Journal of Algebra 211, 99–114 (1999)
Article ID jabr.1998.7579, available online at http://www.idealibrary.com on IDEAL®

A Just-Nonsolvable Torsion-Free Group Defined on the Binary Tree

A. M. Brunner*

Department of Mathematics, University of Wisconsin–Parkside, Kenosha, Wisconsin 53141-2000

Said Sidki†

Departamento de Matemática, Universidade de Brasília, 70910-900 Brasíla-DF, Brazil

and

Ana Cristina Vieira‡

Departamento de Matemática, Universidade de Minas Gerais, 30161901 Belo Horizonte, Minas Gerais, Brazil

Communicated by Efim Zelmanov

Received March 13, 1998

A two-generator torsion-free subgroup of the group of finite-state automorphisms of the binary tree is constructed having the properties of being just-nonsolvable and residually "torsion-free solvable." A presentation is produced for this subgroup in two generators and two relations together with their images under the iterated application of a certain simple substitution. © 1999 Academic Press

Key Words: automorphism groups of one-rooted trees; binary adding machine; finite-state automata; residual properties of groups; just-nonsolvable groups; derived series; lower central series; presentations of groups.

*E-mail: brunner@uwp.edu.

†The author acknowledges support from the Brazilian scientific agencies FAPDF and CNPq. E-mail: sidki@mat.unb.br.

‡E-mail: anacris@tupan.mat.ufmg.br.

0021-8693/99 $30.00

1. INTRODUCTION

The group of automorphisms of rooted trees has attracted interest for being a source of examples of new phenomena in combinatorial group theory, as well as for its connections with other areas such as automata theory [3, 4] and dynamical systems [2]. This group, together with many of its finitely generated subgroups, exhibits rich recursive structures which reflect the recursiveness of the trees themselves [12]. One form the recursiveness takes is in the closure of the subgroup structure under wreath products by cyclic groups having order equal to the valency of the tree. Thus, if L is a group defined on the binary tree, then the wreath product of L by the cyclic group of order 2 is also defined in a canonical manner on the tree.

Of particular interest are the automorphisms of n-ary trees which correspond to finite-state automata defined on an alphabet of size n. These constitute the enumerable *group of finite-state automorphisms*. We have shown in [4] that the integral linear group $\mathrm{GL}(m, Z)$ has a faithful representation into the group of finite-state automorphisms of a one-rooted regular n-ary tree for some n. Besides representing integral linear groups, finite-state automorphisms also represent finitely generated groups which are not linear, such as the finitely generated infinite Burnside groups constructed in [1, 13, 5, 8] and which enjoy diverse additional remarkable properties such as being just-infinite [9] or having intermediate growth [6]. A fractal-like feature common to many of these groups is that they contain subgroups of finite index which are direct products of two or more copies of the same groups.

This paper grew out of investigating torsion-free subgroups generated by finite-state automorphisms of the binary tree. One special finite-state automorphism is the so-called "binary adding machine" which corresponds to addition by 1 in the binary system; in the notation to be explained later, this automorphism is $\tau = (e, \tau)\sigma$. It was proven in [4] that the centralizer of τ in the group of automorphisms of the binary tree is isomorphic to the dyadic integers under addition, and, moreover, any automorphism of the tree with a centralizer subgroup isomorphic to the dyadic integers is conjugate to τ. We observe that in Bass *et al.* [2, p. 116] there is a treatment of the more general q-adic adding machine.

The torsion-free group H which we describe in this paper is generated by $\tau = (e, \tau)\sigma$, and by one of its conjugates $\mu = (e, \mu^{-1})\sigma$. The generators encapsulate in their definitions the permutations they induce on the different levels of the tree. In general, it is a very difficult task fathoming the structure of a group merely from a recursive definition of how its generators act on the tree. Consequently, it is perhaps surprising to see how much detail about the group can be revealed and understood on this basis. We prove

THEOREM 1. *Let H be the subgroup of the automorphisms of the binary tree generated by $\mu = (e, \mu^{-1})\sigma$, $\tau = (e, \tau)\sigma$. Then the following properties hold*:

(i) *H is residually a "torsion-free solvable group," and is just-nonsolvable*;

(ii) *H is residually a "finite 2-group," and every finite quotient $\overline{H}$ of H factors as $\overline{H} = O(\overline{H}) \cdot S$, where $O(\overline{H})$ is a nilpotent Hall 2'-subgroup of $\overline{H}$ having class at most 2 and S is a Sylow 2-subgroup of $\overline{H}$.*

The quotients of the lower central series $\gamma_i(H)$ of H are studied using techniques developed by Vieira in [14]. Let $O^{2'}(H)$ denote the odd-order residual $\bigcap\{K \trianglelefteq H | H/K \text{ has odd order}\}$.

THEOREM 2.

(i) *The cyclic subgroups $\langle \mu \rangle, \langle \tau \rangle$ are self-centralizing in H*;

(ii) *the direct product group $\gamma_2(H) \times \gamma_2(H)$ is a normal subgroup of $\gamma_2(H)$ and the quotient $\gamma_2(H)/(\gamma_2(H) \times \gamma_2(H))$ is infinite cyclic*;

(iii) *the quotient group $H/\gamma_3(H)$ is torsion-free*;

(iv) $\gamma_3(H) = O^{2'}(H)$;

(v) *the central quotients $\gamma_i(H)/\gamma_{i+1}(H)$ have exponents divisors of 8, for all $i \geq 3$.*

The group H admits an elegant presentation with two generators and two relators together with their images under the iterated application of an endomorphism of the free group of rank 2.

THEOREM 3. *Let F be the free group of rank 2 generated by a, b, and let $r = [b, b^a]$, $r' = [b, b^{a^3}]$. Furthermore, let ε be the endomorphism of F determined by ε: $a \to a^2$, $b \to a^2 b^{-1} a^2$. Define the group $L = \langle a, b | \varepsilon^k(r), \varepsilon^k(r'), k \geq 0 \rangle$. Then the map φ: $a \to \tau$, $b \to \lambda (= \tau\mu^{-1})$ extends to an isomorphism from L onto H.*

The preceding presentation is in the same spirit as that given by Lysenok [10] for the Grigorchuk 2-group. The Gupta–Sidki 3-group also has such a presentation; however, the substitutions are more involved [11].

2. PRELIMINARIES

2.1. *Notation*

The one-rooted binary tree T_2 is labeled by the free monoid $\mathcal{Y}$, freely generated by the set $Y = \{0, 1\}$, with identity element $\varnothing$, ordered by the relation: $m \le m'$ if and only if m' is a prefix of m. There is a level function on T_2 arising from $|m|$, the number of syllables in $m \in M$; the root vertex $\varnothing$ has level 0. Let A denote the automorphism group of T_2, and let σ be the automorphism of T_2 interchanging $0m$ and $1m$ for any m in the monoid $\mathscr{Y}$.

