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On some (q, h)-analogues of integral inequalities on discrete time scales

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ABSTRACT

Here we provide some Feng Qi type (q, h)-integral inequalities on discrete time scales, by using analytic and elementary methods in (q, h)-calculus. We show that these inequalities are reduced for $h = 0$ to the Feng Qi type q-integral inequalities on quantum calculus, reduced for $q = 1$ to the Feng Qi type h-integral inequalities on h-calculus and reduced for $q = h = 1$ to the Feng Qi type integral inequalities on difference calculus.

1. Introduction

In [1], Qi studied some interesting integral inequalities and the following open problem was proposed: under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \left[\int_a^b f(x)dx\right]^{t-1}$$

holds for $t > 1$. Many generalization, extension and applications of the above inequality were investigated in recent years, for example see [1–3] and the reference therein. Particularly, in [4–6], the authors have solved the above problem in q-calculus (or quantum calculus) and in h-calculus, respectively.

In this paper, following closely theorems and methods from [5,6], we solve the above mentioned open problem in (q, h)-calculus, which can be reduced to the quantum calculus (the case $h = 0, q > 1$), h-calculus (the case $q = 1, h > 0$) or to the difference calculus (the case $q = h = 1$).

First, we mention several fundamental definitions and results from the calculus on time scales which appears in an excellent introductory text by Bohner and Peterson [7,8] and also the paper [9]. For (q, h)-calculus, we refer to [10].

2. Preliminaries

By a time scale $\mathbb{T}$ we understand any nonempty, closed subset of reals with the ordering inherited from reals. Thus the reals $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the non-negative integers $\mathbb{N}_0$, the h-numbers $h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with fixed $h > 0$, and the q-numbers $q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with fixed $q > 1$ are examples of time scales.

For any $t \in \mathbb{T}$, we define the forward (backward) jump operator by the relation $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}(\rho(t) := \sup\{s \in \mathbb{T} : s < t\})$ and the forward (backward) graininess function $\mu(t) := \sigma(t) - t(\nu(t) := t - \rho(t))$, respectively.

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The symbol $f^{\Delta}(t)(f^{\nabla}(t))$ is called the Δ-derivative (∇-derivative) of $f : \mathbb{T} \to \mathbb{C}$ at $t \in \mathbb{T}^{\kappa}$ ($t \in \mathbb{T}_{\kappa}$) and defined by

$$f^{\Delta}(t) := \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \left(f^{\nabla}(t) := \lim_{s\to t} \frac{f(t) - f(\rho(s))}{t - \rho(s)} \right),$$

respectively. Considering discrete time scales (i.e., such that $\mu(t) \neq 0$ and $\nu(t) \neq 0$ for $t \in \mathbb{T}$) $f^{\Delta}(t)$ and $f^{\nabla}(t)$ exist for all $t \in \mathbb{T}$ and they are given by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \tag{1}$$

and

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}. \tag{2}$$

The Δ-integral of f and the ∇-integral of g over the time scale interval

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\} \cup \mathbb{T}$$

are defined by $\int_a^b f(t)\Delta t := F(b) - F(a)$ and $\int_a^b g(t)\nabla t := G(b) - G(a)$ where $F^{\Delta} = f$ on $\mathbb{T}^{\kappa}$ and $G^{\nabla} = g$ on $\mathbb{T}_{\kappa}$, respectively.

It is known that considering discrete time scales these integrals are exist and can be calculated (provided $a < b$) via the formulae

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t) \tag{3}$$

and

$$\int_a^b g(t)\nabla t = \sum_{t\in(a,b]} \nu(t)g(t). \tag{4}$$

The most significant discrete time scales are those originating from arithmetic and geometric sequence of reals, namely

$$\mathbb{T}_h^{t_0} := \{t_0 + hk : t \in \mathbb{Z}\}, \quad h > 0 \quad \text{and} \quad \mathbb{T}_q^{t_0} := \{t_0 q^k : t \in \mathbb{Z}\} \cup \{0\}, \quad q > 1,$$

respectively, where $t_0 \in \mathbb{R}$. These sets form the basis for the study of h-calculus and q-calculus in the literature. In [10], the authors have introduced the two parameter discrete time scale $\mathbb{T}_{(q,h)}^{t_0}$ generalizing time scales $\mathbb{T}_h^{t_0}$ and $\mathbb{T}_q^{t_0}$. Both time scales are characterized by linearity of the forward (as well as backward) jump operator, because $\sigma(t) = t + h$ or $\sigma(t) = qt$.

