ELSEVIER

Topology and its Applications 74 (1996) 123–140

TOPOLOGY AND ITS APPLICATIONS

Cohomological dimension with respect to perfect groups

Alexander N. Dranishnikov [1], Dušan Repovš [*,2]

Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P.O. Box 2964, 1001 Ljubljana, Slovenia

Received 8 May 1995; revised 29 August 1995

Abstract

We introduce new classes of compact metric spaces: Cannon–Štan'ko, Cainian, and nonabelian compacta. In particular, we investigate compacta of cohomological dimension one with respect to certain classes of nonabelian groups, e.g., perfect groups. We also present a new method of constructing compacta with certain extension properties.

Keywords: Cohomological dimension; Cannon–Štan'ko compactum; Nonabelian compactum; Weakly Cainian compactum; Perfect group; Grope; Eilenberg–MacLane complex; Cell-like map

AMS classification: Primary 55M10; 57Q55, Secondary 54F45; 54C25; 57Q65

We dedicate this paper to the memory of Dr. W.J.R. Mitchell (1950–1994)

1. Introduction

Cohomology theory is defined for arbitrary *abelian* groups. Consequently, one can define cohomological dimension, $\text{c-dim}_G X$, of a compact metric space X for any abelian group G. The standard definition of $\text{c-dim}_G X$ is via the Eilenberg–MacLane complexes $K(G,n)$ (see, e.g., [6,15,25,26,28,33]). If we consider the class $\mathcal{G}$ of *nonabelian* groups G, then one still has well-defined Eilenberg–MacLane complexes $K(G,1)$ [23]. Therefore it makes sense to consider the class of compact metric spaces X with cohomological dimension one for an arbitrary *nonabelian* group $G \in \mathcal{G}$, $\text{c-dim}_G X = 1$. The purpose of our paper is to study the special case when the group $G \in \mathcal{G}$ is *perfect*, the subclass of $\mathcal{G}$ which is often studied by geometric topologists (see, e.g., [1–3]).

* Corresponding author. E-mail: dusan.repovs@uni-lj.si.

1 Current address: University of Florida, Gainesville, FL 32611-2082, USA.

2 Supported in part by the Ministry of Science and Technology of the Republic of Slovenia Research Grants No. P1-0214-101-92 and No. J1-7039/94.

SSDI 0166-8641(96)00050-8

Recall the *Kuratowski* notation $X\tau Y$: it means that for every closed subset X_0 of X and any map $f: X_0 \to Y$ there exists an extension $\overline{f}: X \to Y$ of f over all of X [24]. A compact metric space X is said to be a *Cainian* compactum if for every perfect group G, $X\tau K(G,1)$.

The class $\mathcal{K}$ of all Cainian compacta may play an important role in the celebrated 4-dimensional *Cell-like mapping* problem: *Can cell-like maps on a 4-manifold raise dimension*? (See [26,27].) Namely, we prove that every Cainian compactum is at most 2-dimensional (Corollary 5.6). Next, we show that in every topological 4-manifold M there exists a 1-dimensional Nöbeling net such that every compactum in its complement is necessarily Cainian (Theorem 5.12). So in order to prove that a cell-like image X of a topological 4-manifold M, $f: M \to X$ is at most 4-dimensional (note that X is necessarily a $\mathbb{Z}$-homology 4-manifold), it would suffice to find a similar Nöbeling net in X. Therefore our result might represent a possible first step towards an affirmative solution of the 4-dimensional cell-like mapping problem—the only dimension in which the answer is still unknown. (See [27] for a different approach to this problem.)

One of the most important concepts in geometric topology is the idea of the *grope* which was introduced in the 1970's by Štan'ko [31]. It was also used very effectively by Cannon [1] (whereas the name itself was suggested by D.R. McMillan, Jr.). The gropes were instrumental in the proofs of several key results in both taming theory as well as decomposition theory (cf. [1–3,31]).

It is easy to see that the fundamental group $\Pi = \Pi_1(M)$ of every grope M is *perfect*, i.e., $[\Pi : \Pi] = \Pi$ [1]. In our paper we consider the class of compact metric spaces X which are characterized by the property that $X\tau M^*$ for the so-called *minimal* grope M^* (equivalently, the cohomological dimension of X is at most one with respect to the fundamental group of the minimal grope M^*, $\text{c-dim}_{\Pi_1(M^*)} X \leqslant 1$). We call such spaces *Cannon–Štan'ko* compacta. One of the main results of the present paper is that there exist Cannon–Štan'ko compacta of arbitrary high dimensions (Theorem 3.3). This is quite in contrast with the class of Cainian compacta which cannot be more than 2-dimensional (Corollary 5.6). In fact, we prove that the weakly Cainian compacta are precisely the 2-dimensional Cannon–Štan'ko compacta (Corollary 5.10).

We also introduce and study the class of *nonabelian* compacta X: they are characterized by the property that for every closed subset $X_0 \subset X$ of X and for every map $f: X_0 \to \partial T$ there is an extension $\overline{f}: X \to T$ of X into T, where T is a torus with one hole (and $\partial T = S^1$ is its boundary). We prove that every nonabelian compactum is a Cannon–Štan'ko compactum (cf. Theorem 4.3). These compacta played an important role in the recent solution by the first author of the following problem in dimension theory proposed by Sternfeld [32] (in connection with the *Hilbert's* 13th problem): If a 2-dimensional compactum X is embedded in the product $Y \times Z$ of two 1-dimensional compacta Y and Z, does then X necessarily contain a product $Y_1 \times Z_1$ of two 1-dimensional compacta Y_1 and Z_1? (The answer is *negative* [13]. Another such counterexample was published earlier by Pol [29].)

The present paper has a rather long history. It was started during the visit by the first author to Ljubljana in the Spring of 1989. The preliminary announcement [16] was written during the second author's visit to Moscow in the Spring of 1991 (see also [17]). The main results of this paper were presented by the second author at the *Workshop on Cohomological Dimension Theory* (Knoxville, May 19–21, 1992), at the *Conference on Topology and Its Applications* (Amsterdam, August 15–18, 1994), at the *Conference on Set-Theoretic Topology and Its Applications* (Matsuyama, December 12–16, 1994), and at the *Spring Topology and Southeast Dynamics Conference* (Newark, March 30–April 2, 1995).

2. Constructing compacta with certain extension properties

We shall work in the category of separable metrizable spaces and continuous maps throughout this paper. A *compactum* is a compact metric space. A space X is said to have *cohomological* dimension at most n, $n \in \mathbb{N} \cup \{0\}$, with respect to a group of coefficients G, $\text{c-dim}_G X \leqslant n$ if for every closed subset $A \subset X$ and every map $f: A \to K(G,n)$ of A into the Eilenberg–MacLane complex $K(G,n)$ (see [23] for the definition and properties of Eilenberg–MacLane complexes), there is an extension of f over all of X, i.e., $X\tau K(G,n)$. Equivalently, $K(G,n) \in AE(X)$. (For several equivalent versions of definition of $\text{c-dim}_G X$ see [6,25,33], where its properties are studied in details.)

In [6–9,11,20,21], compacta with differing cohomological and covering dimensions were constructed, using the so-called *Edwards–Walsh* modification of polyhedra. Here we use an *alternative* approach, based on [10]. Whereas, the Edwards–Walsh modification exists only with the Eilenberg–MacLane space $K(R,n)$, where R is a ring with unit, our present approach is valid for an *arbitrary* group G (however, it is not so *geometric*).

A map $f: L \to K$ between polyhedra L and K with triangulations λ and κ, respectively, is said to be *combinatorial with respect to* λ *and* κ, provided $f^{-1}(A)$ is a subpolyhedron of λ for every simplex $A \subset K$. This means that the preimage of every polyhedron A with respect to κ is a polyhedron with respect to λ.

