Cohomological dimension with respect to perfect groups

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Abstract

We introduce new classes of compact metric spaces: Cannon–Štan’ko, Cainian, and nonabelian compacta. In particular, we investigate compacta of cohomological dimension one with respect to certain classes of nonabelian groups, e.g., perfect groups. We also present a new method of constructing compacta with certain extension properties.

Keywords: Cohomological dimension; Cannon–Štan’ko compactum; Nonabelian compactum; Weakly Cainian compactum; Perfect group; Grope; Eilenberg–MacLane complex; Cell-like map

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We dedicate this paper to the memory of Dr. W.J.R. Mitchell (1950–1994)

1. Introduction

Cohomology theory is defined for arbitrary abelian groups. Consequently, one can define cohomological dimension, $c\text{-dim}_G X$, of a compact metric space $X$ for any abelian group $G$. The standard definition of $c\text{-dim}_G X$ is via the Eilenberg–MacLane complexes $K(G, n)$ (see, e.g., [6,15,25,26,28,33]). If we consider the class $G$ of nonabelian groups $G$, then one still has well-defined Eilenberg–MacLane complexes $K(G, 1)$ [23]. Therefore it makes sense to consider the class of compact metric spaces $X$ with cohomological dimension one for an arbitrary nonabelian group $G \in G$, $c\text{-dim}_G X = 1$. The purpose of our paper is to study the special case when the group $G \in G$ is perfect, the subclass of $G$ which is often studied by geometric topologists (see, e.g., [1–3]).

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Recall the Kuratowski notation $X \tau Y$: it means that for every closed subset $X_0$ of $X$ and any map $f : X_0 \to Y$ there exists an extension $\tilde{f} : X \to Y$ of $f$ over all of $X$ [24]. A compact metric space $X$ is said to be a Cainian compactum if for every perfect group $G$, $X \tau K(G, 1)$.

The class $\mathcal{K}$ of all Cainian compacta may play an important role in the celebrated 4-dimensional Cell-like mapping problem: Can cell-like maps on a 4-manifold raise dimension? (See [26,27].) Namely, we prove that every Cainian compactum is at most 2-dimensional (Corollary 5.6). Next, we show that in every topological 4-manifold $M$ there exists a 1-dimensional Nöbeling net such that every compactum in its complement is necessarily Cainian (Theorem 5.12). So in order to prove that a cell-like image $X$ of a topological 4-manifold $M$, $f : M \to X$ is at most 4-dimensional (note that $X$ is necessarily a $\mathbb{Z}$-homology 4-manifold), it would suffice to find a similar Nöbeling net in $X$. Therefore our result might represent a possible first step towards an affirmative solution of the 4-dimensional cell-like mapping problem—the only dimension in which the answer is still unknown. (See [27] for a different approach to this problem.)

One of the most important concepts in geometric topology is the idea of the grope which was introduced in the 1970's by Štan’ko [31]. It was also used very effectively by Cannon [1] (whereas the name itself was suggested by D.R. McMillan, Jr.). The gropes were instrumental in the proofs of several key results in both taming theory as well as decomposition theory (cf. [1-3,31]).

It is easy to see that the fundamental group $\pi = \pi_1(M)$ of every grope $M$ is perfect, i.e., $[\pi : \pi] = \pi [1]$. In our paper we consider the class of compact metric spaces $X$ which are characterized by the property that $X \tau M^*$ for the so-called minimal grope $M^*$ (equivalently, the cohomological dimension of $X$ is at most one with respect to the fundamental group of the minimal grope $M^*$, $\operatorname{c-dim}_{\pi_1(M^*)} X \leq 1$). We call such spaces Cannon–Štan’ko compacta. One of the main results of the present paper is that there exist Cannon–Štan’ko compacta of arbitrary high dimensions (Theorem 3.3). This is quite in contrast with the class of Cainian compacta which cannot be more than 2-dimensional (Corollary 5.6). In fact, we prove that the weakly Cainian compacta are precisely the 2-dimensional Cannon–Štan’ko compacta (Corollary 5.10).

We also introduce and study the class of nonabelian compacta $X$: they are characterized by the property that for every closed subset $X_0 \subset X$ of $X$ and for every map $f : X_0 \to \partial T$ there is an extension $\tilde{f} : X \to T$ of $X$ into $T$, where $T$ is a torus with one hole (and $\partial T = S^1$ is its boundary). We prove that every nonabelian compactum is a Cannon–Štan’ko compactum (cf. Theorem 4.3). These compacta played an important role in the recent solution by the first author of the following problem in dimension theory proposed by Sternfeld [32] (in connection with the Hilbert’s 13th problem): If a 2-dimensional compactum $X$ is embedded in the product $Y \times Z$ of two 1-dimensional compacta $Y$ and $Z$, does then $X$ necessarily contain a product $Y_1 \times Z_1$ of two 1-dimensional compacta $Y_1$ and $Z_1$? (The answer is negative [13]. Another such counterexample was published earlier by Pol [29].)

The present paper has a rather long history. It was started during the visit by the first author to Ljubljana in the Spring of 1989. The preliminary announcement [16] was
written during the second author’s visit to Moscow in the Spring of 1991 (see also [17]). The main results of this paper were presented by the second author at the Workshop on Cohomological Dimension Theory (Knoxville, May 19–21, 1992), at the Conference on Topology and Its Applications (Amsterdam, August 15–18, 1994), at the Conference on Set-Theoretic Topology and Its Applications (Matsuyama, December 12–16, 1994), and at the Spring Topology and Southeast Dynamics Conference (Newark, March 30–April 2, 1995).

2. Constructing compacta with certain extension properties

We shall work in the category of separable metrizable spaces and continuous maps throughout this paper. A compactum is a compact metric space. A space $X$ is said to have cohomological dimension at most $n$, $n \in \mathbb{N} \cup \{0\}$, with respect to a group of coefficients $G$, $\text{c-dim}_G X \leq n$ if for every closed subset $A \subset X$ and every map $f : A \to K(G,n)$ of $A$ into the Eilenberg–MacLane complex $K(G,n)$ (see [23] for the definition and properties of Eilenberg–MacLane complexes), there is an extension of $f$ over all of $X$, i.e., $X \supset K(G,n)$. Equivalently, $K(G,n) \in \mathcal{AE}(X)$. (For several equivalent versions of definition of $\text{c-dim}_G X$ see [6, 25, 33], where its properties are studied in details.)

In [6–9, 11, 20, 21], compacta with differing cohomological and covering dimensions were constructed, using the so-called Edwards–Walsh modification of polyhedra. Here we use an alternative approach, based on [10]. Whereas, the Edwards–Walsh modification exists only with the Eilenberg–MacLane space $K(R,n)$, where $R$ is a ring with unit, our present approach is valid for an arbitrary group $G$ (however, it is not so geometric).

A map $f : L \to K$ between polyhedra $L$ and $K$ with triangulations $\lambda$ and $\kappa$, respectively, is said to be combinatorial with respect to $\lambda$ and $\kappa$, provided $f^{-1}(A)$ is a subpolyhedron of $\lambda$ for every simplex $A \subset K$. This means that the preimage of every polyhedron $A$ with respect to $\kappa$ is a polyhedron with respect to $\lambda$.

