contained in $D$.

Lemma 10. If $H$ is a subgroup of a finite group $G$ and $D$ is a double coset of $H$ in $G$, then a right coset of $H$ in $D$ contains an element of order 3 if and only if every right coset of $H$ in $D$ contains an element of order 3.

Proof. Let $Hg$ be a right coset of $H$ in $D$, where $|g| = 3$. Then any other right coset of $H$ in $D$ is of the form $H(gm)$ for some $m \in H$, and $H(gm)$ contains the element $m^{-1}gm$ of order 3.

Using the rank-22 permutation representation of $J_1$ and the rank-14 permutation representation of $J_3$, we will prove that neither of these groups can be a minimal counterexample to the Hall–Paige conjecture.

Theorem 8. $J_1$ and $J_3$ are not minimal counterexamples to the Hall–Paige conjecture.

Proof. Using the permutation representation of degree 266 for $J_1$ given in [17], we verified the following using magma. In the description given in [17], $J_1$ is generated by two elements, $x$ and $y$, that satisfy the relations $x^2 = y^3 = (xy)^7 = (xy(xyxy^2)^3)^3 = (xy(xyxy2^4)^6xxy(xy^2)^2)^2 = 1$. Let $H$ be the subgroup of $J_1$ generated by $x$ and $(xyxy^2)^4(xy)^{-1}(xy(xyxy^2)^3xy(xyxy^2)^4)$. Then $H$ is a maximal subgroup of $J_1$ of index 1463, and is an HP-group isomorphic to $A_5 \times 2$. The set of double cosets of $H$ in $J_1$ is $D = \{D_i = Hd_iH \mid i = 1, \ldots, 22\}$, where the indices $|D_i : H|$ are 1, 12, 15 twice, 20 twice, 60 nine times, and 120 seven times. Using magma we verified that, for $i = 1, \ldots, 22$, the right coset $Hd_i$ contains an element of order 3, and hence, each $D \in D$ contains an element of order 3. Thus, by Lemma 9, $J_1$ is not a minimal counterexample to the Hall–Paige conjecture.

Using the permutation representation of degree 6156 for $J_3$ given in [17], we verified the following using magma. In the description given in [17], $J_3$ is generated by two elements, $x$ and $y$, that satisfy the relations

$$x^2 = y^3 = (xy)^{19} = [x, y]^9 = ((xy^6(xy^2)^5)^2 = ((xyxyxy^2)^2xyxyxy^2xyxy)^2 = xyxy(xyxy^2)^3xyxy(xyxy^2)^4xy(xyxy^2)^3 = (xyxyxyxyxy^2xyxy^2)^4 = 1,$$

where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$. Let $H$ be the subgroup of $J_3$ generated by $y^2xy$ and $(xy^{-4}(xyxy^2)^3(xy^2)^4).$ Then $H$ is a maximal subgroup of $J_3$ of index 14,688, and is an HP-group isomorphic to $L_2(19)$. The set of double cosets of $H$ in $J_3$ is $D = \{D_i = Hd_iH \mid i = 1, \ldots, 14\}$, where the indices $|D_i : H|$ are 1, 285, 342, 380, 570 twice, 855 twice, 1140 twice, 1710 three times, and 3420. Using magma we verified that, for $i = 1, \ldots, 14$, the right coset $Hd_i$ contains an element of order 3, and hence, each $D \in D$ contains an element of order 3. Thus, by Lemma 9, $J_3$ is not a minimal counterexample to the Hall–Paige conjecture.

To summarize: Wilcox [16] proved that any minimal counterexample to the Hall–Paige conjecture must be either the Tits group or a sporadic simple group, and we have shown that neither the Tits group nor any sporadic simple group can be a minimal counterexample, with the possible exception of $J_4$. Bray [4] has computed the collapsed adjacency matrices for the permutation representation of $J_4$ of degree 3,980,549,947 with point-stabilizer isomorphic to $2_1^{1+12}.3M_{22} : 2$, and from this has determined that $J_4$ is not a minimal counterexample to the Hall–Paige conjecture. Thus, the Hall–Paige conjecture has been proved.
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