For every pair (X, A), every CW complex K, and every map $f: A \to K$, we shall denote the classical *extension* problem:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & K \\ X \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

by $(f_{X,A}, K)$ and we shall call the map $\overline{f}: X \to K$ a *solution* of the extension problem. (This should *not* be confused with the notion of a *resolution* of the extension problem $(f_{X,A}, K)$ introduced below.) Note that by the Borsuk homotopy extension theorem, for every pair $f, f': A \to K$ of homotopic maps, the extension problems $(f_{X,A}, K)$ and $(f'_{X,A}, K)$ are equivalent, i.e., there is an extension $\overline{f}: X \to K$ of f if and only if there is an extension $\overline{f'}: X \to K$ of f'. We shall denote the corresponding *equivalence* classes by $([f]_{X,A}, K)$. A map $g: Y \to X$ is said to be a *resolution* of the extension problem

$(f_{X,A}, K)$ if there is a map $h: Y \to K$ such that $h|_{g^{-1}(A)} = f \circ (g|_{g^{-1}(A)})$, i.e., the following diagram is commutative:

$$\begin{tikzcd}
g^{-1}(A) \arrow[r, "g|"] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook] & K \\
Y \arrow[r, "g"] \arrow[urr, dashed, "h"] & X &
\end{tikzcd}$$

An inverse system $\mathcal{L} = \{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ of polyhedra L_i with triangulations λ_i is said to be K-resolvable, for some CW complex K, provided that for every $i \geqslant 0$, for every finite subpolyhedron $A \subset L_i$ (with respect to the triangulation λ_i) and for every map $f: A \to K$, there exists $k \in \mathbb{N}$ such that the map $q_i^{i+k}: L_{i+k} \to L_i$ is a resolution of the extension problem $(f_{L_i,A}, K)$, i.e., the following diagram is commutative:

$$\begin{tikzcd}
(q_i^{i+k})^{-1}(A) \arrow[r, "q_i^{i+k}|"] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook] & K \\
L_{i+k} \arrow[r, "q_i^{i+k}"] \arrow[urr, dashed] & L_i &
\end{tikzcd}$$

Suppose that $\{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ is an inverse system of polyhedra L_i with triangulations λ_i. We say that the mesh of triangulations $\{\lambda_i\}$ *converges* to zero, $\operatorname{mesh}\{\lambda_i\} \to 0$, if for every $k \in \mathbb{N}$, $\lim_{i\to\infty}\{\operatorname{mesh}(q_k^{k+i}(\lambda_{k+i}))\} = 0$. (This should not be mistaken with the notion $\{\operatorname{mesh}\lambda_i\} \to 0$, which simply means that $\lim_{i\to\infty} \operatorname{mesh}\lambda_i = 0$.)

Lemma 2.1. *Suppose that K is a countable CW complex and that X is a compactum such that $X = \varprojlim\{(L_i, \lambda_i), q_i^{i+1}\}$, where $\mathcal{L} = \{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ is a K-resolvable inverse system of compact polyhedra L_i with triangulations λ_i such that* $\operatorname{mesh}\{\lambda_i\} \to 0$. *Then $X \tau K$.*

Proof. Suppose that for some closed subset $A \subset X$ we have a map $f: A \to K$. We may assume without losing generality, that $X \subset I^\infty$. Since K is an ANR there is an open neighborhood $U \subset I^\infty$ of A in I^∞ and a map $f': U \to K$ such that $f' \mid A = f$. Since $X = \varprojlim\{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ there exists for every integer $i \geqslant 0$, a finite subpolyhedron $A_i \subset L_i$ of L_i with respect to the triangulation λ_i, such that $A = \varprojlim\{(A_i, \lambda_i|_{A_i}), q_i^{i+1}|_{A_{i+1}}\}_{i \geqslant 0}$. Therefore for some integer $i_0 \geqslant 0$, $A_{i_0} \subset U$. Let $f^* = f' \mid A_{i_0}$. Since the inverse system $\{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ is K-resolvable for K, there exists an integer $k \geqslant 0$ such that the map $q_{i_0}^{i_0+k}: A_{i_0+k} \to A_{i_0}$ is the resolution of the extension problem $(f^*_{L_{i_0},A_{i_0}}, K)$. So there is a map $\overline{f^*}: L_{i_0+k} \to K$ such that

$$\overline{f^*} \mid (q_{i_0}^{i_0+k})^{-1}(A_{i_0}) = f^* \circ \left(q_{i_0}^{i_0+k}|_{(q_{i_0}^{i_0+k})^{-1}(A_{i_0})}\right).$$

Since $\operatorname{mesh}\{\lambda_i\} \to 0$, we can choose i_0 big enough so that the composition $\overline{f} = \overline{f^*} \circ (q^{\infty}_{i_0+k}) : X \to K$ solves the extension problem $(f_{X,A}, K)$.

$$\begin{tikzcd}
L_0 & L_1 \arrow[l, "q_0^1"'] & \cdots \arrow[l, "q_1^2"'] & L_{i_0} \arrow[l, "q_{i_0-1}^{i_0}"'] & L_{i_0+k} \arrow[l, "q_{i_0}^{i_0+k}"'] \arrow[dr, dashed, "\overline{f^*}"] & & X \arrow[ll, "q_{i_0+k}^{\infty}"'] \arrow[dl, dashed, "\overline{f}"'] \\
& & & & & K & \\
A_0 \arrow[uu, hook] & A_1 \arrow[l, "q_0^1|A_1"] \arrow[uu, hook] & \cdots \arrow[l, "q_1^2|A_2"] & A_{i_0} \arrow[l, "q_{i_0-1}^{i_0}|A_{i_0}"] \arrow[uu, hook] & A_{i_0+k} \arrow[l, "q_{i_0}^{i_0+k}|A_{i_0+k}"] \arrow[uu, hook] \arrow[ur, dashed, "f^*"] & & A \arrow[ll, "q_{i_0+k}^{\infty}|A"] \arrow[ul, "f"'] \arrow[uu, hook, "\text{incl.}"']
\end{tikzcd}$$

□

Lemma 2.2. *Suppose that K is a CW complex and that X is a compactum. Then for every extension problem $(f_{X,A}, K)$ there exists a resolution $g : Y \to X$ such that for every point $x \in X$, $g^{-1}(x)$ is either contractible or homotopy equivalent to K. Moreover, if X and K are simplicial complexes then Y can be chosen to be a simplicial complex with simplicial map g for some triangulations of X and Y.*

Proof. Given a map $f : A \to K$ consider its extension (which exists since $K \in \mathrm{ANR}$ hence $C(K) \in \mathrm{AR}$) to the cone $C(K)$ over K (i.e., $C(K) = K \times [0,1]/K \times \{1\}$), $\overline{f} : X \to C(K)$ and let $\pi : K \times [0,1] \to C(K)$ be the natural quotient map. Define Y to be the pull-back of the diagram below

$$\begin{tikzcd}
Y \arrow[r, dashed, "h"] \arrow[d, dashed, "g"'] & K \times [0,1] \arrow[d, "\pi"] \\
X \arrow[r, "\overline{f}"] & C(K)
\end{tikzcd}$$

and let $g : Y \to X$ be the projection corresponding to π. Let $\omega : K \times [0,1] \to K$ be the obvious projection. Then

$$\omega \circ (h\,|_{g^{-1}(A)}) = f \circ (g\,|_{g^{-1}(A)})$$

hence g is indeed a resolution of the extension problem. Also, for every $y \in C(K)$, obviously $\pi^{-1}(y) = *$ or K, so for point-inverses of g we have that $g^{-1}(x) \simeq *$ or $\simeq K$.

$$\begin{tikzcd}
g^{-1}(A) \arrow[r, "g|"] \arrow[dd, hook] & A \arrow[r, "f"] \arrow[dd, hook] & K \\
& & K \times [0,1] \arrow[u, "\omega"'] \arrow[d, "\pi"] \\
Y \arrow[r, "g"] \arrow[uurr, dashed, "h"] & X \arrow[r, "\overline{f}"] & C(K)
\end{tikzcd}$$

If X is a polyhedron, then we can replace the extension problem $(f_{X,A}, K)$ by a simplicial one $(f'_{X,A'}, K)$ so that $A \subset A'$ and any resolution of a new problem is also a resolution of the old one. Then the extension $\overline{f}$ can be chosen to be simplicial for some triangulations. In this case the pull-back space Y is a simplicial complex. □