For every pair $(X, A)$, every CW complex $K$, and every map $f : A \to K$, we shall denote the classical extension problem:

$$
\begin{array}{ccc}
A & \longrightarrow & K \\
\uparrow & & \downarrow \\
X & \longrightarrow & \bar{f} \\
\end{array}
$$

by $(f_{X,A}, K)$ and we shall call the map $\bar{f} : X \to K$ a solution of the extension problem. (This should not be confused with the notion of a resolution of the extension problem $(f_{X,A}, K)$ introduced below.) Note that by the Borsuk homotopy extension theorem, for every pair $f, f' : A \to K$ of homotopic maps, the extension problems $(f_{X,A}, K)$ and $(f'_{X,A}, K)$ are equivalent, i.e., there is an extension $\bar{f} : X \to K$ of $f$ if and only if there is an extension $\bar{f}' : X \to K$ of $f'$. We shall denote the corresponding equivalence classes by $([f]_{X,A}, K)$. A map $g : Y \to X$ is said to be a resolution of the extension problem
(f_X, A, K) if there is a map \( h : Y \to K \) such that \( h|_{(g^{-1}(A))} = f \circ (g|_{(g^{-1}(A))}) \), i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
g^{-1}(A) & \xrightarrow{g} & A \\
\downarrow & & \downarrow f \\
Y & \xrightarrow{g} & X \\
\end{array}
\]

An inverse system \( \mathcal{L} = \{(L_i, \lambda_i) : i \geq 0\} \) of polyhedra \( L_i \) with triangulations \( \lambda_i \) is said to be \( K \)-resolvable, for some CW complex \( K \), provided that for every \( i \geq 0 \), for every finite subpolyhedron \( A \subset L_i \) (with respect to the triangulation \( \lambda_i \)) and for every map \( f : A \to K \), there exists \( k \in \mathbb{N} \) such that the map \( q^i_k : L_{i+k} \to L_i \) is a resolution of the extension problem \((f_{L_i}, A, K)\), i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
(q^i_k)^{-1}(A) & \xrightarrow{q^i_k} & A \\
\downarrow & & \downarrow f \\
L_{i+k} & \xrightarrow{q^i_k} & L_i \\
\end{array}
\]

Suppose that \( \{(L_i, \lambda_i) : i \geq 0\} \) is an inverse system of polyhedra \( L_i \) with triangulations \( \lambda_i \). We say that the mesh of triangulations \( \{\lambda_i\} \) converges to zero, \( \text{mesh}\{\lambda_i\} \to 0 \), if for every \( k \in \mathbb{N} \), \( \lim_{i \to \infty} \{\text{mesh}(q^i_k((\lambda_{i+k}))\} = 0 \). (This should not be mistaken with the notion \( \{\text{mesh} \lambda_i\} \to 0 \), which simply means that \( \lim_{i \to \infty} \text{mesh} \lambda_i \to 0 \).)

**Lemma 2.1.** Suppose that \( K \) is a countable CW complex and that \( X \) is a compactum such that \( X = \lim\{(L_i, \lambda_i), q^i_{i+1}\} \), where \( \mathcal{L} = \{(L_i, \lambda_i), q^i_{i+1}\} \) is a \( K \)-resolvable inverse system of compact polyhedra \( L_i \) with triangulations \( \lambda_i \) such that \( \text{mesh}\{\lambda_i\} \to 0 \). Then \( X \tau K \).

**Proof.** Suppose that for some closed subset \( A \subset X \) we have a map \( f : A \to K \). We may assume without losing generality, that \( X \subset I^\infty \). Since \( K \) is an ANR there is an open neighborhood \( U \subset I^\infty \) of \( A \) in \( I^\infty \) and a map \( f' : U \to K \) such that \( f' \mid A = f \). Since \( X = \lim\{(L_i, \lambda_i), q^i_{i+1}\} \) there exists for every integer \( i \geq 0 \), a finite subpolyhedron \( A_i \subset L_i \) of \( L_i \) with respect to the triangulation \( \lambda_i \), such that \( A = \lim\{(A_i, \lambda_i \mid A_i), q^i_{i+1}\}_{i \geq 0} \). Therefore for some integer \( i_0 \geq 0 \), \( A_{i_0} \subset U \). Let \( f^* = f' \mid A_{i_0} \). Since the inverse system \( \{(L_i, \lambda_i), q^i_{i+1}\}_{i \geq 0} \) is \( K \)-resolvable for \( K \), there exists an integer \( k \geq 0 \) such that the map \( q^i_{i+k} : A_{i_0+k} \to A_{i_0} \) is the resolution of the extension problem \((f^i_{L_{i_0}}, A_{i_0}, K)\). So there is a map \( \overline{f^*} : L_{i_0+k} \to K \) such that

\[
\overline{f^*} \mid (q^i_{i+k})^{-1}(A_{i_0}) = f^* \circ (q^i_{i+k} |_{(q^i_{i+k})^{-1}(A_{i_0})}^{-1}).
\]
Since \( \text{mesh}\{\lambda_i\} \to 0 \), we can choose \( i_0 \) big enough so that the composition \( \tilde{f} = \tilde{f}^* \circ (q_{i_0+k}^\infty) : X \to K \) solves the extension problem \( (f_{X,A}, K) \).

Lemma 2.2. Suppose that \( K \) is a CW complex and that \( X \) is a compactum. Then for every extension problem \( (f_{X,A}, K) \) there exists a resolution \( g : Y \to X \) such that for every point \( x \in X \), \( g^{-1}(x) \) is either contractible or homotopy equivalent to \( K \). Moreover, if \( X \) and \( K \) are simplicial complexes then \( Y \) can be chosen to be a simplicial complex with simplicial map \( g \) for some triangulations of \( X \) and \( Y \).

Proof. Given a map \( f : A \to K \) consider its extension (which exists since \( K \in \text{ANR} \) hence \( C(K) \in \text{AR} \)) to the cone \( C(K) \) over \( K \) (i.e., \( C(K) = K \times [0, 1]/K \times \{1\} \)), \( \tilde{f} : X \to C(K) \) and let \( \pi : K \times [0, 1] \to C(K) \) be the natural quotient map. Define \( Y \) to be the pull-back of the diagram below

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & K \times [0, 1] \\
\downarrow{g_1} & & \downarrow{\pi} \\
X & \xrightarrow{\tilde{f}} & C(K)
\end{array}
\]

and let \( g : Y \to X \) be the projection corresponding to \( \pi \). Let \( \omega : K \times [0, 1] \to K \) be the obvious projection. Then

\[
\omega \circ (h \mid_{g^{-1}(A)}) = f \circ (g \mid_{g^{-1}(A)})
\]

hence \( g \) is indeed a resolution of the extension problem. Also, for every \( y \in C(K) \), obviously \( \pi^{-1}(y) = \ast \) or \( K \), so for point-inverses of \( g \) we have that \( g^{-1}(x) \simeq \ast \) or \( \simeq K \).
If $X$ is a polyhedron, then we can replace the extension problem $(f, X, A, K)$ by a simplicial one $(f', X, A', K)$ so that $A \subset A'$ and any resolution of a new problem is also a resolution of the old one. Then the extension $\overline{f}$ can be chosen to be simplicial for some triangulations. In this case the pull-back space $Y$ is a simplicial complex. □

**Lemma 2.3.** Suppose that $K$ is a countable CW complex. Then for every $n \in \mathbb{N}$, there exists an inverse sequence $\mathcal{L} = \{(L_i, \lambda_i, q_i^{i+1})\}_{i \geq 0}$ of countable polyhedra $L_i$ with triangulation $\lambda_i$, such that:

(i) $L_0 \simeq S^n$;
(ii) $\text{mesh}(\lambda_i) \to 0$;
(iii) $\mathcal{L}$ is $K$-resolvable; and
(iv) for every $i \geq 0$ and every point $y \in L_i$, $(q_i^{i+1})^{-1}(y)$ is either contractible or homotopy equivalent to $K$.