For $q > 0, q \neq 1$ we set

$$[k]_q = \frac{q^k - 1}{q - 1}, \quad k \in \mathbb{N}_0.$$

For a given $t_0 \in \mathbb{R}^+$, we define the time scale

$$\mathbb{T}_{(q,h)}^{t_0} := \{t_0 q^k + [k]_q h : k \in \mathbb{Z}\} \cup \left\{ \frac{h}{1-q} \right\},$$

for $q \ge 1, h \ge 0$ and $q + h > 1$. It follows that $\sigma(t) = qt + h$, $\rho(t) = q^{-1}(t - h)$. In general, we have

$$\sigma^k(t) = q^k t + [k]_q h, \qquad \rho^k(t) = q^{-k}(t - [k]_q h) = q^{-k} t + [-k]_q h, \quad k \in \mathbb{Z}^+.$$

The introduction of (q, h)-derivative of $f : \mathbb{T}_{(q,h)}^{t_0} \to \mathbb{C}$ follows naturally from formulae (1) and (2): let $t \in \mathbb{T}_{(q,h)}^{t_0}$. The $\Delta_{(q,h)}$-derivative of f at t is given by

$$f^{\Delta_{q,h}}(t) = \frac{f(qt + h) - f(t)}{(q - 1)t + h}, \tag{5}$$

and the $\nabla_{(q,h)}$-derivative of f at t is given by

$$f^{\nabla_{q,h}}(t) = \frac{f(t) - f(q^{-1}(t - h))}{q^{-1}((q - 1)t + h)}. \tag{6}$$

Let $t \in \mathbb{T}_{(q,h)}^{t_0}$ and $t > t_0$, i.e. there exists $n \in \mathbb{Z}^+$ such that $t = t_0 q^n + [n]_q h$. Then we define the $\Delta_{(q,h)}$-integral by

$$\int_{t_0}^t f(s)\Delta s := ((q - 1)t_0 + h) \sum_{k=0}^{n-1} q^k f(t_0 q^k + [k]_q h). \tag{7}$$

For $\frac{h}{1-q} < a < t$, the $\nabla_{(q,h)}$-integral over the interval $[a, t]$ is defined by

$$\int_a^t f(s)\nabla s := ((1 - q^{-1})t + q^{-1}h) \sum_{k=0}^{n-1} q^{-k} f(q^{-k} t + [-k]_q h). \tag{8}$$

For any function $f:\mathbb{T}_{(q,h)}^{t_0}\to\mathbb{C}$, we have

$$\left(\int_{t_0}^{t} f(s)\Delta s\right)^{\Delta_{(q,h)}}=\left(\int_{t_0}^{t} f(s)\nabla s\right)^{\nabla_{(q,h)}}=f(t).$$

3. Delta (q,h)-integral inequalities

In this section, we give some Feng Qi type delta (q,h)-integral inequalities on discrete time scale $\mathbb{T}_{(q,h)}^{t_0}$. We begin with the following useful lemma.