Lemma 2.3. *Suppose that K is a countable CW complex. Then for every $n \in \mathbb{N}$, there exists an inverse sequence $\mathcal{L} = \{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ of countable polyhedra L_i with triangulation λ_i, such that:*

(i) $L_0 \simeq S^n$;

(ii) $\mathrm{mesh}\{\lambda_i\} \to 0$;

(iii) *$\mathcal{L}$ is K-resolvable; and*

(iv) *for every $i \geqslant 0$ and every point $y \in L_i$, $(q_i^{i+1})^{-1}(y)$ is either contractible or homotopy equivalent to K.*

Proof. The proof consists of an inductive construction. Define λ_0 to be any triangulation of S^n with mesh $\lambda_0 < 1$. There exist only countably many different extension problems $\alpha^0 = \{(f_{L_0,A}, K)\}$ with finite subpolyhedron A. (Indeed, K^X is a separable metric space hence there exists a countable dense subset $\{f_i\}_{i \in \mathbb{N}} \subset K^X$. Since K is an ANR it follows that every $f \in K^X$ is homotopic to some f_i.) We shall enumerate them by even integers: α_2^0, α_4^0, etc. Let $\alpha_i = \alpha_0^i$, for every $i \in 2\mathbb{Z}$.

Apply Lemma 2.2 to get a map $q_0^1 : L_1 \to L_0$ with the property (iv) and such that q_0^1 resolves the extension problem $\alpha_2 = \alpha_2^0$, i.e.,

$$\begin{tikzcd}
(q_0^1)^{-1}(A) \arrow[r, "q_0^1|"] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook] & K \\
L_1 \arrow[r, "q_0^1"'] \arrow[urr, dashed] & L_0 &
\end{tikzcd}$$

Choose any triangulation λ_1 of L_1 such that mesh $\lambda_1 < 1/2$ and mesh $q_0^1(\lambda_1) < 1/2$. Again, there are only countably many different extension problems $\alpha^1 = \{(f_{L_1,A}, K)\}$. We enumerate them by odd numbers, divisible by 3: $\alpha_3^1, \alpha_9^1, \alpha_{15}^1, \alpha_{21}^1$, etc. Define $\alpha_i = \alpha_i^k$, $k \in \{0,1\}$, $i \in 2\mathbb{Z} \cup 3\mathbb{Z}$. Apply Lemma 2.2 to get a map $q_1^2 : L_2 \to L_1$ with the property (iv) and such that q_1^2 resolves the extension problem $\alpha_3 = \alpha_3^1$, i.e.,

$$\begin{tikzcd}
(q_1^2)^{-1}(A) \arrow[r, "q_1^2|"] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook] & K \\
L_2 \arrow[r, "q_1^2"'] \arrow[urr, dashed] & L_1 &
\end{tikzcd}$$

Choose any triangulation λ_2 of L_2 such that mesh $\lambda_2 < 1/4$, mesh $q_1^2(\lambda_2) < 1/4$, and mesh $q_0^2(\lambda_2) < 1/4$. Again, there are only countably many different extension problems $\alpha^2 = \{(f_{L_2,A}, K)\}$. We enumerate them by integers, divisible by 5 but not divisible by 2 or 3: $\alpha_5^2, \alpha_{25}^2, \alpha_{35}^2, \alpha_{55}^2$, etc. Define $\alpha_i = \alpha_i^k$, $k \in \{0,1,2\}$, $i \in 2\mathbb{Z} \cup 3\mathbb{Z} \cup 5\mathbb{Z}$.

Consider $\alpha_4 = \alpha_4^0 = (f_{L_0,A}, K)$. It defines an extension problem

$$(q_0^2)^{-1}(\alpha_4) = \left(f \circ (q_0^2|)_{L_2,(q_0^2)^{-1}(A)}, K\right).$$

Apply Lemma 2.2 to resolve the extension problem $(q_0^2)^{-1}(\alpha_4)$ by the map $q_2^3 : L_3 \to L_2$:

$$\begin{tikzcd}
(q_2^3)^{-1}((q_0^2)^{-1}(A)) \arrow[r, "q_2^3|"] \arrow[d, hook] & (q_0^2)^{-1}(A) \arrow[r, "q_0^2|"] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook] & K \\
L_3 \arrow[r, "q_2^3"] \arrow[rrru, dashed] \arrow[rr, bend right, "q_0^3"'] & L_2 \arrow[r, "q_0^2"] & L_0 &
\end{tikzcd}$$

Note that then the map q_0^3 resolves the original extension problem α_4. We can proceed inductively, to construct a system $\{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$. All equivalence classes of extension problems $([f]_{L_i,A}, K)$ will be enumerated by $\mathbb{N}\backslash\{1\}$: $\alpha_2, \alpha_3, \alpha_4$, etc.

Clearly, the property (ii) holds by the construction. Next, for every $i \in \mathbb{N}\backslash\{1\}$, every extension problem α_i is resolved by the map $q_k^{i-1} : L_{i-1} \to L_k$, where $\alpha_i = \alpha_i^k$, hence the property (iii) holds. Finally, property (iv) holds by Lemma 2.2 and the construction. $\square$

A generalized cohomology theory h^* is said to be *continuous* if for every countable CW complex W with a compact stratification $W_1 \subset W_2 \subset \cdots \subset W_i \subset \cdots$ there is an equality $h^*(W) = \varprojlim h^*(W_i)$.

Theorem 2.4. *Let K be a countable CW complex and h be a nontrivial homology theory (respectively, nontrivial continuous cohomology theory) with $\widetilde{h}_*(K) = 0$ (respectively, $\widetilde{h}^*(K) = 0$). Then for every integer $n \geqslant 1$, there is a compactum X of dimension $\dim X \geqslant n$ with the property $X\tau K$.*

Proof. We'll give a proof for cohomology (for homology it is analogous). Apply Lemma 2.3 to obtain an inverse system $\mathcal{L} = \{(L_i, \lambda_i), q_i^{i+1}\}_{i \geqslant 0}$ of polyhedra L_i with triangulations λ_i, such that (i) $L_0 \simeq S^n$; (ii) $\operatorname{mesh}\{\lambda_i\} \to 0$; (iii) $\mathcal{L}$ is K-resolvable; and (iv) For every $y \in L_i$ and every i, $(q_i^{i+1})^{-1}(y) \simeq$ point or $\simeq K$.

We shall construct inductively a K-resolvable inverse system

$$\mathcal{Q} = \{(Q_i, \lambda_i|_{Q_i}), q_i^{i+1}|_{Q_i}\}_{i \geqslant 0}$$

of compact subpolyhedra Q_i of L_i with respect to triangulations λ_i. Define $Q_0 = L_0$ and consider any $\gamma_0 = \gamma \neq 0 \in h^*(L_0)$. It follows by the property (iv) above that $(q_0^1)^* : h^*(L_0) \to h^*(L_1)$ is an isomorphism [18]. Since by hypothesis, h^* is continuous, the nonzero element $\gamma_1 = (q_0^1)^*(\gamma_0)$ lives in some compact subpolyhedron Q_1 of L_1 with respect to the triangulation λ_1. Repeat this argument for γ_1 to get a compact subpolyhedron Q_2 of L_2 with respect to the triangulation λ_2 with $(q_1^2)^*(\gamma_1) \neq 0$ living in Q_2, etc. Let $X = \varprojlim\{(Q_i, \lambda_i|_{Q_i}), q_i^{i+1}|_{Q_i}\}_{i \geqslant 0}$. Clearly, $(q_1^\infty)^*(\gamma_0) \neq 0$ therefore $q_1^\infty : X \to L_0 \simeq S^n$ is essential so $\dim X \geqslant n$. Also, by Lemma 2.1, $X\tau K$. $\square$

Proposition 2.5. *For every compacta X, Y and Z the following implication holds: $X\tau(Y \vee Z) \Rightarrow X\tau Y$.*

Proof. Straightforward (cf., e.g., [12]). □

By using Proposition 2.5 it is possible to generalize Theorem 2.4 in a straightforward manner to the following:

Theorem 2.6. *Let $\{K_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of countable CW complexes, acyclic with respect to a nontrivial homology (respectively, continuous cohomology) theory h. Then for every integer $n \geqslant 1$, there exists a compactum X of dimension $\dim X \geqslant n$, having the property $X\tau K_i$ for every $i \in \mathbb{N}$.*

3. Cannon–Štan'ko compacta

We begin by briefly recalling the construction of a *grope* M. (For more about gropes see [1].) One defines M as the direct limit $M = \varinjlim\{L_i, p^i_{i+1}\}_{i\geqslant 0}$ of a direct system of compact 2-dimensional polyhedra L_i and the injective bonding maps $p^i_{i+1} : L_i \to L_{i+1}$. The polyhedron L_n is called the nth *stage* of the grope construction. Here, L_0 is an oriented compact surface S_g of genus $g > 0$ with an open disk deleted. Let $A_0 \subset S_g$ be a set of $2g$ circles which generate the 1-dimensional homology of the surface S_g. The complex L_{n+1} is then obtained from L_n for every $n \geqslant 0$, by attaching for every circle $a \in A_n$, an oriented compact surface S_{g_a} of genus g_a, with an open disk deleted, by identifying the boundary ∂S_{g_a} of the surface S_{g_a} with the circle $a \in A_n$. The generators of homology of S_{g_a} then determine the set of $2g_a$ circles $A_{n+1} \subset S_{g_a}$ which generate the 1-dimensional homology of the surface S_{g_a}.

In particular, we shall need the so-called *minimal* grope $M^* = \varinjlim\{L^*_i, p^i_{i+1}\}_{i\geqslant 0}$ which is distinguished by the fact that the genus of L_0 is one and that for every $i \geqslant 0$, we attach only two 1-handles to each 1-handle pair of generators of the 1-dimensional homology of the complex L^*_i.

Definition 3.1. A compactum X is said to be a *Cannon–Štan'ko* compactum provided that for the minimal grope M^*, $X\tau M^*$. Equivalently, for the minimal grope M^*, $X\tau K(\Pi_1(M^*), 1)$, i.e., $\text{c-dim}_{\Pi_1(M^*)} X \leqslant 1$.

Examples 3.2. Every compactum of dimension $\leqslant 1$ is evidently also a Cannon–Štan'ko compactum. The *Pontryagin* disk $\mathbb{D}^2$, introduced in [27], is an example of a 2-dimensional Cannon–Štan'ko compactum. By glueing two copies of $\mathbb{D}^2$ along the boundary $\partial\mathbb{D}^2 = S^1$ one gets the so-called *Riemannian surface of infinite local genus* $\mathbb{S}^2$ which is an example of a homogeneous 2-dimensional Cannon–Štan'ko compactum [13].

We shall now prove that there actually exist examples of Cannon–Štan'ko compacta of arbitrary high dimensions:

Theorem 3.3. *For every integer $n \geqslant 1$, there exists an n-dimensional Cannon–Štan'ko compactum.*

Proof. By Theorem 2.4 it suffices to find a nontrivial homology theory h_* such that $\widetilde{h_*}(M^*) = 0$. For h_* we take the singular homology theory with integer coefficients, $h_* = H_*(\_\_;\mathbb{Z})$. Clearly, for the minimal grope $M^* = \varinjlim\{L_i^*, p_{i+1}^i\}_{i\geqslant 0}$, all bonding maps $p_{i+1}^i : L_i^* \to L_{i+1}^*$ induce the zero-homomorphisms $(p_{i+1}^i)_* : H_*(L_i^*;\mathbb{Z}) \to H_*(L_{i+1}^*;\mathbb{Z})$ since:

(i) For every 1-cycle $\gamma \in H_1(L_i^*;\mathbb{Z})$, $(p_{i+1}^i)_*(\gamma)$ bounds a 2-chain in $H_*(L_{i+1}^*;\mathbb{Z})$, hence $H_1(M^*;\mathbb{Z}) = \varinjlim\{H_1(L_i^*;\mathbb{Z}), (p_{i+1}^i)_*\}_{i\geqslant 0} = 0$; and

(ii) For every $j \geqslant 2$, $H_j(L_i^*;\mathbb{Z}) = 0$ since L_i^* is homotopically just a bouquet of finitely many circles S^1, thus it follows again that $H_j(M^*;\mathbb{Z}) = 0$.

Finally, since h_* is an ordinary homology theory, we may assume that in the proof of Theorem 2.4, in the construction of the inverse system $\mathcal{Q} = \{(Q_i, \lambda_i|_{Q_i}), q_i^{i+1}\}_{i\geqslant 0}$, the subpolyhedra Q_i of L_i^* are all n-dimensional. We can achieve this by starting with an n-dimensional cycle $\gamma \in h_*(L_0^*)$ and then the n-dimensional homology will always live already in the n-skeleton $Q_i^{(n)}$ of the polyhedron Q_i. □

If instead of Theorem 2.4 one uses Theorem 2.6, the same proof yields the following stronger statement:

Theorem 3.4. *Let $\mathcal{M} = \{M_k\}_{k\in\mathbb{N}}$ be an arbitrary (at most countable) collection of gropes. Then for every integer $n \geqslant 1$, there exists an n-dimensional compactum X such that for every index $k \in \mathbb{N}$, $X\tau M_k$.*

4. Nonabelian compacta

Definition 4.1. Let $T = (S^1 \times S^1)\backslash \operatorname{Int} B$ be a 2-torus with a hole (obtained by removing the disk B) and denote its boundary by ∂T (hence $\partial T = S^1$). A compactum X is said to be *nonabelian* if for every closed subset $A \subset X$ of X and every continuous map $f : A \to \partial T$ there exists a continuous map $\tilde{f} : X \to T$ such that $\tilde{f}|_A = f$. We shall denote this extension property by $X\tau(T, \partial T)$.

Examples 4.2. Every compactum of dimension $\leqslant 1$ is evidently also nonabelian. An example of a 2-dimensional nonabelian compactum is the classical Pontryagin mod 2 "surface" [30], i.e., the inverse limit of an inverse system of modifications of the 2-sphere where disks are replaced by Möbius bands (observe that $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$ is homeomorphic to $\mathbb{R}P^2\#$ (1-handle)) [13].

Theorem 4.3. *For every integer $n \geqslant 1$, every n-dimensional nonabelian compactum is also a Cannon–Štan'ko compactum.*

We shall first need to prove two lemmas. As before, let S_g be the 2-sphere with $g > 0$ orientable 1-handles and one hole, and let $\mathbb{D}^2$ be the Pontryagin disk [27] (recall that $\partial\mathbb{D}^2 = S^1$).

Lemma 4.4. *For every compactum X, the following statements are equivalent:*

(1) *X is nonabelian;*

(2) *for every $g > 0$, $X\tau(S_g, \partial S_g)$, where S_g is a compact orientable surface of genus g and with one hole; and*

(3) *$X\tau(\mathbb{D}^2, \partial\mathbb{D}^2)$.*

Proof. (1) $\Rightarrow$ (2) Take in S_g a bouquet $A = \bigvee_{i=1}^{g-1} A_i$ of $g-1$ arcs such that for every i:

(i) $\partial A_i = \{a, a_i\}$;

(ii) $A_i \cap A_j = \{a\}$, for every $i \neq j$;

(iii) $\partial A_i \cap S_g = \partial A_i \cap \partial S_g = \partial A_i$; and

(iv) if one cuts S_g along A then each of the resulting pieces $E_1, \ldots, E_g$ is a disk with just one of the g 1-handles H_i.

By shrinking the bouquet A to the point a one gets a cell-like map $f: S_g \to Y$ of S_g onto a bouquet $Y = \bigvee_{i=1}^{g} D_i$, where for every i, $D_i = f(E_i)$ is a 2-torus with a hole. (Note that $f(\partial E_i) \subset \partial D_i$, for every i. For the definition and the main properties of *cell-like* maps see, e.g., the survey [26].) By collapsing every D_i onto an interval $I_i \subset D_i$ one furthermore gets a surjective map $g: Y \to Z$ of Y onto a bouquet $Z = \bigvee_{i=1}^{g} I_i$.