**Proof.** The proof consists of an inductive construction. Define $\lambda_0$ to be any triangulation of $S^n$ with mesh $\lambda_0 < 1$. There exist only countably many different extension problems $\alpha^0 = \{(f_{L_0}, A, K)\}$ with finite subpolyhedron $A$. (Indeed, $K^X$ is a separable metric space hence there exists a countable dense subset $\{f_i\}_{i \in \mathbb{N}} \subseteq K^X$. Since $K$ is an ANR it follows that every $f \in K^X$ is homotopic to some $f_i$.) We shall enumerate them by even integers: $\alpha_0^0, \alpha_0^1, \alpha_0^2, \ldots$. Let $\alpha_i = \alpha_0^i$, for every $i \in 2\mathbb{Z}$.

Apply Lemma 2.2 to get a map $q_0^1 : L_1 \to L_0$ with the property (iv) and such that $q_0^1$ resolves the extension problem $\alpha_2 = \alpha_0^2$, i.e.,

\[
(q_0^1)^{-1}(A) \xrightarrow{q_0^1} A \xrightarrow{f} K
\]

Choose any triangulation $\lambda_1$ of $L_1$ such that mesh $\lambda_1 < 1/2$ and mesh $q_0^1(\lambda_1) < 1/2$. Again, there are only countably many different extension problems $\alpha_1 = \{(f_{L_1}, A, K)\}$. We enumerate them by odd numbers, divisible by 3: $\alpha_1^0, \alpha_1^1, \alpha_1^2, \alpha_1^3, \alpha_1^4, \ldots$. Define $\alpha_i = \alpha_0^i$, $k \in \{0, 1, 2\}$, $i \in 2\mathbb{Z} \cup 3\mathbb{Z}$. Apply Lemma 2.2 to get a map $q_1^2 : L_2 \to L_1$ with the property (iv) and such that $q_1^2$ resolves the extension problem $\alpha_3 = \alpha_1^3$, i.e.,

\[
(q_1^2)^{-1}(A) \xrightarrow{q_1^2} A \xrightarrow{f} K
\]

Choose any triangulation $\lambda_2$ of $L_2$ such that mesh $\lambda_2 < 1/4$, mesh $q_1^2(\lambda_2) < 1/4$, and mesh $q_0^1(\lambda_2) < 1/4$. Again, there are only countably many different extension problems $\alpha_2 = \{(f_{L_2}, A, K)\}$. We enumerate them by integers, divisible by 5 but not divisible by 2 or 3: $\alpha_2^0, \alpha_2^1, \alpha_2^2, \alpha_2^3, \alpha_2^4, \alpha_2^5, \ldots$. Define $\alpha_i = \alpha_0^i$, $k \in \{0, 1, 2\}$, $i \in 2\mathbb{Z} \cup 3\mathbb{Z} \cup 5\mathbb{Z}$.
Consider $\alpha_4 = \alpha_4^0 = (f_{L_0}, A, K)$. It defines an extension problem
\[(q_0^2)^{-1}(\alpha_4) = \left( f \circ (q_0^2)^{-1}(A), K \right). \]

Apply Lemma 2.2 to resolve the extension problem $(q_0^2)^{-1}(\alpha_4)$ by the map $q_3^2 : L_3 \to L_2$:
\[
\begin{array}{cccccc}
L_3 & \xrightarrow{q_3^2} & L_2 & \xrightarrow{q_3^1} & A & \xrightarrow{f} & K \\
\downarrow & & \downarrow & & \downarrow & & \\
L_0 & & L_0 & & \end{array}
\]

Note that then the map $q_0^3$ resolves the original extension problem $\alpha_4$. We can proceed inductively, to construct a system $\{(L_i, \lambda_i, q_i^{i+1})\}_{i \geq 0}$. All equivalence classes of extension problems $([f]_{L_i}, A, K)$ will be enumerated by $\mathbb{N}\setminus\{1\}$: $\alpha_2, \alpha_3, \alpha_4$, etc.

Clearly, the property (ii) holds by the construction. Next, for every $i \in \mathbb{N}\setminus\{1\}$, every extension problem $\alpha_i$ is resolved by the map $q_k^{-1} : L_{i-1} \to L_k$, where $\alpha_i = \alpha_k^0$, hence the property (iii) holds. Finally, property (iv) holds by Lemma 2.2 and the construction. \[\square\]

A generalized cohomology theory $h^*$ is said to be continuous if for every countable CW complex $W$ with a compact stratification $W_1 \subset W_2 \subset \cdots \subset W_i \subset \cdots$ there is an equality $h^*(W) = \lim_i h^*(W_i)$.

**Theorem 2.4.** Let $K$ be a countable CW complex and $h$ be a nontrivial homology theory (respectively, nontrivial continuous cohomology theory) with $\tilde{h}_*(K) = 0$ (respectively, $\tilde{h}^*(K) = 0$). Then for every integer $n \geq 1$, there is a compactum $X$ of dimension $\dim X \geq n$ with the property $X \not\simeq K$.

**Proof.** We'll give a proof for cohomology (for homology it is analogous). Apply Lemma 2.3 to obtain an inverse system $\mathcal{L} = \{(L_i, \lambda_i, q_i^{i+1})\}_{i \geq 0}$ of polyhedra $L_i$ with triangulations $\lambda_i$ such that (i) $L_0 \simeq S^n$; (ii) mesh $\{\lambda_i\} \to 0$; (iii) $\mathcal{L}$ is $K$-resolvable; and (iv) For every $y \in L_i$ and every $i$, $(q_i^{i+1})^{-1}(y) \simeq \text{point or } \simeq K$.

We shall construct inductively a $K$-resolvable inverse system
\[
\mathcal{Q} = \{(Q_i, \lambda_i|_{Q_i}), q_i^{i+1}|_{Q_i}\}_{i \geq 0}
\]
of compact subpolyhedra $Q_i$ of $L_i$ with respect to triangulations $\lambda_i$. Define $Q_0 = L_0$ and consider any $\gamma_0 = \gamma \neq 0 \in h^*(L_0)$. It follows by the property (iv) above that $(q_0^1)^* : h^*(L_0) \to h^*(L_1)$ is an isomorphism [18]. Since by hypothesis, $h^*$ is continuous, the nonzero element $\gamma_1 = (q_0^1)^*(\gamma_0)$ lives in some compact subpolyhedron $Q_1$ of $L_1$ with respect to the triangulation $\lambda_1$. Repeat this argument for $\gamma_1$ to get a compact subpolyhedron $Q_2$ of $L_2$ with respect to the triangulation $\lambda_2$ with $(q_1^2)^*(\gamma_1) \neq 0$ living in $Q_2$, etc. Let $X = \lim_i\{(Q_i, \lambda_i|_{Q_i}), q_i^{i+1}|_{Q_i}\}_{i \geq 0}$. Clearly, $(q_0^\infty)^*(\gamma_0) \neq 0$ therefore $q_0^\infty : X \to L_0 \simeq S^n$ is essential so $\dim X \geq n$. Also, by Lemma 2.1, $X \not\simeq K$. \[\square\]
Proposition 2.5. For every compacta $X$, $Y$ and $Z$ the following implication holds: $X \tau (Y \lor Z) \Rightarrow X \tau Y$.