Lemma 3.1. *Let $p\ge 1$ be a real number, G be a non-negative increasing function on $\mathbb{T}_{(q,h)}^{t_0}$ and $t\in\mathbb{T}_{(q,h)}^{t_0}$. Then we have*

$$p\,G^{p-1}(t)G^{\Delta_{(q,h)}}(t)\le[G^p(t)]^{\Delta_{(q,h)}}\le p\,G^{p-1}(qt+h)G^{\Delta_{(q,h)}}(t). \tag{9}$$

Proof. By (5), we have

$$[G^p(t)]^{\Delta_{q,h}}=\frac{G^p(qt+h)-G^p(t)}{q't+h}=\frac{p}{q't+h}\int_{G(t)}^{G(qt+h)}u^{p-1}du, \tag{10}$$

where $q'=q-1$. Since G is a non-negative increasing function, we have

$$G^{p-1}(t)[G(qt+h)-G(t)]\le\int_{G(t)}^{G(qt+h)}u^{p-1}du\le G^{p-1}(qt+h)[G(qt+h)-G(t)].$$

Hence, according to relation (10), we obtain

$$p\,G^{p-1}(t)G^{\Delta_{(q,h)}}(t)\le[G^p(t)]^{\Delta_{(q,h)}}\le p\,G^{p-1}(qt+h)G^{\Delta_{(q,h)}}(t).\quad\square$$

Now, we prove the Feng Qi type delta (q,h)-integral inequalities on discrete time scales.

Theorem 3.2. *If $f:[a,b]_{\mathbb{T}_{(q,h)}^{t_0}}\to\mathbb{C}$ is a non-negative increasing function and satisfies*

$$f^{\alpha-2}(t)f^{\Delta_{(q,h)}}(t)\ge(\alpha-2)q(q^2t+[2]_qh-a)^{\alpha-3}f^{\alpha-2}(q^2t+[2]_qh),$$

for $t\in[a,b]_{\mathbb{T}_{(q,h)}^{t_0}}$ and $\alpha\ge 3$, then

$$\int_a^b f^{\alpha}(u)\Delta u\ge\left(\int_a^b f(u)\Delta u\right)^{\alpha-1}.$$

Proof. For $t\in[a,b]_{\mathbb{T}_{(q,h)}^{t_0}}$, let $g(t)=\int_a^t f(u)\Delta u$ and

$$F(t)=\int_a^t f^{\alpha}(u)\Delta u-\left(\int_a^b f(u)\Delta u\right)^{\alpha-1}.$$

We have

$$F^{\Delta_{(q,h)}}(t)=f^{\alpha}(t)-[g^{\alpha-1}(t)]^{\Delta_{(q,h)}}.$$

Since f and g increase in $[a,b]_{\mathbb{T}_{(q,h)}^{t_0}}$, by virtue of Lemma 3.1, it follows that

$$\begin{aligned}F^{\Delta_{(q,h)}}(t)&\ge f^{\alpha}(t)-(\alpha-1)g^{\alpha-2}(qt+h)g^{\Delta_{(q,h)}}(t)\\&=f^{\alpha}(t)-(\alpha-1)g^{\alpha-2}(qt+h)f(t)=f(t)G(t)\end{aligned}$$

where $G(t)=f^{\alpha-1}(t)-(\alpha-1)g^{\alpha-2}(qt+h)$.

On the other hand, we have

$$G^{\Delta_{(q,h)}}(t)=[f^{\alpha-1}(t)]^{\Delta_{(q,h)}}-(\alpha-1)[g^{\alpha-2}(qt+h)]^{\Delta_{(q,h)}}.$$

Applying Lemma 3.1 again, we get

$$\begin{aligned}G^{\Delta_{(q,h)}}(t)&\ge(\alpha-1)f^{\alpha-2}(t)f^{\Delta_{(q,h)}}(t)-(\alpha-1)(\alpha-2)g^{\alpha-3}(q^2t+[2]_qh)g^{\Delta_{(q,h)}}(qt+h)\\&=(\alpha-1)f^{\alpha-2}(t)f^{\Delta_{(q,h)}}(t)-(\alpha-1)(\alpha-2)g^{\alpha-3}(q^2t+[2]_qh)qf(qt+h).\end{aligned}$$

Since f is a non-negative increasing function, we have

$$g(q^2t+[2]_qh)=\int_a^{q^2t+[2]_qh} f(u)\Delta u \le (q^2t+[2]_qh-a)f(q^2t+[2]_qh).$$