An automorphism α which fixes the vertices labeled by $0, 1$ is represented as an ordered pair (α_0, α_1) where α_i is the automorphism of the subtree headed by $i \in \{0, 1\}$. Since these subtrees are (standardly) isomorphic to T_2, we identify the α_i's with automorphisms of T_2. Therefore, a general automorphism α may be represented inductively as $\alpha = (\alpha_0, \alpha_1) \cdot \sigma^{i_\phi}$, where $i_\phi = 0, 1$ modulo 2, and similarly $\alpha_0 = (\alpha_{00}, \alpha_{01})\sigma^{i_0}$, $\alpha_1 = (\alpha_{10}, \alpha_{11})\sigma^{i_1}$, and so on. Multiplication is determined by $\sigma \cdot (\alpha_0, \alpha_1) \cdot \sigma = (\alpha_1, \alpha_0)$ and $(\alpha_0, \alpha_1) \cdot (\beta_0, \beta_1) = (\alpha_0 \beta_0, \alpha_1 \beta_1)$. Successive developments of α produce for every $u \in \mathscr{Y}$ an automorphism $\alpha_u = \alpha_u(\alpha)$ of the tree, together with a permutation $\sigma_u = \sigma_u(\alpha)$ of Y.

Define the following involutory automorphisms of T_2: $\sigma_0 = \sigma$, $\sigma_1 = (\sigma, e)$, $\sigma_2 = (\sigma_1, e) = ((\sigma, e), (e, e))$, and, inductively, $\sigma_{i+1} = (\sigma_i, e)$. Then the group generated by $\{\sigma_i | 0 \le i \le k\}$ is the wreath product $\wr_k C_2 = (\ldots \wr C_2) \wr C_2$ of the cyclic group C_2, iterated k times, while the $\{\sigma_i | i \ge 0\}$ is the infinitely iterated restricted wreath product $\wr_\infty C_2$. We observe that A induces on the kth-level vertices a permutation group isomorphic to $\wr_{k-1} C_2$ for all $k \ge 1$. Let A_k denote *kth-level stabilizer subgroup* of A; that is, A_k is the kernel of the action of A on the kth-level vertices. Then $A/A_k \cong \wr_{k-1} C_2$, and $\bigcap\{A_k | k \ge 0\}$ is trivial. In particular, A is residually "a finite 2-group."

Given $\alpha \in A$, the set $Q(\alpha) = \{\alpha_u: u \in \mathscr{Y}\}$ is called the *set of states* of α. A state α_u is said to be *inactive* if $i_u = 0$; otherwise it is *active*. It is possible to interpret the automorphism α as an automaton with alphabet Y: the set of states is $Q(\alpha)$; when the automaton is in state β, the output function is given by $y \to z = (y)\sigma_\phi(\beta)$, the image of y under $\sigma_{\phi(\beta)}$; the state transition function is $\beta \to \beta_z$. We call α a *finite-state automorphism* if the set of states $Q(\alpha)$ is finite.

For an automorphism α of the binary tree, we let $\alpha^{(1)}$ denote (α, α) and, inductively, $\alpha^{(k+1)} = (\alpha^{(k)})^{(1)}$. Then $\alpha^{(k)} \in A_k$.

2.2. *First Facts about the Group H*

(i) Define $\lambda = \tau\mu^{-1}$. Then, as $\tau = (e, \tau)\sigma$, $\mu = (e, \mu^{-1})\sigma$, $\lambda = (e, \tau)\sigma \cdot (\mu, e)\sigma = (e, \tau)(e, \mu) = (e, \tau\mu) = (e, \tau\lambda^{-1}\tau)$. Therefore,

$$\lambda = (e, \tau\lambda^{-1}\tau).$$

Clearly, $H = \langle \tau, \lambda \rangle$. Then $H_1 = H \cap A_1$ contains the elements $\lambda = (e, \tau\lambda^{-1}\tau)$, $\tau^2 = (\tau, \tau) = \tau^{(1)}$. Therefore, $H = H_1\langle \tau \rangle$, $[H : H_1] = 2$, and

H_1 projects in its first and second coordinates onto H. That is, H_1 is a subdirect product of $\times_2 H = H \times H$ seen as a subgroup of A. Our group H itself is a subgroup of $(\times_2 H)\langle \sigma \rangle$ and the embedding can be continued to produce the chain of subgroups

$$H \leq \left(\bigtimes_{2} H \right) \cdot \langle \sigma \rangle \leq \cdots \leq \left(\bigtimes_{2^k} H \right) \cdot \langle \sigma_i | 0 \leq i \leq k \rangle.$$

(ii) Let λ_i denote the conjugates λ^{τ^i} of λ, for i any integer. Also define $r = [\lambda, \lambda_1]$, $r' = [\lambda, \lambda_3]$. Then $\lambda_1 = (\lambda^{-1}\tau^2, e)$, $\lambda_3 = (\lambda^{-1}\tau^2, e)^{\tau^2} = (\lambda^{-\tau}\tau^2, e)$, and $r = e = r'$ follow. Indeed, we observe that $[\lambda_i, \lambda_j] = e$ whenever i, j have opposite parities. Let Λ be the normal closure of λ in H, $\Lambda_1 = \langle \lambda_i | i \text{ is odd} \rangle$, and $\Lambda_2 = \langle \lambda_i | i \text{ is even} \rangle$. Therefore,

$$H_1 = \Lambda\langle \tau^2 \rangle, \qquad \Lambda = \Lambda_1 + \Lambda_2.$$

(iii) We calculate $\tau^{-2}\lambda\tau^{-2} = (\tau^{-2}, \lambda^{-1})$, thus separating λ from τ in the second coordinate. We can separate λ at successively lower levels: $\tau^2\lambda\tau^2 = (\tau^2, (\tau^2, \lambda))$, $\tau^{-6}\lambda\tau^{-6} = (\tau^{-6}, (\tau^{-2}, (\tau^{-2}, \lambda^{-1})))$, and so on.