Proof. Straightforward (cf., e.g., [12]).

By using Proposition 2.5 it is possible to generalize Theorem 2.4 in a straightforward manner to the following:

Theorem 2.6. Let $\{K_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence of countable CW complexes, acyclic with respect to a nontrivial homology (respectively, continuous cohomology) theory $h$. Then for every integer $n \geq 1$, there exists a compactum $X$ of dimension $\dim X \geq n$, having the property $X \tau K_i$ for every $i \in \mathbb{N}$.

3. Cannon–Stan’ko compacta

We begin by briefly recalling the construction of a grope $M$. (For more about gropes see [1].) One defines $M$ as the direct limit $M = \lim \{L_i, p_{i+1}^i\}_{i \geq 0}$ of a direct system of compact 2-dimensional polyhedra $L_i$ and the injective bonding maps $p_{i+1}^i : L_i \rightarrow L_{i+1}$.

The polyhedron $L_n$ is called the $n$th stage of the grope construction. Here, $L_0$ is an oriented compact surface $S^g$ of genus $g > 0$ with an open disk deleted. Let $A_0 \subset S^g$ be a set of $2g$ circles which generate the 1-dimensional homology of the surface $S^g$. The complex $L_{n+1}$ is then obtained from $L_n$ for every $n \geq 0$, by attaching for every circle $a \in A_n$, an oriented compact surface $S^g_a$, with an open disk deleted, by identifying the boundary $\partial S^g_a$ of the surface $S^g_a$ with the circle $a \in A_n$. The generators of homology of $S^g_a$ then determine the set of $2g_a$ circles $A_{n+1} \subset S^g_a$ which generate the 1-dimensional homology of the surface $S^g_a$.

In particular, we shall need the so-called minimal grope $M^* = \lim \{L_i^*, p_{i+1}^i\}_{i \geq 0}$ which is distinguished by the fact that the genus of $L_0$ is one and that for every $i \geq 0$, we attach only two 1-handles to each 1-handle pair of generators of the 1-dimensional homology of the complex $L_i^*$.

Definition 3.1. A compactum $X$ is said to be a Cannon–Stan’ko compactum provided that for the minimal grope $M^*$, $X \tau M^*$. Equivalently, for the minimal grope $M^*$, $\chi_1(M^*)$, i.e., $\dim H_1(M^*) X \leq 1$.

Examples 3.2. Every compactum of dimension $\leq 1$ is evidently also a Cannon–Stan’ko compactum. The Pontryagin disk $D^2$, introduced in [27], is an example of a 2-dimensional Cannon–Stan’ko compactum. By gluing two copies of $D^2$ along the boundary $\partial D^2 = S^1$ one gets the so-called Riemannian surface of infinite local genus $S^2$ which is an example of a homogeneous 2-dimensional Cannon–Stan’ko compactum [13].

We shall now prove that there actually exist examples of Cannon–Stan’ko compacta of arbitrary high dimensions:
Theorem 3.3. For every integer \( n \geq 1 \), there exists an \( n \)-dimensional Cannon–Štan’ko compactum.

Proof. By Theorem 2.4 it suffices to find a nontrivial homology theory \( h_* \) such that \( h_*(M^*) = 0 \). For \( h_* \) we take the singular homology theory with integer coefficients, \( h_* = H_* (\_, \mathbb{Z}) \). Clearly, for the minimal grope \( M^* = \lim \{ L_i^* ; p_i^{i+1} \}_{i \geq 0} \), all bonding maps \( p_i^{i+1} : L_i^* \rightarrow L_{i+1}^* \) induce the zero-homomorphisms \( (p_i^{i+1})_* : H_*( L_i^* ; \mathbb{Z} ) \rightarrow H_*( L_{i+1}^* ; \mathbb{Z} ) \) since:

(i) For every 1-cycle \( \gamma \in H_1 ( L_i^* ; \mathbb{Z} ) \), \( (p_i^{i+1})_* (\gamma) \) bounds a 2-chain in \( H_2 ( L_i^* ; \mathbb{Z} ) \), hence \( H_1 ( M^* ; \mathbb{Z} ) = \lim \{ H_1 ( L_i^* ; \mathbb{Z} ) , (p_i^{i+1})_* \} \}_{i \geq 0} = 0 \); and

(ii) For every \( j \geq 2 \), \( H_j ( L_i^* ; \mathbb{Z} ) = 0 \) since \( L_i^* \) is homotopically just a bouquet of finitely many circles \( S^1 \), thus it follows again that \( H_j ( M^* ; \mathbb{Z} ) = 0 \).

Finally, since \( h_* \) is an ordinary homology theory, we may assume that in the proof of Theorem 2.4, in the construction of the inverse system \( \mathcal{Q} = \{ (Q_i, \lambda_i | Q_i) , q_{i+1}^{i+1} \}_{i \geq 0} \), the subpolyhedra \( Q_i \) of \( L_i^* \) are all \( n \)-dimensional. We can achieve this by starting with an \( n \)-dimensional cycle \( \gamma \in h_* ( L_0^* ) \) and then the \( n \)-dimensional homology will always live already in the \( n \)-skeleton \( Q_i^{(n)} \) of the polyhedron \( Q_i \). \( \square \)

If instead of Theorem 2.4 one uses Theorem 2.6, the same proof yields the following stronger statement:

Theorem 3.4. Let \( \mathcal{M} = \{ M_k \}_{k \in \mathbb{N}} \) be an arbitrary (at most countable) collection of gropes. Then for every integer \( n \geq 1 \), there exists an \( n \)-dimensional compactum \( X \) such that for every index \( k \in \mathbb{N} \), \( X \cap M_k \).

4. Nonabelian compacta

Definition 4.1. Let \( T = ( S^1 \times S^1 ) \setminus \text{int} B \) be a 2-torus with a hole (obtained by removing the disk \( B \)) and denote its boundary by \( \partial T \) (hence \( \partial T = S^1 \)). A compactum \( X \) is said to be nonabelian if for every closed subset \( A \subset X \) of \( X \) and every continuous map \( f : A \rightarrow \partial T \) there exists a continuous map \( \tilde{f} : X \rightarrow T \) such that \( \tilde{f}|_A = f \). We shall denote this extension property by \( X \tau ( T , \partial T ) \).

Examples 4.2. Every compactum of dimension \( \leq 1 \) is evidently also nonabelian. An example of a 2-dimensional nonabelian compactum is the classical Pontryagin mod 2 "surface" \[ 30 \], i.e., the inverse limit of an inverse system of modifications of the 2-sphere where disks are replaced by Möbius bands (observe that \( \mathbb{R} P^2 \# \mathbb{R} P^2 \# \mathbb{R} P^2 \) is homeomorphic to \( \mathbb{R} P^2 \# (1\text{-handle}) \) \[ 13 \].