Take now any nonabelian compactum X, an arbitrary closed subset $A \subset X$ and any continuous map $\varphi: A \to \partial S_g$. Then the compositum $\psi = g \circ f \circ \varphi: A \to Z$ gives a continuous map of A into Z. Since Z is obviously an absolute retract (AR), the map ψ can be extended to a continuous map on X, $\overline{\psi}: X \to Z$.

For every $k \in \{1, \ldots, g\}$, consider the restriction

$$\xi_k = f \circ \varphi|_{\overline{\psi}^{-1}(I_k) \cap (A \cup \overline{\psi}^{-1}(a))} : \overline{\psi}^{-1}(I_k) \cap \left(A \cup \overline{\psi}^{-1}(a)\right) \to \partial D_k.$$

Note that $\overline{\psi}^{-1}(a)$ separates X into components the closures of which are $\psi^{-1}(I_k)$, $k \in \{1, \ldots, g\}$. Since by hypothesis, X is nonabelian, so is $\overline{\psi}^{-1}(I_k)$ hence there is an extension over all of $\overline{\psi}^{-1}(I_k)$, $\overline{\xi}_k: \overline{\psi}^{-1}(I_k) \to D_k$. Let $\xi = \bigcup_{k=1}^{g} \overline{\xi}_k: X \to Y$. Clearly, ξ is well-defined and continuous. Since $f: S_g \to Y$ is cell-like there is a lifting $\overline{\xi}: X \to S_g$ which is up to homotopy, an extension of $\varphi: A \to \partial S_g$ over X, i.e., $\overline{\xi}|_A \simeq \varphi$.

(2) $\Rightarrow$ (1) This is obvious—consider the case when $g = 1$.

(3) $\Rightarrow$ (1) Suppose that $f: A \to \partial T$ is a continuous map from a closed subset $A \subset X$ of a compactum X with the property $X\tau(\mathbb{D}^2, \partial\mathbb{D}^2)$. Recall the inverse limit construction of the Pontryagin disk $\mathbb{D}^2$ from [27]: $\mathbb{D} = \varprojlim\{K_i, p_i^{i+1}\}_{i \geqslant 0}$ where K_0 is a 2-disk and K_{i+1} is a Pontryagin-like modification of K_i ($i \geqslant 0$) except that instead of Möbius bands we attach orientable 1-handles. In particular, for every $i \geqslant 1$, K_i is a compact orientable surface of some finite genus $g_i > 0$ and $g_{i+1} > g_i$. Also, we can identify $\partial K_i \equiv S^1 \equiv \partial\mathbb{D}^2$ for every i. Therefore there is a continuous map $\varphi: (\mathbb{D}^2, \partial\mathbb{D}^2) \to (T, \partial T)$ obtained as the compositum of the canonical projection $p_1^\infty: (\mathbb{D}^2, \partial\mathbb{D}^2) \to (K_1, \partial K_1)$, followed by the obvious onto map $\beta: (K_1, \partial K_1) \to (T, \partial T)$ which simply kills the extra handles.

By hypothesis, the map $(\varphi|_{\partial\mathbb{D}^2})^{-1} \circ f : A \to \partial\mathbb{D}^2$ extends over all of X to a map $g : X \to \mathbb{D}^2$. Obviously, the composition $\overline{f} = \varphi \circ g : X \to T$ is then the desired extension of f over all of X. This verifies that X is indeed nonabelian.

(1) $\Rightarrow$ (3) Represent again $\mathbb{D}^2$ as the inverse limit, $\mathbb{D}^2 = \varprojlim\{K_i, p_i^{i+1}\}_{i\geqslant 0}$ of 2-disks-with-handles K_i, i.e., $K_i = S_{g_i}$ for some $g_i \geqslant 0$. Suppose that X is a nonabelian compactum and pick any closed subset $A \subset X$ and any continuous map $\varphi : A \to \partial\mathbb{D}^2$. Since we've already verified that the assertions (1) and (2) of the lemma are equivalent and since one can identify $\partial K_i \equiv \partial\mathbb{D}^2$, it follows that there is an extension of φ to a map $\varphi_i : X \to K_i$ for every $i \geqslant 0$.

Fix an $i_0 \geqslant 0$ and consider any 2-simplex $\sigma \in K_{i_0}^{(2)}$. Look at the pair $(\varphi_{i_0}^{-1}(\sigma), \psi_{i_0}^{-1}(\partial\sigma))$. Then one can lift the map

$$\varphi_{i_0}|_{\varphi_{i_0}^{-1}(\sigma)} : \varphi_{i_0}^{-1}(\sigma) \to K_{i_0}$$

to $(p_{i_0}^{i_0+1})^{-1}(\sigma) \subset K_{i_0+1}$ (since X is nonabelian and since $(p_{i_0}^{i_0+1})^{-1}(\sigma)$ is a disk with one 1-handle), so by doing this operation for every $\sigma \in K_{i_0}^{(2)}$, we get a lifting of the map $\varphi_{i_0} : X \to K_{i_0}$ to K_{i_0+1}, i.e., a map $\psi_{i_0+1} : X \to K_{i_0+1}$.

$$\begin{tikzcd}
K_{i_0} & K_{i_0+1} \arrow[l] & \mathbb{D}^2 \arrow[l] \\
 & X \arrow[ul, "\varphi_{i_0}"] \arrow[u, dashed, "\psi_{i_0+1}"] \arrow[ur, dashed, "\psi"'] & \\
 & A \arrow[u, "\text{incl.}"] \arrow[uul, "\varphi_{i_0}|A"] \arrow[uur, dashed, "\varphi"'] &
\end{tikzcd}$$

Provided that $i_0 \geqslant 0$ is chosen big enough, we may assume that $\varphi_{i_0} \simeq p_{i_0}^{i_0+1} \circ \psi_{i_0+1}$ via some $(\frac{1}{2})$-homotopy. Therefore, a map $\psi = \lim_{n\to\infty} \psi_{i_0+n} : X \to \mathbb{D}^2$, where for every $k \geqslant 0$, $\psi_{i_0+k+1} : X \to K_{i_0+k+1}$ is a lifting of the map $\psi_{i_0+k} : X \to K_{i_0+k}$ such that

$$\psi_{i_0+k} \simeq p_{i_0+k}^{i_0+k+1} \circ \psi_{i_0+k+1}$$

via some (2^{-k})-homotopy, is well-defined and continuous, and it is, up to a homotopy, an extension of the given map $\varphi : A \to \mathbb{D}^2$. (If necessary, one can choose an appropriate subsequence of the sets of indices $\{i_0, i_0 + 1, i_0 + 2, \ldots\}$.) This verifies that indeed $X\tau(\mathbb{D}^2, \partial\mathbb{D}^2)$.

This completes the proof of the lemma. $\square$

Lemma 4.5. *Let M be any grope. Then for an arbitrary compactum X, the following statements are equivalent:*

(1) $X\tau M$*; and*

(2) $X\tau(M, \partial M)$.

Proof.

(1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let X be any compactum and take any closed subset $A \subset X$ and any continuous map $f : A \to M$. Represent M as the direct limit of finite stages of the grope construction, $M = \varinjlim\{L_i, p^i_{i+1}\}_{i\geqslant 0}$. Since A is compact, its image $\overline{\varphi}(A)$ lies in some finite stage of M, i.e., for some $i \geqslant 1$, $\overline{\varphi}(A) \subset L_i$, where $\overline{\varphi} = p^i_\infty \circ \varphi$ and $p^i_\infty : L_i \to M$ is the canonical inclusion.