(iv) The relation r can be produced at the first level as follows:

$$\tau^{-2}\lambda\tau^{-2} = (\tau^{-2}, \lambda^{-1}), \qquad (\tau^{-2}\lambda\tau^{-2})^{\tau^2} = \tau^{-4}\lambda = (\tau^{-2}, \lambda_1^{-1}),$$

$$r'' = \left[\tau^{-2}\lambda\tau^{-2}, (\tau^{-2}\lambda\tau^{-2})^{\tau^2}\right] = [\tau^{-2}\lambda\tau^{-2}, \tau^{-4}\lambda] = \left(e, [\lambda^{-1}, \lambda_1^{-1}]\right) = e.$$

Therefore,

$$r'' = \left(e, [\lambda, \lambda_1]^{\lambda_1^{-1}\lambda^{-1}}\right) = (e, r)^{\tau^{-4}\lambda\tau^{-2}\lambda\tau^{-2}}.$$

(v) Let F be the free group freely generated by a, b. A word $w \in F$ is written as $w = w(a, b) = a^{i_1}b^{j_1}a^{i_2}b^{j_2} \cdots a^{i_s}b^{j_s}$ for some integer $s \geq 0$, and some integers i_t, j_t, $1 \leq t \leq s$. Let w be a reduced word. We will use the b-length $|w|_b = \Sigma\{|j_h| \,|1 \leq h \leq s\}$ in analyzing word combinatorics. This length function is especially suitable since $w(\tau, \lambda) = \tau^{i_1}\lambda^{j_1}\tau^{i_2}\lambda^{j_2} \cdots \tau^{i_s}\lambda^{j_s} = (w_0(\tau, \lambda), w_1(\tau, \lambda))\sigma^i$ where $i = \Sigma\{|i_h| \,|1 \leq h \leq s\}$ modulo 2, and where $w_0(a, b), w_1(a, b)$ are words such that $|w_0|_b + |w_1|_b \leq |w|_b$.

(vi) *The group* Λ *is six generated.* We note that

$$r'' = [\lambda_2\tau^{-4}, \lambda_4\tau^{-4}] = (\lambda_2\tau^{-4})^{-1}(\lambda_4\tau^{-4})^{-1}\lambda_2\tau^{-4}\lambda_4\tau^{-4}$$
$$= \lambda_{-2}^{-1}\lambda_{-4}^{-1}\lambda_{-6}\lambda_0 = e.$$

On conjugating this last equation by τ^6, we obtain $\lambda_4^{-1}\lambda_2^{-1}\lambda_0\lambda_6 = e$, and so

$$\lambda_6 = \lambda_0^{-1}\lambda_2\lambda_4, \qquad \Lambda = \langle \lambda_i | 0 \leq i \leq 5 \rangle.$$

(vii) *The generators* τ *and* μ *are conjugate by* $\theta = \theta^{(2)}\sigma^{(1)}$. Since $\theta^2 = (\theta^2)^{(2)}$ and its states $\theta^2, (\theta^2)^{(1)}$ are inactive, it follows that $\theta^2 = e$.

We note that

$$\theta\tau\theta = (e, \sigma\theta^{(1)}\tau\sigma\theta^{(1)})\sigma,$$

$$\mu^{-1}\theta\tau\theta = (\mu, e)\sigma(e, \sigma\theta^{(1)}\tau\sigma\theta^{(1)})\sigma = (\mu\sigma\theta^{(1)}\tau\sigma\theta^{(1)}, e)$$

$$= (\mu, e)\sigma(e, \sigma\theta^{(1)}\tau\sigma\theta^{(1)})\sigma = (\mu\sigma\theta^{(1)}\tau\sigma\theta^{(1)}, e),$$

where the first coordinate is

$$\mu\sigma\theta^{(1)}\tau\sigma\theta^{(1)} = (e, \mu^{-1})(\theta^2, \theta\tau\theta) = (e, \mu^{-1}\theta\tau\theta).$$

Thus, $\mu^{-1}\theta\tau\theta = ((e, \mu^{-1}\theta\tau\theta), e)$, and $\mu^{-1}\theta\tau\theta = e$ follows.

2.3. *Decomposition of the Subgroups* H', $\gamma_3(H)$, H''

Let H' denote the first derived subgroup, H'' the second derived subgroup, and, more generally, $H^{(k)}$ the kth derived subgroup of H. Furthermore, define $c = [\lambda, \tau] \in H'$ and the sequence of tree automorphisms $c_1 = (c, e)$, $c_{i+1} = (c_i, e)$ for $i \geq 1$.

PROPOSITION 4. *The derived group* H' *factors as* $H' = (\times_2 H')\langle c \rangle$, *and* H' *is the normal closure of* $\langle c_1, c \rangle$ *in* H.

Proof. From $c = (\lambda^{-1}\tau^2, \tau^{-1}\lambda\tau^{-1})$, $c^\tau = (\tau^{-2}\lambda, \lambda^{-1}\tau^2)$, we calculate

$$cc^\tau = (e, \tau^{-1}\lambda\tau^{-1}\lambda^{-1}\tau^2) = (e, \lambda^\tau \lambda^{-\tau^2}) = (e, c^{-\tau}).$$

Therefore,

$$cc^\tau = (e, c^{-\tau}).$$

From this we conclude that $c_1 \in H'$, for

$$\tau^{(1)}\tau^{-1}\tau^{(1)}(cc^\tau)^{-1}\tau^{-(1)}\tau\tau^{-(1)} = (e, c^\tau)^{\tau^{-(1)}\tau\tau^{-(1)}} = c_1.$$

As the normal closure of c_1 in H_1 is $H' \times \{e\}$, we conclude that $H' \times \{e\} \leq H'$ and H' is the normal closure of $\langle c_1, c \rangle$ in H. Hence, $\times_2 H'$ is a normal subgroup of H, $c^\tau = c^{-1}$ modulo $\times_2 H'$, and we obtain the required factorization of H'. ∎

PROPOSITION 5. *The third term of the lower central series* $\gamma_3(H)$ *factors as*

$$\gamma_3(H) = \left(\times_2 \gamma_3(H) \right) \langle [c, \tau^2] \rangle \langle [c, \tau] \rangle$$

and $[c, \tau^2] \equiv (c^{-1}, c)$ *modulo* $\times_2 \gamma_3(H)$.

Proof. From $c_1 = (c, e) \in H'$ and $\tau^2 = (\tau, \tau)$, we obtain

$$[c_1, \tau^2] = ([c, \tau], e) \in \gamma_3(H).$$

The group $\gamma_3(H)$ is the normal closure of $[c, \tau]$ in H, since $[c, \lambda] = e$. We conclude that the normal closure of $[c_1, \tau^2]$ in H_1 is $\gamma_3(H) \times \{e\}$ and that this is a subgroup of $\gamma_3(H)$. Therefore,

$$\underset{2}{\times} \gamma_3(H) \leq \gamma_3(H).$$

The assertion $[c, \tau^2] \equiv (c^{-1}, c)$ modulo $\times_2 \gamma_3(H)$ follows from

$$[c, \tau^2] = ([\lambda^{-1}\tau^2, \tau], [\tau^{-1}\lambda\tau^{-1}, \tau]) \equiv ([\lambda, \tau]^{-1}, [\lambda, \tau]) \equiv (c^{-1}, c)$$

modulo $\times_2 \gamma_3(H)$. We note that modulo $\times_2 \gamma_3(H)$, the element (c^{-1}, c) of $\gamma_3(H)$ is inverted by τ and is centralized by $\lambda, c, [c, \tau]$.