Theorem 4.3. For every integer \( n \geq 1 \), every \( n \)-dimensional nonabelian compactum is also a Cannon–Štan’ko compactum.
We shall first need to prove two lemmas. As before, let $S_g$ be the 2-sphere with $g > 0$ orientable 1-handles and one hole, and let $D^2$ be the Pontryagin disk [27] (recall that $\partial D^2 = S^1$).

**Lemma 4.4.** For every compactum $X$, the following statements are equivalent:

1. $X$ is nonabelian;
2. for every $g > 0$, $X \tau(S_g, \partial S_g)$, where $S_g$ is a compact orientable surface of genus $g$ and with one hole; and
3. $X \tau(D^2, \partial D^2)$.

**Proof.** (1) $\Rightarrow$ (2) Take in $S_g$ a bouquet $A = \bigvee_{i=1}^{g-1} A_i$ of $g - 1$ arcs such that for every $i$:
   
   (i) $\partial A_i = \{a, a_i\};$
   (ii) $A_i \cap A_j = \{a\}$, for every $i \neq j$;
   (iii) $\partial A_i \cap S_g = \partial A_i \cap \partial S_g = \partial A_i$; and
   (iv) if one cuts $S_g$ along $A$ then each of the resulting pieces $E_1, \ldots, E_g$ is a disk with just one of the $g$ 1-handles $H_i$.

By shrinking the bouquet $A$ to the point $a$ one gets a cell-like map $f : S_g \rightarrow Y$ of $S_g$ onto a bouquet $Y = \bigvee_{i=1}^{g} D_i$, where for every $i$, $D_i = f(E_i)$ is a 2-torus with a hole. (Note that $f(\partial E_i) \subset \partial D_i$, for every $i$. For the definition and the main properties of cell-like maps see, e.g., the survey [26].) By collapsing every $D_i$ onto an interval $I_i \subset D_i$ one furthermore gets a surjective map $g : Y \rightarrow Z$ of $Y$ onto a bouquet $Z = \bigvee_{i=1}^{g} I_i$.

Take now any nonabelian compactum $X$, an arbitrary closed subset $A \subset X$ and any continuous map $\varphi : A \rightarrow \partial S_g$. Then the compositum $\psi = g \circ f \circ \varphi : A \rightarrow Z$ gives a continuous map of $A$ into $Z$. Since $Z$ is obviously an absolute retract (AR), the map $\psi$ can be extended to a continuous map on $X$, $\bar{\psi} : X \rightarrow Z$.

For every $k \in \{1, \ldots, g\}$, consider the restriction

$$\xi_k = f \circ \varphi|_{\psi^{-1}(k) \cap \{A \cup \psi^{-1}(a)\}} : \psi^{-1}(I_k) \cap (A \cup \psi^{-1}(a)) \rightarrow \partial D_k.$$

Note that $\psi^{-1}(a)$ separates $X$ into components the closures of which are $\psi^{-1}(I_k)$, $k \in \{1, \ldots, g\}$. Since by hypothesis, $X$ is nonabelian, so is $\psi^{-1}(I_k)$ hence there is an extension over all of $\psi^{-1}(I_k)$, $\xi_k : \psi^{-1}(I_k) \rightarrow D_k$. Let $\xi = \bigcup_{k=1}^{g} \xi_k : X \rightarrow Y$. Clearly, $\xi$ is well-defined and continuous. Since $f : S_g \rightarrow Y$ is cell-like there is a lifting $\tilde{\xi} : X \rightarrow S_g$ which is up to homotopy, an extension of $\varphi : A \rightarrow \partial S_g$ over $X$, i.e., $\tilde{\xi}|_A \simeq \varphi$.

(2) $\Rightarrow$ (1) This is obvious—consider the case when $g = 1$.

(3) $\Rightarrow$ (1) Suppose that $f : A \rightarrow \partial T$ is a continuous map from a closed subset $A \subset X$ of a compactum $X$ with the property $X \tau(D^2, \partial D^2)$. Recall the inverse limit construction of the Pontryagin disk $D^2$ from [27]: $D = \lim_{\leftarrow i \geq 0} \{K_i, p^i_{i+1}\}$ where $K_0$ is a 2-disc and $K_{i+1}$ is a Pontryagin-like modification of $K_i$ ($i \geq 0$) except that instead of Möbius bands we attach orientable 1-handles. In particular, for every $i \geq 1$, $K_i$ is a compact orientable surface of some finite genus $g_i > 0$ and $g_{i+1} > g_i$. Also, we can identify $\partial K_i \equiv S^1 \equiv \partial D^2$ for every $i$. Therefore there is a continuous map $\varphi : (D^2, \partial D^2) \rightarrow (T, \partial T)$ obtained as the compositum of the canonical projection $p_\infty : (D^2, \partial D^2) \rightarrow (K_1, \partial K_1)$, followed by the obvious onto map $\beta : (K_1, \partial K_1) \rightarrow (T, \partial T)$ which simply kills the extra handles.
By hypothesis, the map \((\varphi|_{\partial D^2})^{-1} \circ f : A \to \partial D^2\) extends over all of \(X\) to a map \(g : X \to D^2\). Obviously, the composition \(\tilde{f} = \varphi \circ g : X \to T\) is then the desired extension of \(f\) over all of \(X\). This verifies that \(X\) is indeed nonabelian.

(1) \(\Rightarrow\) (3) Represent again \(D^2\) as the inverse limit, \(D^2 = \lim_{\leftarrow} \{K_i, p_{i+1}^{-1}\}_{i \geq 0}\) of 2-disks-with-handles \(K_i\), i.e., \(K_i = S_{g_i}\) for some \(g_i \geq 0\). Suppose that \(X\) is a nonabelian compactum and pick any closed subset \(A \subset X\) and any continuous map \(\varphi : A \to \partial D^2\). Since we've already verified that the assertions (1) and (2) of the lemma are equivalent and since one can identify \(\partial K_i = \partial D^2\), it follows that there is an extension of \(\varphi\) to a map \(\varphi_i : X \to K_i\) for every \(i \geq 0\).