Now, L_i is of the homotopy type of a finite bouquet of circles, i.e., $L_i \simeq Y = \bigvee_{j=1}^{k(i)} S^1_j$, and moreover, $Y \subset Z = \bigvee_{j=1}^{k(i)} N_j$, where N_j are gropes and $\partial N_j = S_j$, for every $j \in \{1, \ldots, k(i)\}$. We have a deformation retraction $r : L_i \to Y$ which we can follow by the map $g : Z \to W = \bigvee_{j=1}^{n(i)} I_j$ which collapses each grope N_j onto the segment I_j. The composition $\psi = g \circ r \circ \overline{\varphi} : A \to W$ can now be extended over X to give $\overline{\psi} : X \to W$. This map approximately lifts to $\overline{\overline{\psi}} : X \to Z$ since g is cell-like. Since $Z \simeq M$ and since it suffices to solve our extension problem only up to homotopy, the proof is completed. $\square$

$$\begin{tikzcd}
A \arrow[rr, "\overline{\varphi}"] \arrow[d, "\text{incl.}"'] \arrow[dr, "\varphi"'] & & L_i \arrow[dd, "r"] \\
X \arrow[r, dashed] \arrow[d, "\overline{\psi}"'] \arrow[dr, "\overline{\overline{\psi}}"'] & M \arrow[ur, "p^i_\infty"'] & \\
W & Z \arrow[l, "g"'] \arrow[u, dashed] & Y \arrow[l, "\text{incl.}"']
\end{tikzcd}$$

Proof of Theorem 4.3. If X is nonabelian then $X\tau(S_1, \partial S_1)$, so by Lemma 4.4, $X\tau(S_g, \partial S_g)$, for every $g > 0$, hence $X\tau(M, \partial M)$, for every grope M (in particular for the minimal grope M^*) since obviously $(S_1, \partial S_1) \subset (M, \partial M)$, thus by Lemma 4.4, $X\tau M$. $\square$

Obviously, the proof of Theorem 4.3 yields the following stronger statement:

Theorem 4.6. *For every integer $n \geqslant 1$, every n-dimensional nonabelian compactum X has the property $X\tau M$, for every grope M.*

Proposition 4.7. *Let X be a compactum, K a polyhedron and (L, L') a polyhedral pair. Suppose that the property $X\tau(K * L, K * L')$ holds, where $*$ denotes the join of polyhedra. Then there is an F_σ-set $Z \subset X$ (respectively G_δ-set $W \subset X$) such that $Z\tau K$ (respectively $W\tau K$) and $(X\backslash Z)\tau(L, L')$ (respectively $(X\backslash W)\tau(L, L')$).*

Proof. This proposition is a relative version of Corollary 2 from [14] and the proof is the same (see the proofs of Theorem 1 and Corollary 2 in [14]). $\square$

Theorem 4.8. *Let X be an arbitrary compactum. Then there exists a 0-dimensional F_σ-set $Z \subset X$ (respectively G_δ-set $W \subset X$) such that every compactum $Y \subset X\backslash Z$ (respectively $Y \subset X\backslash W$) is nonabelian.*

Proof. Let X be an arbitrary compactum and define $K = S^0$, where S^0 is the 0-dimensional sphere, and let $(L, L') = (T, \partial T)$. Then $(K * L, K * L') = (\Sigma T, \Sigma(\partial T))$ and the inclusion-induced homomorphism $\Pi_*(\Sigma(\partial T)) \to \Pi_*(\Sigma T)$ is trivial, hence the property $X\tau(K * L, K * L')$ holds. Apply now Proposition 4.7 to obtain an F_σ-set $Z \subset X$ (respectively G_δ-set $W \subset X$) such that $Z\tau K$ (respectively $W\tau K$) and $(X\backslash Z)\tau(L, L')$ (respectively $(X\backslash W)\tau(L, L')$). In particular, $Z\tau S^0$ (respectively $W\tau S^0$) implies that $\dim Z = 0$ (respectively $\dim W = 0$), whereas the other property of Z (respectively W) clearly implies that every compactum $Y \subset X\backslash Z$ (respectively $Y \subset X\backslash W$) is nonabelian. □

Corollary 4.9. *For every integer $n \geqslant 0$, there exists an n-dimensional nonabelian compactum.*

Proof. Given any integer $n \geqslant 0$, consider the following compactum $X = I^{n+1}$, i.e., the $(n+1)$-dimensional cube. By Theorem 4.8, there exists a G_δ-set $W \subset X$ such that $\dim W = 0$ and every compactum $Y \subset X\backslash W$ is nonabelian. By the Urysohn–Menger sum formula, $\dim X\backslash W \geqslant n$ hence if we write the F_σ-set $X\backslash W$ as the countable union of compacta $C_i \subset X\backslash W$, $X\backslash W = \bigcup\{C_i \mid i \in \mathbb{N}\}$ then by the Countable sum theorem there is at least one $i_0 \in \mathbb{N}$ such that $\dim C_{i_0} = n$. Since C_{i_0} is also nonabelian this proves the assertion. □

5. Cainian compacta

Definition 5.1. A compactum X is said to be *Cainian* provided that for every perfect group Π, $X\tau K(\Pi, 1)$. Equivalently, c-$\dim_\Pi X \leqslant 1$.

Examples 5.2. Every compactum of dimension $\leqslant 1$ is Cainian. The Pontryagin disk $\mathbb{D}^2$ and the Riemannian surface of infinite local genus $\mathbb{S}^2$ are examples of 2-dimensional Cainian compacta.

Let us verify this for $\mathbb{D}^2$. So let Π be a perfect group and choose any closed subset $A \subset \mathbb{D}^2$ and any continuous map $f: A \to K(\Pi, 1)$. Represent $(\mathbb{D}^2, A)$ as the inverse limit of polyhedral pairs (assume all lie in I^∞):

$$(\mathbb{D}^2, A) = \varprojlim \{(D_i, A_i), (p_i^{i+1}, p_i^{i+1}|_{A_{i+1}})\}_{i \geqslant 0}.$$

Since $K(\Pi, 1)$ is an ANR, there is an integer $i_0 \in \mathbb{N}$ such that f can be extended over A_{i_0} to give a map $\overline{f}_{i_0}: A_i \to K(\Pi, 1)$ such that $\overline{f}_{i_0} \circ (p_{i_0}^\infty|_A) \simeq f$. Clearly, $\overline{f}_{i_0}$ can be extended over the 1-skeleton $D_{i_0}^{(1)}$ of D_{i_0} to give a map $g_{i_0}: A_{i_0} \cup D_{i_0}^{(1)} \to K(\Pi, 1)$. This gives an extension up to homotopy, $\overline{g}: A \cup \mathbb{D}^{(1)} \to K(\Pi, 1)$.

Pick now any 2-simplex $\sigma_j \in D_{i_0}^{(2)} \backslash A_{i_0}^{(2)}$ and consider $\gamma_{\sigma_j} = g_{i_0}(\partial\sigma_j)$ as the element of $\Pi_1(K(\Pi, 1)) = \Pi$. Then γ_{σ_j} is a product of commutators, $\gamma_{\sigma_j} = c_1 \cdots c_{t(j)}$ since Π is a perfect group.

Consider $(p_{i_0}^\infty)^{-1}(\sigma_j) \subset \mathbb{D}^2$. We have a well-defined extension of $f: A \to K(\Pi, 1)$ over $A \cup (p_{i_0}^\infty)^{-1}(\partial\sigma_j)$. We wish to extend over $(p_{i_0}^\infty)^{-1}(\sigma_j)$. We procede as follows: Let

$i_1 \geqslant i_0$ be the smallest index such that $(p_{i_0}^{i_1})^{-1}(\sigma_j)$ is a disk with $\geqslant t_j$ handles. Clearly, then there is a map

$$g_{i_1} : (p_{i_0}^{i_1})^{-1}(\sigma_j) \to K(\Pi, 1)$$

because all c_i's can be represented by (orientable) 1-handles (and we can kill extra 1-handles if necessary). We now do this for all 2-simplices $\sigma_1, \ldots, \sigma_{n(i_0)} \in D_{i_0}^{(2)} \backslash A_{i_0}^{(2)}$ and let $i_{\max} = \max\{i_1, \ldots, i_{n(i_0)}\}$. Then clearly, one gets a well-defined map $g_{\max} : D_{i_{\max}} \to K(\Pi, 1)$ such that it is an extension up to homotopy, of $f : A \to K(\Pi, 1)$ and so the composition

$$\overline{g} = (g_{\max} \circ p_{i_{\max}}^{\infty}) : \mathbb{D}^2 \to K(\Pi, 1)$$

is again up to homotopy, the desired extension.