Let $\hat{D}(H')$ be the normal closure of (c^{-1}, c) in H. Then

$$\hat{D}(H') \leq \gamma_3(H),$$

$$\left(\underset{2}{\times} \gamma_3(H)\right)\hat{D}(H') = \left(\underset{2}{\times} \gamma_3(H)\right)\langle (c^{-1}, c)\rangle.$$

In order to complete the description of $\gamma_3(H)$, we will compute the commutators $[c, \tau, \lambda], [c, \tau, \tau]$ modulo $\times_2 \gamma_3(H)$. Another form of $c^\tau = c^{-1}(e, c^{-\tau})$ is

$$[c, \tau] = c^{-2}(e, c^{-\tau}).$$

Since $[c, \lambda] = e$, we have

$$\begin{aligned} [c, \tau, \lambda] &= [c^{-2}(e, c^{-\tau}), \lambda] \equiv [c^{-2}(e, c^{-1}), \lambda] \\ &\equiv [c^{-2}, \lambda]^{(e, c^{-1})}[(e, c^{-1}), \lambda] \equiv [(e, c^{-1}), \lambda] \\ &\equiv (e, [c^{-1}, \tau\lambda^{-1}\tau]) \equiv e \end{aligned}$$

modulo $\times_2 \gamma_3(H)$. Hence,

$$[c, \tau] = c^{-2}(e, c^{-1}) \qquad \text{modulo } \underset{2}{\times} \gamma_3(H),$$

$$[c, \tau, \lambda] \in \underset{2}{\times} \gamma_3(H).$$

By commutator calculus $[c, \tau^2] = [c, \tau]^2[c, \tau, \tau]^{[c, \tau]}$. Since, by the previous proposition, $[c, \tau^2] \equiv (c^{-1}, c)$ modulo $\times_2 \gamma_3(H)$, we find that $[c, \tau, \tau]^{[c, \tau]} \equiv [c, \tau]^{-2}(c^{-1}, c)$, and, therefore,

$$[c, \tau, \tau] \equiv [c, \tau]^{-2}(c^{-1}, c) \qquad \text{modulo } \underset{2}{\times} \gamma_3(H).$$

With this we arrive at the required factorization of $\gamma_3(H)$. ∎

PROPOSITION 6. *The second term of the derived series H'' factors as*

$$H'' = \underset{2}{\times}\left\{\left(\underset{2}{\times}\gamma_3(H)\right)\langle (c^{-1},c)\rangle\right\}.$$

Proof. We conclude from the factorization of the first term of the derived series

$$H' = \left(\underset{2}{\times} H'\right)\langle c\rangle$$

that H'' is the normal closure of $[\times_2 H', c]$ in H. Therefore, using Proposition 4, H'' is the normal closure of $[c_1, c], [c_2, c]$ in H. First, we calculate $[c_2, c] = ([c_1, \lambda^{-1}\tau^2], e) = (([c,\tau], e), e)$ and conclude that $H'' \geq \times_4 \gamma_3(H)$. Now we calculate $[c_1, c]$ in H modulo $\times_4 \gamma_3(H)$,

$$[c_1, c] = ([c, \lambda^{-1}\tau^2], e) = ([c,\tau^2], e) \equiv ((c^{-1}, c), e)$$

from which the factorization of H'' follows. ∎

2.4. *Centralizers*

LEMMA 7. *The centralizer of the derived group H' in H is trivial.*

Proof. Let $w \in F$ have least b-length $|w|_b$ such that $w(\tau, \lambda) = (w_0, w_1)\sigma^i$ centralizes H'. Since $c_1 = (c, e) \in H'$, clearly, $i = 0$, and, therefore, w_0, w_1 also centralize H'. Hence, $|w|_b = 0$, and $w(\tau, \lambda)$ is a power of τ. Since $c_i = (c_{i-1}, e) \in H'$, we conclude that $w = e$. ∎

LEMMA 8. *The generators τ, μ are self-centralizing in H.*

Proof. We will prove the assertion for τ. The case for μ will be analogous. Suppose $C_H(\tau) \neq \langle\tau\rangle$. Let $w \in F$ have least nonzero b-length such that $w(\tau, \lambda) = (w_0, w_1)\sigma^i$ centralizes τ. Since $|w|_b \neq 0$, it follows that $|w_0|_b, |w_1|_b < |w|_b$. If $i = 1$, then $w(\tau, \lambda)\tau = (w_0\tau, w_1)$ commutes with $\tau^2 = (\tau, \tau)$. Therefore, w_0, w_1 commute with τ and so $w_0 = \tau^j$, $w_1 = \tau^k$ for some integers j, k. Now conjugation of w by τ shows that $k = j + 1$ and, therefore, $w = \tau^{2j+1}$, a contradiction. The proof for the case $w(\tau, \lambda) = (w_0, w_1)$ proceeds similarly. ∎

3. DESCENDING CENTRAL SERIES

PROPOSITION 9. *The commutator quotient H/H' is torsion-free of rank 2 and $H'/\gamma_3(H)$ is torsion-free cyclic.*

Proof.