Fix an \(i_0 \geq 0\) and consider any 2-simplex \(\sigma \in K_{i_0}^{(2)}\). Look at the pair \((\varphi_{i_0}^{-1}(\sigma), \psi_{i_0+1}^{-1}(\partial \sigma))\). Then one can lift the map

\[
\varphi_{i_0} \mid \varphi_{i_0}^{-1}(\sigma) : \varphi_{i_0}^{-1}(\sigma) \to K_{i_0}
\]

to \((p_{i_0}^{i_0+1})^{-1}(\sigma) \subset K_{i_0+1}\) (since \(X\) is nonabelian and since \((p_{i_0}^{i_0+1})^{-1}(\sigma)\) is a disk with one 1-handle), so by doing this operation for every \(\sigma \in K_{i_0}^{(2)}\), we get a lifting of the map \(\varphi_{i_0} : X \to K_{i_0}\) to \(K_{i_0+1}\), i.e., a map \(\psi_{i_0+1} : X \to K_{i_0+1}\).

\[
\begin{array}{c}
K_{i_0} \\
\psi_{i_0} \\
\varphi_{i_0}
\end{array} \quad \begin{array}{c}
K_{i_0+1} \\
\psi_{i_0+1} \\
\varphi_{i_0+1}
\end{array} \quad \begin{array}{c}
D^2 \\
\psi \\
\varphi
\end{array}
\]

\[
A
\]

Provided that \(i_0 \geq 0\) is chosen big enough, we may assume that \(\varphi_{i_0} \simeq p_{i_0}^{i_0+1} \circ \psi_{i_0+1}\) via some \((\frac{1}{2})\)-homotopy. Therefore, a map \(\psi = \lim_{n \to \infty} \psi_{i_0+n} : X \to D^2\), where for every \(k \geq 0\), \(\psi_{i_0+k+1} : X \to K_{i_0+k+1}\) is a lifting of the map \(\psi_{i_0+k} : X \to K_{i_0+k}\) such that

\[
\psi_{i_0+k} \simeq p_{i_0+k}^{i_0+k+1} \circ \psi_{i_0+k+1}
\]

via some \((2^{-k})\)-homotopy, is well-defined and continuous, and it is, up to a homotopy, an extension of the given map \(\varphi : A \to D^2\). (If necessary, one can choose an appropriate subsequence of the sets of indices \(\{i_0, i_0 + 1, i_0 + 2, \ldots\}\).) This verifies that indeed \(X \tau(D^2, \partial D^2)\).

This completes the proof of the lemma. \(\square\)

**Lemma 4.5.** Let \(M\) be any grope. Then for an arbitrary compactum \(X\), the following statements are equivalent:

1. \(X \tau M\); and
2. \(X \tau(M, \partial M)\).

**Proof.**

(1) \(\Rightarrow\) (2) Obvious.
(2) ⇒ (1) Let $X$ be any compactum and take any closed subset $A \subset X$ and any continuous map $f : A \to M$. Represent $M$ as the direct limit of finite stages of the grope construction, $M = \lim_{\to} \{L_i, p^i_{i+1}\}_{i \geq 0}$. Since $A$ is compact, its image $\varphi(A)$ lies in some finite stage of $M$, i.e., for some $i \geq 1$, $\varphi(A) \subset L_i$, where $\varphi = p^i_\infty \circ \varphi$ and $p^i_\infty : L_i \to M$ is the canonical inclusion.

Now, $L_i$ is of the homotopy type of a finite bouquet of circles, i.e., $L_i \simeq Y = \bigvee_{j=1}^{k(i)} S_j$, and moreover, $Y \subset Z = \bigvee_{j=1}^{k(i)} N_j$, where $N_j$ are gropes and $\partial N_j = S_j$, for every $j \in \{1, \ldots, k(i)\}$. We have a deformation retraction $r : L_i \to Y$ which we can follow by $\gamma_n(\cdot)$ the map $g : Z \to W = \bigvee_{j=1}^{k(i)} I_j$ which collapses each grope $N_j$ onto the segment $I_j$. The composition $\psi = g \circ r \circ \varphi : A \to W$ can now be extended over $X$ to give $\varphi : X \to W$.

This map approximately lifts to $\psi : X \to Z$ since $g$ is cell-like. Since $Z \simeq M$ and since it suffices to solve our extension problem only up to homotopy, the proof is completed. 

\[
\begin{array}{c}
A \\
\downarrow \text{incl.} \\
X \\
\downarrow \text{incl.} \\
W \\
\end{array} 
\xrightarrow{\varphi} 
\begin{array}{c}
L_i \\
\downarrow p^i_\infty \\
M \\
\downarrow r \\
Z \\
\downarrow \psi \\
W \\
\end{array} 
\xrightarrow{\psi} 
\begin{array}{c}
A \\
\downarrow \text{incl.} \\
X \\
\downarrow \text{incl.} \\
W \\
\end{array}
\]

**Proof of Theorem 4.3.** If $X$ is nonabelian then $X \tau(S_1, \partial S_1)$, so by Lemma 4.4, $X \tau(S_g, \partial S_g)$, for every $g > 0$, hence $X \tau(M, \partial M)$, for every grope $M$ (in particular for the minimal grope $M^*$) since obviously $(S_1, \partial S_1) \subset (M, \partial M)$, thus by Lemma 4.4, $X \tau M$. 

Obviously, the proof of Theorem 4.3 yields the following stronger statement:

**Theorem 4.6.** For every integer $n \geq 1$, every $n$-dimensional nonabelian compactum $X$ has the property $X \tau M$, for every grope $M$.

**Proposition 4.7.** Let $X$ be a compactum, $K$ a polyhedron and $(L, L')$ a polyhedral pair. Suppose that the property $X \tau (K \ast L, K \ast L')$ holds, where $\ast$ denotes the join of polyhedra. Then there is an $F_\sigma$-set $Z \subset X$ (respectively $G_\delta$-set $W \subset X$) such that $Z \tau K$ (respectively $W \tau K$) and $(X \setminus Z) \tau (L, L')$ (respectively $(X \setminus W) \tau (L, L')$).

**Proof.** This proposition is a relative version of Corollary 2 from [14] and the proof is the same (see the proofs of Theorem 1 and Corollary 2 in [14]).

**Theorem 4.8.** Let $X$ be an arbitrary compactum. Then there exists a 0-dimensional $F_\sigma$-set $Z \subset X$ (respectively $G_\delta$-set $W \subset X$) such that every compactum $Y \subset X \setminus Z$ (respectively $Y \subset X \setminus W$) is nonabelian.
Proof. Let $X$ be an arbitrary compactum and define $K = S^0$, where $S^0$ is the 0-dimensional sphere, and let $(L, L') = (T, \partial T)$. Then $(K * L, K * L') = (\Sigma T, \Sigma \partial T))$ and the inclusion-induced homomorphism $\pi_*(\Sigma \partial T)) \to \pi_*(\Sigma T)$ is trivial, hence the property $X' \tau(K * L, K * L')$ holds. Apply now Proposition 4.7 to obtain an $F_\sigma$-set $Z \subset X$ (respectively $G_\sigma$-set $W \subset X$) such that $Z'K \subset X \setminus Z$ (respectively $W'K \subset X \setminus W$) and 

$$(X \setminus Z)'T(L, L')$$

In particular, $Z' \tau S^0$ (respectively $W' \tau S^0$) implies that $\dim Z = 0$ (respectively $\dim W = 0$), whereas the other property of $Z$ (respectively $W$) clearly implies that every compactum $Y \subset X \setminus Z$ (respectively $Y \subset X \setminus W$) is nonabelian. \[\] 

Corollary 4.9. For every integer $n \geq 0$, there exists an $n$-dimensional nonabelian compactum.