Definition 5.3. Let $N = N_1 \cup N_2$ be the (boundary connected) sum of two copies $N_1 = M^* = N_2$ of the minimal grope M^* along its boundary circle $\partial M^* = S^1$. A compactum X is said to be *weakly* Cainian provided that for the fundamental group $\Pi = \Pi_1(N)$, $X \tau K(\Pi, 1)$. Equivalently, c-$\dim_\Pi X \leqslant 1$.

Remark 5.4. Let $\mathcal{W}$ be the class of all perfect groups Π such that $H_2(\Pi) \cong \mathbb{Z}$ and $H_q(\Pi) = 0$, for all $q \neq 2$. Note that obviously, $\Pi = \Pi_1(N) \in \mathcal{W}$.

Theorem 5.5. *Every weakly Cainian compactum is at most 2-dimensional.*

Proof. If one glues together two minimal gropes N_1 and N_2 along the boundary then it follows by [5, Lemma (1.18)] (also by [4, Aspherical Pasting Lemma 4]) that the resulting space N is *aspherical*. Also Π is perfect since $H_1(N) = 0$, for we have killed (in fact, twice) the only generator of $H_1(S^1) \cong \mathbb{Z}$, therefore $N = K(\Pi, 1)$.

Next, the suspension of N, ΣN is a homotopy 3-sphere since it is the union of two suspensions of a grope glued along the suspension of a circle, i.e., it is the union of two homotopy 3-balls glued along their boundary.

Suppose now that X is a weakly Cainian compactum. Then $X \tau N$ hence by [12], $(X \times I)\tau(\Sigma N)$ and consequently, $(X \times I)\tau S^3$ so $\dim(X \times I) \leqslant 3$ hence $\dim X \leqslant 2$. □

Corollary 5.6. *Every Cainian compactum is at most 2-dimensional.*

Remark 5.7. In their recent preprint, J. Dydak and K. Yokoi renamed Cainian compacta as compacta of *perfect cohomological dimension one*. In particular, they obtained an *alternative* proof of our Corollary 5.6 (cf. [19, Corollary (3.7)]).

Theorem 5.8. *Every weakly Cainian compactum is a Cannon–Štan'ko compactum.*

Proof. Suppose that X is a weakly Cainian compactum and let M^* be the minimal grope. Then $X = K(\Pi, 1)$ where Π is the fundamental group of the space $N = N_1 \cup N_2$, obtained by glueing two copies of M^* along $\partial M^* = S^1$. Let $A \subset X$ be any closed subset and consider any map $f:A \to N_1 \subset N_1 \cup N_2$. Since X is weakly Cainian, there exists an extension $\overline{f}:X \to N_1 \cup N_2$. Let $\varphi: N_1 \cup N_2 \to N_1$ be the obvious "flip". Then $\varphi \circ \overline{f}: X \to N_1$ is the desired extension of the map f over all of X. □

Theorem 5.9. *Every* 2*-dimensional Cannon–Štan'ko compactum is a weakly Cainian compactum.*

Proof. Suppose that X is a 2-dimensional Cannon–Štan'ko compactum. Then in particular, $X\tau M^*$, for the minimal grope M^*. We must prove that $X\tau K(\Pi_1(N),1)$, where $N = N_1 \cup N_2$, $N_1 = N_2 = M^*$ and $\partial N_1 = \partial N_2 = S^1$.

Suppose now that we have a closed subset $A \subset X$ and a map $f:A \to N$. Then the pull-back of S^1 under f separates A. Let $C' = f^{-1}(S^1)$. Since $\dim X = 2$ there exists a subset C such that $C \cup C'$ separates X, C separates $X \backslash A$, and $\dim C = 1$. Denote the components of $X \backslash C$ by X_1 and X_2, i.e., $X_1 \supset f^{-1}(N_1)$ and $X_2 \supset f^{-1}(N_2)$.

First, extend $f|_{C'}$ over C (we can do this since $\dim C = 1$ and we are mapping to S^1), i.e., extend $f|_{C'}: C' \to S^1$ to $\overline{f}: C \cup C' \to S^1$. Now solve the extension problem separately for each "half" space X_1 and X_2, using the hypothesis that $X\tau M^*$. (Note that every subcompactum of a Cannon–Štan'ko compactum is also a Cannon–Štan'ko compactum. Indeed, given such a subcompactum $X_0 \subset X$ use the extension for all X and take its restriction onto X_0.) This gives extensions $F_i: X_i \to N_i$, $i \in \{1,2\}$, such that for every i, $F_i|_{C \cup C'} = \overline{f}: C \cup C' \to \partial N_i$. Thus

$$F = F_1 \cup F_2 : X = X_1 \cup X_2 \to N = N_1 \cup N_2$$

is then the desired extension of the map f. □

Corollary 5.10 (Characterization of weakly Cainian compacta). *Weakly Cainian compacta are precisely the* 2*-dimensional Cannon–Štan'ko compacta.*

Theorem 5.11. *Every* 2*-dimensional nonabelian compactum is Cainian.*

Proof. Let X be an arbitrary 2-dimensional nonabelian compactum. Let Π be an arbitrary perfect group, $A \subset X$ any closed subset of X and $f:A \to K(\Pi,1)$ any continuous map.

Represent the compactum X as the inverse limit of compact 2-dimensional polyhedral pairs $(L_i, A_i), (X,A) = \varprojlim\{(L_i,A_i),(p_i^{i+1}, p_i^{i+1}|_{A_{i+1}})\}_{i \geqslant 0}$. We may assume that $L_i, X \subset I^\infty =$ the Hilbert cube. Since $K(\Pi,1)$ is an ANR, there exists an integer $i_0 \geqslant 1$ such that f factors through A_i, i.e., the diagram below is commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p_i^\infty|_A"'] & K(\Pi,1) \\ A_i \arrow[ur, "f_i"'] & \end{tikzcd}$$

It is easy to extend f_i over the 1-skeleton $L_i^{(1)}$ of L_i, $\overline{f}_i: A_i \cup L_i^{(1)} \to K(\Pi,1)$.

Pick any 2-simplex $\sigma \in L_i^{(2)}$. Then $\alpha = \overline{f}_i(\partial\sigma) \in \Pi$ and α is a product of commutators, $\alpha = [\alpha_1, \beta_1] \cdot [\alpha_2, \beta_2] \cdots [\alpha_k, \beta_k]$, since Π is perfect. Glue onto σ (in I^∞) k orientable 1-handles $H_1, H_2, \ldots, H_k$, one for each commutator $[\alpha_i, \beta_i]$. Next, each α_i and β_i is itself a product of commutators, e.g., $\alpha_i = [\gamma_1^i \delta_1^i] \cdot [\gamma_2^i \delta_2^i] \cdots [\gamma_l^i \delta_l^i]$, so again we can glue onto H_i the corresponding l orientable 1-handles $G_1^i, \ldots, G_l^i$, etc. This infinite process ends in glueing of k gropes $M_1, M_2, \ldots, M_k$ onto σ. Denote by $\widetilde{\sigma}$ the resulting new complex, $\widetilde{\sigma} = \sigma\#\{M_i \mid 1 \leqslant i \leqslant k\}$. Then $\overline{f}_i|_{\partial\sigma} : \partial\sigma \to K(\Pi, 1)$ can be extended to a continuous map $\tilde{f}_i|_{\widetilde{\sigma}} : \widetilde{\sigma} \to K(\Pi, 1)$.