(i) *H/H' is a torsion-free group of rank* 2. Let i, j be integers such that $u = \lambda^i \tau^j \in H'$. Since $u \in H_1$, it follows that $j = 2j_1$. Therefore, $u = (\tau^{j_1}, c' \lambda^{-i} \tau^{2i+j_1})$ for some $c' \in H'$. On the other hand, from the decomposition of H', there exist an integer k and $c'', c''' \in H'$ such that $u = c^k = (c'' \lambda^{-k} \tau^{2k}, c''' \lambda^k \tau^{-2k})$. Therefore,

$$(\tau^{j_1}, c' \lambda^{-i} \tau^{2i+j_1}) = (c'' \lambda^{-k} \tau^{2k}, c''' \lambda^k \tau^{-2k})$$

from which we conclude

$$\lambda^{-k} \tau^{2k-j_1}, \lambda^{k+i} \tau^{-(2i+2k+j_1)} \in H'.$$

On using $\lambda^i \tau^j \in H'$ with the later conclusions, we produce $\tau^{2(i+j)} \in H'$. Thus, we have

$$\lambda^i \tau^j, \tau^{2(i+j)}, \lambda^{-k} \tau^{2k-j_1} \in H'.$$

The fact $\lambda^{-k} \tau^{2k-j_1} \in H'$ implies that $2k - j_1$ is even, and, therefore, j_1 is also even; $j = 4j_2$. On substituting $\lambda^i \tau^j \in H'$ by $\tau^{2(i+j)} \in H'$ in the previous argument, we conclude that $2(i+j)$ is a multiple of 4, and so $i = 2i_1$. Therefore, on considering $\lambda^{-k} \tau^{2k-j_1} \in H'$, we conclude that $k = 2k_1$. Hence, $\lambda^{-k} \tau^{2k-j_1} = \lambda^{-2k_1} \tau^{2(2k_1 - j_2)}$. Again, there exists an integer $k' = 2k_1'$ such that $\lambda^{-k'} \tau^{2k' - (2k_1 - j_2)} \in H'$. Therefore, $2k' - (2k_1 - j_2)$ is even, and so $j = 8j_3$. Going back to $\tau^{2(i+j)} \in H'$, we conclude that $i = 4i_2$. This procedure may be repeated to prove that $2^s | j$ and $2^{s-1} | i$ for any $s \geq 1$. Hence, $j = i = 0$.

(ii) *$H'/\gamma_3(H)$ is torsion-free.* Clearly, $c \notin \gamma_3(H)$. We have from the proof of Proposition 4 that $cc^\tau = (e, c^{-\tau})$ and, therefore,

$$c^2 = (e, c^{-1}) \qquad \text{modulo } \gamma_3(H)$$

from which we conclude $c^2 = c^{2\tau} = (c^{-1}, e)$ modulo $\gamma_3(H)$.

Let k be the smallest nonnegative integer such that $c^{2k} \in \gamma_3(H)$. Then $(c^k, e) \in \gamma_3(H)$, and, from the decomposition $\gamma_3(H) = (\times_2 \gamma_3(H)) \langle (c^{-1}, c) \rangle \langle [c, \tau] \rangle$, we obtain $(c^k, e) = (c', c'')(c^{-1}, c)^l [c, \tau]^m$ for some $c', c'' \in \gamma_3(H)$ and l, m integers. Since $[c, \tau] = c^{-2}(e, c^{-\tau})$, $c = (\lambda^{-1}\tau^2, \tau^{-1}\lambda\tau^{-1})$, we obtain $(\tau^{-2}\lambda)^{2m} \in H'$ and, therefore, $m = 0$ follows. Hence, $c^{k+l}, c^l \in \gamma_3(H)$, and we conclude that $c^k \in \gamma_3(H)$. ∎

PROPOSITION 10. *The central quotient $\gamma_i(H)/\gamma_{i+1}(H)$ has exponent a divisor of* 8 *for all $i \geq 3$.*

Proof. It is sufficient to prove the assertion for $\gamma_3(H)/\gamma_4(H)$.

We develop the relation $r'' = [\tau^{-2}\lambda\tau^{-2}, \tau^{-4}\lambda] = e$ in terms of commutators: $e = [(\tau^{-4}\lambda)^{-1} \cdot \tau^{-2}\lambda\tau^{-2}, \tau^{-4}\lambda] = [[\lambda, \tau^{-2}], \tau^{-4}\lambda] = [\lambda, \tau^{-2}, \lambda] [\lambda, \tau^{-2}, \tau^{-4}]^\lambda$, and so we arrive at the dependence equation in $\gamma_3(H)$:

$$[\lambda, \tau^{-2}, \tau^{-4}] = [\lambda, \tau^{-2}, \lambda^{-1}].$$

On considering this equation modulo $\gamma_4(H)$, we obtain $[\lambda, \tau, \tau]^8 \equiv [\lambda, \tau, \lambda]^2 = e$. Therefore,

$$[c, \tau]^8 \equiv e \qquad \text{modulo } \gamma_4(H),$$

and $\gamma_3(H)/\gamma_4(H)$ is cyclic of exponent a divisor of 8. ∎

4. TORSION-FREENESS AND JUST-NONSOLVABILITY

PROPOSITION 11. *The group H is residually "torsion-free solvable" and is nonsolvable.*

Proof. First, we will prove that, for all $k \geq 0$, $(\times_{2^k} H')/(\times_{2^{k+1}} H')$ *is* torsion-free. For the case $k = 0$, we recall the decomposition $H' = (H' \times H')\langle c \rangle$ and we suppose that $c^n \in H' \times H'$ for some integer n. Then, as $c^n = ((\lambda^{-1}\tau^2)^n, (\tau^{-1}\lambda\tau^{-1})^n)$, we conclude that $(\lambda^{-1}\tau^2)^n \in H'$, and $n = 0$ follows from the fact that H/H' is torsion-free. The general case is treated similarly.

We conclude from the preceding argument that $H/(\times_{2^k} H')$ is a torsion-free group. Since $H \leq (\times_{2^k} H) \cdot (\wr_k C_2)$, it follows that $H/(\times_{2^k} H')$ is a homomorphic image of the solvable group $(\times_{2^k} H/H') \cdot (\wr_k C_2)$. Also, since $\times_{2^k} H' \leq H_k$, the kth stabilizer subgroup, $\bigcap\{\times_{2^k} H' | k \geq 0\} \leq \bigcap\{H_k | k \geq 0\} = \{e\}$. Hence, H is residually a "torsion-free solvable" group.

Clearly, H'' is nontrivial. Since $H'' \geq \times_2 \hat{D}(H')$, we have that, for all $k \geq 1$, $H^{(k+1)} \geq (\times_2 \hat{D}(H'))^{(k)}$ which is a subdirect product of $\times_4 H^{(k)}$. By induction on k, we conclude that H is nonsolvable. ∎

PROPOSITION 12. *The group H is just-nonsolvable.*

Proof. Let $w \in H$, $w \neq e$. We will show that the normal closure W of any nontrivial element w in H contains a term of the derived series of H. There are two cases to consider depending on whether or not $w \in H_1$.