Proof. Given any integer $n \geq 0$, consider the following compactum $X = I^{n+1}$, i.e., the $(n + 1)$-dimensional cube. By Theorem 4.8, there exists a $G_\sigma$-set $W \subset X$ such that $\dim W = 0$ and every compactum $Y \subset X \setminus W$ is nonabelian. By the Urysohn–Menger sum formula, $\dim X \setminus W \geq n$ hence if we write the $F_\sigma$-set $X \setminus W$ as the countable union of compacta $C_i \subset X \setminus W$, $X \setminus W = \bigcup \{C_i \mid i \in \mathbb{N}\}$ then by the Countable sum theorem there is at least one $i_0 \in \mathbb{N}$ such that $\dim C_{i_0} = n$. Since $C_{i_0}$ is also nonabelian this proves the assertion. \[\]

5. Cainerian compacta

Definition 5.1. A compactum $X$ is said to be Cainerian provided that for every perfect group $\Pi$, $X'\in K(\Pi, 1)$. Equivalently, $c-dim_\Pi X \leq 1$.

Examples 5.2. Every compactum of dimension $\leq 1$ is Cainerian. The Pontryagin disk $D^2$ and the Riemannian surface of infinite local genus $S^2$ are examples of 2-dimensional Cainerian compacta.

Let us verify this for $D^2$. So let $\Pi$ be a perfect group and choose any closed subset $A \subset D^2$ and any continuous map $f : A \to K(\Pi, 1)$. Represent $(D^2, A)$ as the inverse limit of polyhedral pairs (assume all lie in $I^\infty$):

$$(D^2, A) = \lim_{\leftarrow} \{(D_i, A_i), (p_i^{i+1}, p_i^{i+1}|_{A_i+i})\}_{i \geq 0}.$$ 

Since $K(\Pi, 1)$ is an ANR, there is an integer $i_0 \in \mathbb{N}$ such that $f$ can be extended over $A_{i_0}$ to give a map $\overline{f}_{i_0} : A_{i} \to K(\Pi, 1)$ such that $\overline{f}_{i_0} \circ (p_{i_0}^{i+1}|_A) \simeq f$. Clearly, $\overline{f}_{i_0}$ can be extended over the 1-skeleton $D_{i_0}^{(1)}$ of $D_{i_0}$ to give a map $g_{i_0} : A_{i_0} \cup D_{i_0}^{(1)} \to K(\Pi, 1)$. This gives an extension up to homotopy, $\gamma : A \cup D^{(1)} \to K(\Pi, 1)$.

Pick any 2-simplex $\sigma_j \in D_{i_0}^{(2)} \setminus A_{i_0}$ and consider $\gamma_{\sigma_j} = g_{i_0}(\partial \sigma_j)$ as the element of $\pi_1(K(\Pi, 1))$. Then $\gamma_{\sigma_j}$ is a product of commutators, $\gamma_{\sigma_j} = c_1 \cdots c_{i(j)}$ since $\Pi$ is a perfect group.

Consider $(p_{\sigma_j}^{m})^{-1}(\sigma_j) \subset D^2$. We have a well-defined extension of $f : A \to K(\Pi, 1)$ over $A \cup (p_{\sigma_j}^{m})^{-1}(\partial \sigma_j)$. We wish to extend over $(p_{\sigma_j}^{m})^{-1}(\sigma_j)$. We proceed as follows: Let
i_1 \geq i_0 \text{ be the smallest index such that } (p^i_{i_0})^{-1}(\sigma_j) \text{ is a disk with } \geq t_j \text{ handles. Clearly, then there is a map }

\[ g_{i_1} : (p^i_{i_0})^{-1}(\sigma_j) \to K(\Pi, 1) \]

because all \( c_i \)'s can be represented by (orientable) 1-handles (and we can kill extra 1-handles if necessary). We now do this for all 2-simplices \( \sigma, \ldots, \sigma_{n(i_0)} \in D^{(2)} \setminus A^{(2)}_0 \) and let \( i_{\max} = \max\{i_1, \ldots, i_{n(i_0)}\} \). Then clearly, one gets a well-defined map \( \gamma_{\max} : D_{i_{\max}} \to K(\Pi, 1) \) such that it is an extension up to homotopy, of \( f : A \to K(\Pi, 1) \) and so the composition

\[ \gamma = (\gamma_{\max} \circ p^\infty_{i_{\max}}) : \mathbb{D}^2 \to K(\Pi, 1) \]

is again up to homotopy, the desired extension.

**Definition 5.3.** Let \( N = N_1 \cup N_2 \) be the (boundary connected) sum of two copies \( N_1 = M^* = N_2 \) of the minimal grope \( M^* \) along its boundary circle \( \partial M^* = S^1 \). A compactum \( X \) is said to be weakly Cainian provided that for the fundamental group \( \Pi = \Pi_1(N), \chi_{\Pi, K(\Pi, 1)}. \) Equivalently, \( c\dim_{\Pi} X \leq 1. \)

**Remark 5.4.** Let \( W \) be the class of all perfect groups \( \Pi \) such that \( H_2(\Pi) \cong \mathbb{Z} \) and \( H_q(\Pi) = 0, \) for all \( q \neq 2. \) Note that obviously, \( \Pi = \Pi_1(N) \in W. \)

**Theorem 5.5.** Every weakly Cainian compactum is at most 2-dimensional.

**Proof.** If one glues together two minimal gropes \( N_1 \) and \( N_2 \) along the boundary then it follows by [5, Lemma (1.18)] (also by [4, Aspherical Pasting Lemma 4]) that the resulting space \( N \) is aspherical. Also \( \Pi \) is perfect since \( H_1(N) = 0, \) for we have killed (in fact, twice) the only generator of \( H_1(S^1) \cong \mathbb{Z}, \) therefore \( N = K(\Pi, 1). \)

Next, the suspension of \( N, \Sigma N \) is a homotopy 3-sphere since it is the union of two suspensions of a grope glued along the suspension of a circle, i.e., it is the union of two homotopy 3-balls glued along their boundary.

Suppose now that \( X \) is a weakly Cainian compactum. Then \( X \tau N \) hence by [12], \( (X \times I) \tau (\Sigma N) \) and consequently, \( (X \times I) \tau S^3 \) so \( \dim(X \times I) \leq 3 \) hence \( \dim X \leq 2. \)

**Corollary 5.6.** Every Cainian compactum is at most 2-dimensional.

**Remark 5.7.** In their recent preprint, J. Dydak and K. Yokoi renamed Cainian compacta as compacta of perfect cohomological dimension one. In particular, they obtained an alternative proof of our Corollary 5.6 (cf. [19, Corollary (3.7)]).

**Theorem 5.8.** Every weakly Cainian compactum is a Cannon–Štan’ko compactum.

**Proof.** Suppose that \( X \) is a weakly Cainian compactum and let \( M^* \) be the minimal grope. Then \( X = K(\Pi, 1) \) where \( \Pi \) is the fundamental group of the space \( N = N_1 \cup N_2, \) obtained by glueing two copies of \( M^* \) along \( \partial M^* = S^1. \) Let \( A \subset X \) be any closed
subset and consider any map \( f : A \to N_1 \subset N_1 \cup N_2 \). Since \( X \) is weakly Cainian, there exists an extension \( \overline{f} : X \to N_1 \cup N_2 \). Let \( \varphi : N_1 \cup N_2 \to N_1 \) be the obvious "flip". Then \( \varphi \circ \overline{f} : X \to N_1 \) is the desired extension of the map \( f \) over all of \( X \).