As a result, doing this construction simplex by simplex for all 2-simplex $\sigma \in L_i^{(2)}$, we get an extension $\tilde{f}_i : \widetilde{L}_i \to K(\Pi, 1)$ over the modified complex:

$$\widetilde{L}_i = \left(L \setminus \bigcup_{\sigma \in L_i^{(2)} \setminus A_i} \operatorname{Int} \sigma \right) \cup \left(\bigcup_{\sigma \in L_i^{(2)} \setminus A_i} \widetilde{\sigma} \right),$$

so that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "p_i^\infty|_A"'] & & K(\Pi, 1) \\
A_i \arrow[urr, "f_i"] \arrow[r, "\text{incl.}"] & A_i \cup L_i^{(1)} \arrow[ur, "\overline{f}_i"] \arrow[r, "\text{incl.}"] & \widetilde{L}_i \arrow[u, "\tilde{f}_i"']
\end{tikzcd}$$

Finally, since by hypothesis, $X \tau K(\Pi_1(M), 1)$ we can get up to homotopy, an extension of f over X via $\tilde{f}_i$, $F : X \to K(\Pi, 1)$. In summary, we have the following diagram (commutative up to homotopy):

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\text{incl.}"] \arrow[ddd, "p_i^\infty|_A"'] & & K(\Pi, 1) \\
 & X \arrow[d, "p_i^\infty"'] \arrow[ur, dashed, "F"] & \\
 & L_i \arrow[uur, dashed, "\overline{f}_i"'] & \\
A_i \arrow[ur, "\text{incl.}"] \arrow[r, "\text{incl.}"] & A_i \cup L_i^{(1)} \arrow[u] \arrow[uuur] \arrow[r, "\text{incl.}"] & \widetilde{L}_i \arrow[uuu, "\tilde{f}_i"']
\end{tikzcd}$$

□

Theorem 5.12. *There exists a countable collection $\{S_i\}_{i \in \mathbb{N}}$ of 1-dimensional compacta S_i in B^4 such that every compactum $Z \subset B^4 \setminus \bigcup_{i \in \mathbb{N}} S_i$ is Cainian.*

Proof. By Theorem 4.8, there exists a 0-dimensional F_σ-set $Z_1 \subset B^4$ such that every compactum $Y \subset B^4 \setminus Z_1$ is nonabelian. Moreover, there exists a (standard) 1-dimensional Nöbeling net $Z_2 \subset B^4$, i.e., N is an F_σ-set and every compactum $Y \subset B^4 \setminus Z_2$ is at most 2-dimensional (see, e.g., [22]). Then the union $Z = Z_1 \cup Z_2$ is a countable collection of

1-dimensional compacta and obviously, every compactum $Y \subset B^4 \backslash Z$ is 2-dimensional and nonabelian. Hence by Theorem 5.11, Y is also Cainian. □

Acknowledgements

Both authors wish to acknowledge the hospitality of the Cornell University and the University of Tennessee at Knoxville where revised versions of this manuscript were prepared in the Winter and in the Summer of 1993, respectively. We also wish to acknowledge comments from R.J. Daverman, J. Dydak and the referee.

References

[1] J.W. Cannon, The recognition problem for topological manifolds: What is a topological manifold? Bull. Amer. Math. Soc. 84 (1978) 832–866.
[2] J.W. Cannon, J.L. Bryant and R.C. Lacher, The structure of generalized manifolds having nonmanifold set of trivial dimension, in: J.C. Cantrell Jr., ed., Geometric Topology (Academic Press, New York, 1979) 261–303.
[3] R.J. Daverman, Decompositions of Manifolds (Academic Press, Orlando, 1986).
[4] R.J. Daverman, Hereditary aspherical compacta and cell-like maps, Topol. Appl. 41 (1991) 247–254.
[5] M.W. Davis and T. Januszkiewicz, Hyperbolization of polyhedra, J. Diff. Geom. 34 (1991) 347–388.
[6] A.N. Dranishnikov, Homological dimension theory, Uspehi Mat. Nauk 43(4) (1988) 11–55 (in Russian); English translation in: Russian Math. Surveys 43(4) (1988) 11–63.
[7] A.N. Dranishnikov, The Bockstein inequalities in homological dimension theory, Dokl. Akad. Nauk SSSR 299(5) (1988) (in Russian); English translation in: Soviet Math. Dokl. 37(2) (1988) 505–509.
[8] A.N. Dranishnikov, On realization of dimension functions I, Sibirsk. Mat. Zh. 29 (1988) 32–38 (in Russian).
[9] A.N. Dranishnikov, On problem of P.S. Alexandroff, Mat. Sb. 135 (177)(4) (1988) 551–557 (in Russian); English translation in: Math. USSR-Sb. 63 (1988) 539–545.
[10] A.N. Dranishnikov, Alternative construction of compacta with different dimensions, in: D. Repovš, ed., Proc. Grad. Workshop Math. and Its Appl. in Soc. Sci., Ljubljana, 1991 (DMFA Slovenije, Ljubljana, 1991) 33–36.
[11] A.N. Dranishnikov, On realization of dimension functions III, Sibirsk. Mat. Zh. 32 (1991) 179–181 (in Russian).
[12] A.N. Dranishnikov, On intersections of compacta in Euclidean space II, Proc. Amer. Math. Soc. 113 (1991) 1149–1154.
[13] A.N. Dranishnikov, On a problem of Y. Sternfeld, Glasnik Mat. 27 (47) (1992) 365–368.
[14] A.N. Dranishnikov, On the mapping intersection problem, Pacif. J. Math., to appear.
[15] A.N. Dranishnikov, J. Dydak and J.J. Walsh, Cohomological dimension theory and its applications, in preparation.
[16] A.N. Dranishnikov and D. Repovš, Cohomological dimension with respect to perfect groups. Preliminary report, University of Ljubljana, 1991.
[17] A.N. Dranishnikov and D. Repovš, Cohomological dimension with respect to perfect groups, Abstr. Amer. Math. Soc. 13 (1992) 341–342.

[18] J. Dydak and G. Kozlowski, Vietoris–Begle theorem and spectra, Proc. Amer. Math. Soc. 113 (1991) 587–592.
[19] J. Dydak and K. Yokoi, Hereditarily aspherical compacta, Proc. Amer. Math. Soc., to appear.
[20] J. Dydak and J.J. Walsh, Complexes that arise in cohomological dimension theory: A unified approach, J. London Math. Soc. 48(2) (1993) 329–347.
[21] J. Dydak and J.J. Walsh, Infinite dimensional compacta having cohomological dimension two: An application of the Sullivan Conjecture, Topology 32 (1993) 93–104.
[22] R. Engelking, Dimension Theory (North-Holland, Amsterdam, 1978).
[23] D.B. Fuks, Eilenberg–MacLane complexes, Uspehi Mat. Nauk 21(5) (1966) 213–215 (in Russian); English translation in: Russian Math. Surveys 21(5) (1966) 205–207.
[24] K. Kuratowski, Topology, Vol. I (PWN, Warsaw, 1968).
[25] V.I. Kuz'minov, Homological dimension theory, Uspehi Mat. Nauk 23(5) (1968) 3–49 (in Russian); English translation in: Russian Math. Surveys 23(5) (1968) 1–45.
[26] W.J.R. Mitchell and D. Repovš, The topology of cell-like mappings, in: Proc. Conf. Diff. Geom. and Topol., Cala Gonone, 1988, Suppl. Rend. Sem. Fac. Sci. Univ. Cagliari 58 (1988) 265–300.
[27] W.J.R. Mitchell, D. Repovš and E.V. Ščepin, On 1-cycles and the finite dimensionality of homology 4-manifolds, Topology 31 (1992) 605–623.
[28] J.-I. Nagata, Modern Dimension Theory (North-Holland, Amsterdam, 1965).
[29] R. Pol, A 2-dimensional compactum in the product of two 1-dimensional compacta which does not contain any rectangle, Topol. Proc. 16 (1991) 133–135.
[30] L.S. Pontryagin, Sur une hypothèse fondamentale de la théorie de la dimension, C. R. Acad. Paris 190 (1930) 1105–1107.
[31] M.A. Štan'ko, Embeddings of compacta in Euclidean space, Mat. Sb. 83(2) (1970) 234–255 (in Russian); English translation in: Math. USSR-Sb. 10 (1970) 234–254.
[32] Y. Sternfeld, Stability and dimension—A counterexample to a conjecture of Chogoshvili, Trans. Amer. Math. Soc. 340 (1993) 243–251.
[33] J.J. Walsh, Dimension, cohomological dimension, and cell-like mappings, in: S. Mardešić and J. Segal, eds., Proc. Conf. Shape Theory and Geom. Topol., Dubrovnik, 1981, Lecture Notes Math. 870 (Springer, Berlin, 1981) 105–118.