Case 1. Let $w = (w_0, w_1)\sigma$. Then a direct calculation shows that

$$[w, \lambda] = \left(w_1^{-1}\tau^{-1}\lambda\tau w_1, \tau\lambda^{-1}\tau\right),$$

$$[w, \lambda, \lambda_2] = \left(e, [\tau\lambda^{-1}\tau, \lambda^{-1}\tau^2]\right) = \left(e, \lambda_0\lambda_3\lambda_2^{-1}\lambda_1^{-1}\right).$$

Let $\nu = \lambda_0\lambda_3\lambda_2^{-1}\lambda_1^{-1}$ and let N be the normal closure of ν in H. Then, in Λ modulo N, $\lambda_3 \equiv \lambda_0^{-1}\lambda_1\lambda_2 = \lambda_0^{-1}\lambda_2\lambda_1$, and so $[\lambda_3, \lambda_1] \equiv e$. Therefore, Λ/N is abelian of rank at most 3, H/N is metabelian, and $H'' \leq N$. From the embedding $H \leq (H \times H)\langle \sigma \rangle$, we conclude that $N \times N \leq W$, $H'' \times H'' \leq W$, and H is solvable.

Case 2. Let $w = (w_0, w_1)$. We may assume $w_1 \neq e$. Let W_1 be the normal closure of w_1 in H. Therefore, by Lemma 7, $[w_1, H']$ is nontrivial. Now, $W \geq [(w_0, w_1), (e, H')] = (e, [w_1, H'])$, and so $W \geq (e, [W_1, H'])$. Clearly, if W_1 contains a derived subgroup $H^{(l)}$ for some l, then W would contain $H^{(l+1)} \times H^{(l+1)}$, and, as $H/(H^{(l+1)} \times H^{(l+1)})$ is solvable, the case would be done. The desired conclusion is reached as there exists a minimum level where w is active. ∎

5. FINITE QUOTIENTS OF H

PROPOSITION 13.

(i) *The odd-order residual* $O^{2'}(H)$ *is equal to* $\gamma_3(H)$.

(ii) *Any finite quotient* $\overline{H}$ *of* H *decomposes as* $\overline{H} = O(\overline{H}) \cdot S$ *where* $O(\overline{H})$ *is a normal Hall* $2'$*-group of* $\overline{H}$, *which is nilpotent of class at most* 2, *and where* S *is a* 2*-group.*

Proof.

(i) Let n be an odd number and let N be the normal closure of τ^n. Then, in H modulo N, $\lambda_n = \lambda_0$ and, therefore, $\Lambda_1 = \Lambda_2 = \Lambda$, and Λ is an abelian group generated by $\lambda_0, \lambda_2, \lambda_4$. The equation $\lambda_0^{-1}\lambda_2\lambda_4\lambda_6^{-1} = e$ translates additively to $\lambda_0 \cdot (-1 + \tau^2 + \tau^4 - \tau^6) = 0$.

Define the polynomials $l(x) = -1 + x^2 + x^4 - x^6$ and $m(x) = -1 + x^n$. Then $l(x), m(x)$ factor as $l(x) = (-1 + x)l_1(x)$, where $l_1(x) = (1 - x)(1 + x)^2(1 + x^2)$, and $m(x) = -1 + x^n = (-1 + x)m_1(x)$. Let GCD denote the greatest common divisor operator applied to polynomials, with integer coefficients. Then, since $\text{GCD}(1 + x, m_1(x)) = 1$, $\text{GCD}(1 + x^2, m_1(x)) = 1$, $\text{GCD}((1 - x), m_1(x)) = n$, we conclude that $\text{GCD}(l(x), m(x)) = n(-1 + x)$.

Let $a(x), b(x)$ be polynomials over the integers such that

$$a(x)(1 + x)^2(1 + x^2) + b(x)m_1(x) = 1.$$

Then, on multiplying the preceding by $(1 - x)^2$, we obtain

$$-a(x)l(x) - b(x)m(x) = (1 - x)^2.$$

Therefore, the following equations hold:

$$\lambda_0 \cdot n(-1 + \tau) = 0, \qquad \lambda_0(1 - \tau)^2 = 0$$

in H modulo N. Returning to multiplicative notation,

$$N\lambda_0^n = N\lambda_2^n = N\lambda_4^n,$$

$$\gamma_3(H) \leq N.$$

Now we reach the conclusion $O^{2'}(H) = \gamma_3(H)$ since, easily, $\bigcap\{H^k | k \text{ odd}\} \le \gamma_3(H)$.

(ii) Let $n = 2^k n'$, n' odd, M the normal closure of τ^{2^k}, and N the normal closure of τ^n in H.

We assert that $M = (\times_{2^k} H')\langle \tau^{2^k} \rangle$. Since $\tau^{2^k} = \tau^{(k)}$ and there exist integers $i, j, \dots, m$, such that $\xi = (\tau^i, (\tau^j, (\dots, (\tau^m, \lambda)\dots))) \in H$ where λ is isolated at the kth level, we produce $[\tau^{2^k}, \xi] = (e, e, \dots, e, [\tau, \lambda]) \in H_k$. Therefore, M contains the normal subgroup $\times_{2^k} H'$. It is easy to see, by induction on k, that $[\tau^{2^k}, \lambda] \in \times_{2^k} H'$. With this, our assertion is established.

Now $\tau^n = (\tau^{2^k})^{n'} = (\tau^{n'})^{(k)}$, and it follows from part (i) that N contains $\times_{2^k} \gamma_3(H)$. Since

$$M \Big/ \Big(\underset{2^k}{\times} \gamma_3(H) \Big) = \Big(\underset{2^k}{\times} H' / \gamma_3(H) \Big) \cdot \Big\langle \Big(\underset{2^k}{\times} \gamma_3(H) \Big) \tau^{(k)} \Big\rangle$$

is clearly a class 2 nilpotent group, therefore M/N is also nilpotent of class at most 2.

Let $\tilde{N} = H^n$ be the group generated by $\{h^n | h \in H\}$ and let $\tilde{M} = H^{2^k}$. Then $H/\tilde{M}$ is a finitely generated solvable group of exponent 2^k, and so it is a finite 2-group. Thus, $\tilde{M}$ is a finitely generated group. Now $\tilde{M}/\tilde{N}$ is a finitely generated nilpotent group of class at most 2, and has odd exponent n'. Therefore, $\tilde{M}/\tilde{N}$ is a finite group of odd order, and $H/\tilde{N}$ is a finite group having the type of structure as had been affirmed.

Since any finite quotient group of H is a quotient of H/H^n for some integer n, the proof follows. ∎

6. A PRESENTATION FOR H

Let K be the subgroup of H_1 generated by τ^2, λ. Also, let $\rho: K \to H$ be the projection on the second coordinate. Then, as $\tau^2 = (\tau, \tau)$, $\lambda = (e, \tau\lambda^{-1}\tau)$, we have $\rho: \tau^2 \to \tau$, $\lambda \to \tau\lambda^{-1}\tau$ and, therefore, it is an epimorphism. It is actually an isomorphism. However, first we need to prove the following.