**Theorem 5.9.** Every 2-dimensional Cannon–Stan’ko compactum is a weakly Cainian compactum.

**Proof.** Suppose that \( X \) is a 2-dimensional Cannon–Stan’ko compactum. Then in particular, \( X \cap M^* \), for the minimal grope \( M^* \). We must prove that \( X \cap K(P_i, 1) \), where \( N = N_1 \cup N_2 \), \( N_1 = N_2 = M^* \) and \( \partial N_1 = \partial N_2 = S^1 \).

Suppose now that we have a closed subset \( A \subset X \) and a map \( f : A \to N_1 \cup N_2 \). Then the pull-back of \( S^1 \) under \( f \) separates \( A \). Let \( C' = f^{-1}(S^1) \). Since \( \dim X = 2 \) there exists a subset \( C \) such that \( C \cup C' \) separates \( X \), \( C \) separates \( X \setminus A \), and \( \dim C = 1 \). Denote the components of \( X \setminus C \) by \( X_1 \) and \( X_2 \), i.e., \( X_1 \supset f^{-1}(N_1) \) and \( X_2 \supset f^{-1}(N_2) \).

First, extend \( f|_{C'} \) over \( C \) (we can do this since \( \dim C = 1 \) and we are mapping to \( S^1 \)), i.e., extend \( f|_{C'} : C' \to S^1 \) to \( \overline{f} : C \cup C' \to S^1 \). Now solve the extension problem separately for each "half" space \( X_1 \) and \( X_2 \), using the hypothesis that \( X \cap M^* \). (Note that every subcompactum of a Cannon–Stan’ko compactum is also a Cannon–Stan’ko compactum. Indeed, given such a subcompactum \( X_0 \subset X \) use the extension for all \( X \) and take its restriction onto \( X_0 \).) This gives extensions \( F_i : X_i \to N_i, i \in \{1,2\} \), such that for every \( i \), \( F_i|_{C \cup C'} = \overline{f} : C \cup C' \to \partial N_i \). Thus

\[
F = F_1 \cup F_2 : X = X_1 \cup X_2 \to N = N_1 \cup N_2
\]
is then the desired extension of the map \( f \). \( \square \)

**Corollary 5.10** (Characterization of weakly Cainian compacta). Weakly Cainian compacta are precisely the 2-dimensional Cannon–Stan’ko compacta.

**Theorem 5.11.** Every 2-dimensional nonabelian compactum is Cainian.

**Proof.** Let \( X \) be an arbitrary 2-dimensional nonabelian compactum. Let \( P_i \) be an arbitrary perfect group, \( A \subset X \) any closed subset of \( X \) and \( f : A \to K(P_i, 1) \) any continuous map.

Represent the compactum \( X \) as the inverse limit of compact 2-dimensional polyhedral pairs \( (L_i, A_i), (X, A) = \lim{(L_i, A_i), (p_i^{i+1}, p_i^{i+1}|_{A_i+1})}_{i \geq 0} \). We may assume that \( L_i, X \subset I^\infty = \) the Hilbert cube. Since \( K(P_i, 1) \) is an ANR, there exists an integer \( i_0 \geq 1 \) such that \( f \) factors through \( A_i \), i.e., the diagram below is commutative:

\[
\begin{array}{ccc}
A & \longrightarrow & K(P_i, 1) \\
\downarrow & & \downarrow f_i \\
A_i & \longrightarrow & A_i
\end{array}
\]

It is easy to extend \( f_i \) over the 1-skeleton \( L_i^{(1)} \) of \( L_i \), \( \overline{f}_i : A_i \cup L_i^{(1)} \to K(P_i, 1) \).
Pick any 2-simplex \( \sigma \in \mathcal{L}_2 \). Then \( \alpha = \overline{f}_i(\partial \sigma) \in \Pi \) and \( \alpha \) is a product of commutators, \( \alpha = [\alpha_1, \beta_1] \cdot [\alpha_2, \beta_2] \cdots [\alpha_k, \beta_k] \), since \( \Pi \) is perfect. Glue onto \( \sigma \) (in \( I^\infty \)) \( k \) orientable 1-handles \( H_1, H_2, \ldots, H_k \), one for each commutator \( [\alpha_i, \beta_i] \). Next, each \( \alpha_i \) and \( \beta_i \) is itself a product of commutators, e.g., \( \alpha_i = [\gamma_1 \delta_1] \cdot [\gamma_2 \delta_2] \cdots [\gamma_i \delta_i] \), so again we can glue onto \( H_i \) the corresponding \( l \) orientable 1-handles \( G_1^i, \ldots, G_l^i \), etc. This infinite process ends in gluing of \( k \) gropes \( M_1, M_2, \ldots, M_k \) onto \( \sigma \). Denote by \( \overline{\sigma} \) the resulting complex, \( \overline{\sigma} = \sigma \# \{ M_i \mid 1 \leq i \leq k \} \). Then \( \overline{f}_i(\partial \sigma) : \partial \sigma \to K(\Pi, 1) \) can be extended to a continuous map \( \overline{f}_i : \overline{\sigma} \to K(\Pi, 1) \).

As a result, doing this construction simplex by simplex for all 2-simplex \( \sigma \in \mathcal{L}_2 \), we get an extension \( \overline{f}_i : \overline{L}_i \to K(\Pi, 1) \) over the modified complex:

\[
\overline{L}_i = \left( \mathcal{L} \setminus \bigcup_{\sigma \in \mathcal{L}_2 \setminus A_i} \text{Int} \, \sigma \right) \cup \left( \bigcup_{\sigma \in \mathcal{L}_2 \setminus A_i} \overline{\sigma} \right),
\]

so that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{f} & K(\Pi, 1) \\
p \searrow^{A_i} & \downarrow^{i} & \downarrow^{f_i} \\
A_i & \xrightarrow{\text{incl.}} & \overline{L}_i
\end{array}
\]

Finally, since by hypothesis, \( X \cap K(\Pi(M), 1) \) we can get up to homotopy, an extension of \( f \) over \( X \) via \( \overline{f}_i, F : X \to K(\Pi, 1) \). In summary, we have the following diagram (commutative up to homotopy):

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{f} & K(\Pi, 1) \\
p \searrow^{A_i} & \downarrow^{i} & \downarrow^{\overline{f}_i} \\
A_i & \xrightarrow{\text{incl.}} & \overline{L}_i
\end{array}
\]

**Theorem 5.12.** There exists a countable collection \( \{ S_i \}_{i \in \mathbb{N}} \) of 1-dimensional compacta \( S_i \) in \( B^4 \) such that every compactum \( Z \subset B^4 \setminus \bigcup_{i \in \mathbb{N}} S_i \) is Cainian.

**Proof.** By Theorem 4.8, there exists a 0-dimensional \( F_\sigma \)-set \( Z_1 \subset B^4 \) such that every compactum \( Y \subset B^4 \setminus Z_1 \) is nonabelian. Moreover, there exists a (standard) 1-dimensional No\" beling net \( Z_2 \subset B^4 \), i.e., \( N \) is an \( F_\sigma \)-set and every compactum \( Y \subset B^4 \setminus Z_2 \) is at most 2-dimensional (see, e.g., [22]). Then the union \( Z = Z_1 \cup Z_2 \) is a countable collection of
1-dimensional compacta and obviously, every compactum $Y \subset B^4 \setminus Z$ is 2-dimensional and nonabelian. Hence by Theorem 5.11, $Y$ is also Cainian. □

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