LEMMA 14. *Let s be an integer, $s \ge 1$. Then the tree automorphisms defined by $(e, \tau)^s, (\tau, e)^s$ do not belong to H.*

Proof. Since $\tau^{2s} = (\tau^s, \tau^s) \in H$, it is sufficient to prove the assertion for $\alpha = (e, \tau)^s$. Suppose that $\alpha \in H$. Since H/H' is 2-generated, there exist integers i, j such that $\alpha\lambda^i\tau^j \in H'$. Therefore, $j = 2j'$,

$$\alpha\lambda^i\tau^j = \left(\tau^{j'}, \tau^s(\tau\lambda^{-1}\tau)^i \tau^{j'}\right),$$

and, from the decomposition $H' = (H' \times H')\langle c \rangle$, we obtain the second expression

$$\alpha \lambda^i \tau^j = (c', c'')c^k$$

for some $c', c'' \in H'$ and some integer k. However, on substituting

$$c^k = \left((\lambda^{-1}\tau^2)^k, (\tau^{-1}\lambda\tau^{-1})^k \right)$$

in the previous equation and on comparing first coordinates, we arrive at $\tau^{j'} = c'(\lambda^{-1}\tau^2)^k$. Since H/H' is freely generated by $H'\lambda, H'\tau$, we conclude that $k = 0 = j' = j$. Thus, $\alpha\lambda^i = (e, \tau^{s'}(\tau\lambda^{-1}\tau)^i) = (c', c'')$, $\tau^{s'+2i}\lambda^{-i} \in H'$, and $i = 0$ follows. ∎

Let w be a word in τ^2, λ. Then $w = w(\tau^2, \lambda) = (w(\tau, e), \rho(w))$. If $\rho(w) = e$, then $w = (w(\tau, e), e) = (\tau^s, e)$ for some integer s. Thus, (τ^s, e) is an element of H, and $s = 0$ by Lemma 14; hence, $w = e$. Define ε: $H \to K$ to be the inverse map of ρ. Then ε: $\tau \to \tau^2$, $\lambda \to \tau^2\lambda^{-1}\tau^2$ determines an isomorphism from H onto K.

PROPOSITION 15. *Let $r = [b, b^a]$, $r' = [b, b^{a^3}]$ be words in the free group F freely generated by a, b. Also consider the endomorphism of F determined by ε: $a \to a^2$, $b \to a^2b^{-1}a^2$, and define the group*

$$L = \langle a, b | \varepsilon^k(r), \varepsilon^k(r'), \ k \geq 0 \rangle.$$

Then the map φ: $a \to \tau$, $b \to \lambda$ extends to a homomorphism from L onto H.

Proof. Explicitly, ε^k: $a \to a^{2^k}$, $b \to a^{l_k}b^{\delta_k}a^{l_k}$, where $\delta_k = (-1)^k$, $l_k = \frac{2}{3}(2^k - \delta_k)$. In order to write down $\varepsilon^k(r), \varepsilon^k(r')$, we define $b_s = b^{a^s}$. Then

$$\varepsilon^k(b) = b_{-l_k}^{-1} a^{2l_k},$$

$$\varepsilon^k(r) = b_{l_k}^{-\delta_k} b_{3l_k+2^k}^{-\delta_k} b_{3l_k}^{\delta_k} b_{l_k+2^k}^{\delta_k},$$

$$\varepsilon^k(r') = b_{l_k}^{-\delta_k} b_{3(l_k+2^k)}^{-\delta_k} b_{3l_k}^{\delta_k} b_{l_k+3\cdot 2^k}^{\delta_k}$$

for all $k \geq 0$. We note that, for $k = 1$, $\delta_k = -1$, $l_k = 2$, and, thus,

$$\varepsilon(r) = b_2 b_8 b_6^{-1} b_4^{-1}, \qquad \varepsilon(r') = b_2 b_{12} b_6^{-1} b_8^{-1}.$$

Let B be the normal closure of b in L, $B_1 = \langle b_k | k \text{ odd} \rangle$, $B_2 = \langle b_m | m \text{ even} \rangle$. The relation $r = e$ means that b_0 commutes with b_1, and, by conjugation by a, we conclude that b_1 commutes with b_0, b_2. Now $r' = e$ implies that b_1 commutes with b_4 as well. Since $\varepsilon(r) = b_2 b_8 b_6^{-1} b_4^{-1} = e$, we have $b_0 b_6 b_4^{-1} b_2^{-1} = e$, and, therefore, b_1 commutes with b_0, b_2, b_4, b_6.

Furthermore, since $b_2 b_8 b_6^{-1} b_4^{-1} = e$, we deduce that b_1 commutes with b_0, b_2, b_4, b_6, b_8. It is clear that on repeating this argument we obtain that b_1 commutes with b_i for all i even. Thus, L factors as $L = B\langle a \rangle$, and $B = B_1 + B_2$.

An element $w(a, b)$ of L can be written as $w(a, b) = w_1(a, b) w_2(a, b) a^n$ where $w_1(a, b) = b_{k_1}^{i_1} b_{k_2}^{i_2} \cdots b_{k_s}^{i_s} \in B_1$, k_t odd for $1 \le t \le s$, and where $w_2(a, b) = b_{m_1}^{j_1} b_{m_2}^{j_2} \cdots b_{m_q}^{j_q} \in B_2$, m_p even for $1 \le p \le q$; note that $|w_1|_b + |w_2|_b = |w|_b$. On applying the φ map, we have $\varphi w(a, b) = (w_2'(\tau, \lambda), w_1'(\tau, \lambda))\tau^n$. Here $w_1'(a, b) = (b_{k_1'}^{-1} a^2)^{i_1} (b_{k_2'}^{-1} a^2)^{i_2} \cdots (b_{k_s'}^{-1} a^2)^{i_s}$, where $k_h' = k_h/2 - 1$ for $1 \le h \le s$, and $w_2'(a, b) = (b_{m_1'}^{-1} a^2)^{j_1} (b_{m_2'}^{-1} a^2)^{j_2} \cdots (b_{m_q'}^{-1} a^2)^{j_q}$, where $m_h' = (m_h - 1)/2$ for $1 \le h \le q$. Thus, $|w_1'|_b = |w_1|_b$, $|w_2'|_b = |w_2|_b$.

Suppose that $w(a, b)$ is a nontrivial element in the kernel of φ, and let it be of minimum b-length $|w|_b$. Then $\varphi w(a, b) = (w_2'(\tau, \lambda), w_1'(\tau, \lambda))\tau^n = e$, and so $n = 2n_0$, and $(w_2'(\tau, \lambda)\tau^{n_0}, w_1'(\tau, \lambda)\tau^{n_0}) = e$. Hence, $w_1'(a, b)a^{n_0}$, $w_2'(a, b)a^{n_0}$ are also in the kernel of φ, and

$$|w_1'(a, b)a^{n_0}|_b = |w_1|_b, \qquad |w_2'(a, b)a^{n_0}|_b = |w_2|_b.$$

If both w_1, w_2 have b-lengths shorter than w, then $w_1'(a, b)a^{n_0} = e = w_2'(a, b)a^{n_0}$, $w_1(a, b) = e = w_2(a, b)$, and, consequently, $n = 0$, $w = e$, a contradiction. Thus, we may assume $w(a, b) = w_2(a, b)a^n$, $q > 1$. Then $n = 2n_0$, and both $w_2(a, b)a^n, w_2'(a, b)a^{n_0}$ are in the kernel of φ. Now we replace in the preceding argument w by

$$w_2'(a, b)a^{n_0} = \left(b_{m_1'}^{-1} a^2\right)^{j_1} \left(b_{m_2'}^{-1} a^2\right)^{j_2} \cdots \left(b_{m_q'}^{-1} a^2\right)^{j_q} a^{n_0} = w_2''(a, b)a^{n_1},$$

where $n_1 = 2j_1 + \cdots + 2j_q + n_0$. Thus, again, $w_2''(a, b) \in B_1$ or B_2. That is, m_h' has the same parity as m_1' for all $1 \le h \le q$. Hence,

$$m_h' - m_1' = \frac{m_h - 1}{2} - \frac{m_1 - 1}{2} = \frac{m_h - m_1}{2}$$

is even, and so $4 | m_h - m_1$. If $j_1 > 1$ then, since $(b_{m_1'}^{-1} a^2)^{j_1} = b_{m_1'}^{-1} b_{m_1'-2}^{-1} \ldots$, the consecutive indices cannot be congruent modulo 4. Therefore, $|j_h| = 1$ for all $1 \le h \le q$. As $q > 1$, it follows that $j_{h+1} = -j_h$, and q is even. We may assume $j_1 = 1$. Thus, $w_2'(a, b)a^{n_0} = b_{m_1'}^{-1} b_{m_2'} \cdots b_{m_{q'}}^{-1} b_{m_{q'+1}'} a^{n_0}$. Once more, we obtain that $2 | n_0, 4 | m_h' - m_1'$. Therefore, $8 | m_h - m_1$. A repetition of this argument produces $m_h = m_1$, $n = 0$. Hence, $w_2(a, b) = b_{m_1}^{j_1} b_{m_2}^{j_2} \cdots b_{m_q}^{j_q} = e$, $a^n = e$, $w = e$, a contradiction. ∎

7. FINAL COMMENTS

(i) On changing b by its conjugate $\dot{b} = b^{a^2}$ in the previous presentation for H, the definition of the endomorphism becomes ε: $a \to a^2$, $\dot{b} \to \dot{b}^{-1}a^4$. The new relators may still be chosen to be $\dot{r} = [\dot{b}, \dot{b}^a]$, $\dot{r}' = [\dot{b}, \dot{b}^{a^3}]$ together with their images under repeated applications of ε.

(ii) It was shown in Section 2 that conjugation by $\theta = \theta^{(2)}\sigma^{(1)}$ is an automorphism of H of order 2 which interchanges τ and μ. We note that conjugation by $\delta = \delta^{(1)}\sigma$ inverts both τ and μ. It appears that the 4-group $\langle \delta, \theta \rangle$ is the group of outer automorphisms of H.

(iii) Define R_l to be the normal closure of $\langle \varepsilon^k(r), \varepsilon^k(r') | 0 \leq k \leq l \rangle$ in the free group F. It can be proven that $\{R_l | l \geq 0\}$ is a strictly ascending chain of subgroups and, therefore, H is not finitely presentable.

(iv) The proof of Proposition 14 provides an algorithm for solving the word problem in H. It would be interesting to see if the approach of Wilson and Zalesskii [15] leads to the solution of the conjugacy problem for this group.

(v) A torsion-free group which is an extension of an abelian group of infinite rank by the Grigorchuk 2-group has been defined to act on a tree with infinite valency and was shown to have intermediate growth [7]. This raises the question about the growth function of our group.

REFERENCES

1. S. V. Aleshin, Finite automata and the Burnside problem for periodic groups, *Math. Notes* **11** (1972), 199–203.
2. H. Bass, M. Otero-Espinar, D. Rockmore, and C. P. L. Tresser, "Cyclic Renormalization and Automorphism Groups of Rooted Trees," Lecture Notes in Mathematics, Vol. 1621, Springer-Verlag, Berlin, 1995.
3. A. M. Brunner and S. Sidki, On the automorphism group of the one-rooted binary tree, *J. Algebra* **195** (1997), 465–486.
4. A. M. Brunner and S. Sidki, The generation of GL(n, Z) by finite automata, *Internat. J. Algebra Comput.* **8**, (1998), 127–139.
5. R. I. Grigorchuk, On the Burnside problem for periodic groups, *Funct. Anal. Appl.* **14** (1980), 41–43.
6. R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), 939–985.
7. R. I. Grigorchuk, On degrees of growth of p-groups and torsion-free groups, *Mat. Sb.* **126** (1985), 194–214.
8. N. Gupta and S. Sidki, On the Burnside problem for periodic groups, *Math. Z.* **182** (1983), 385–388.
9. N. Gupta and S. Sidki, Extensions of groups by tree automorphisms, "Contributions to Group Theory," Contemporary Mathematics, Vol. 33, 232–246, Amer. Math. Soc., Providence, 1984.

10. I. G. Lysenok, A system of defining relations for a Grigorchuk group, *Math. Notes* **25** (1985), 784–792.
11. S. Sidki, On a 2-generated infinite 3-group: the presentation problem, *J. Algebra* **110** (1987), 13–23.
12. S. Sidki, "Regular Trees and Their Automorphisms," Monografias de Matematica, Vol. 56, IMPA, Rio de Janeiro, 1998.
13. V. I. Suschanskii, Periodic p-groups of permutations and the unrestricted Burnside problem, *Soviet Math. Dokl.* **20** (1979), 776–770.
14. A. C. Vieira, On the lower central series and the derived series of the Gupta–Sidki 3-group, *Comm. Algebra* **26** (1998), 1319–1333.
15. J. S. Wilson and P. A. Zalesskii, Conjugacy separability of certain torsion groups, *Arch. Math.* **68** (1997), 441–449.