Hence,

$$\begin{aligned} G^{\Delta_{(q,h)}}(t) &\ge (\alpha-1)f^{\alpha-2}(t)f^{\Delta_{(q,h)}}(t)-(\alpha-1)(\alpha-2)q(q^2t+[2]_qh-a)^{\alpha-3}f^{\alpha-3}(q^2t+[2]_qh)f(qt+h) \\ &\ge (\alpha-1)f^{\alpha-2}(t)f^{\Delta_{(q,h)}}(t)-(\alpha-1)(\alpha-2)q(q^2t+[2]_qh-a)^{\alpha-3}f^{\alpha-2}(q^2t+[2]_qh) \\ &= (\alpha-1)\{f^{\alpha-2}(t)f^{\Delta_{(q,h)}}(t)-(\alpha-2)q(q^2t+[2]_qh-a)^{\alpha-3}f^{\alpha-2}(q^2t+[2]_qh)\}\ge 0. \end{aligned}$$

We conclude that G is an increasing function. Since $G(t)\ge G(a)\ge 0$, it follows that $F^{\Delta_{(q,h)}}(t)=f(t)G(t)\ge 0$. Hence, F is increasing and since $F(t)\ge F(a)=0$ this concludes the proof of Theorem 3.2. □

Theorem 3.3. *Let $f:[a,b]_{\mathbb{T}^{t_0}_{(q,h)}}\to\mathbb{C}$ is a non-negative increasing function satisfying*

$$f^{\beta}(t)f^{\Delta_{(q,h)}}(t)\ge \frac{\beta q(q^2t+[2]_qh-a)^{\beta-1}}{(b-a)^{\beta-1}}f^{\beta}(q^2t+[2]_qh),$$

for $t\in[a,b]_{\mathbb{T}^{t_0}_{(q,h)}}$ and $\beta\ge 3$. Then

$$\int_a^b f^{\beta+2}(u)\Delta u \ge \frac{1}{(b-a)^{\beta-1}}\left(\int_a^b f(u)\Delta u\right)^{\beta+1}.$$

Proof. For each $t\in[a,b]_{\mathbb{T}^{t_0}_{(q,h)}}$, let $g(t)=\int_a^t f(u)\Delta u$ and

$$F(t)=\int_a^t f^{\beta+2}(u)\Delta u-\frac{1}{(b-a)^{\beta-1}}\left(\int_a^t f(u)\Delta u\right)^{\beta+1}.$$

Then, we have

$$F^{\Delta_{(q,h)}}(t)=f^{\beta+2}(t)-\frac{1}{(b-a)^{\beta-1}}[g^{\beta+1}(t)]^{\Delta_{(q,h)}}.$$

Since f and g increase in $[a,b]_{\mathbb{T}^{t_0}_{(q,h)}}$, by virtue of Lemma 3.1, we have

$$\begin{aligned} F^{\Delta_{(q,h)}}(t) &\ge f^{\beta+2}(t)-\frac{(\beta+1)}{(b-a)^{\beta-1}}g^{\beta}(qt+h)g^{\Delta_{(q,h)}}(t) \\ &= f^{\beta+2}(t)-\frac{(\beta+1)}{(b-a)^{\beta-1}}g^{\beta}(qt+h)f(t)=f(t)H(t) \end{aligned}$$

where $H(t)=f^{\beta+1}(t)-\frac{(\beta+1)}{(b-a)^{\beta-1}}g^{\beta}(qt+h)$.

On the other hand, we have

$$H^{\Delta_{(q,h)}}(t)=[f^{\beta+1}(t)]^{\Delta_{(q,h)}}-\frac{(\beta+1)}{(b-a)^{\beta-1}}[g^{\beta}(qt+h)]^{\Delta_{(q,h)}}.$$

Applying Lemma 3.1 again, we get

$$\begin{aligned} H^{\Delta_{(q,h)}}(t) &\ge (\beta+1)f^{\beta}(t)f^{\Delta_{(q,h)}}(t)-\frac{(\beta+1)\beta}{(b-a)^{\beta-1}}g^{\beta-1}(q^2t+[2]_qh)g^{\Delta_{(q,h)}}(qt+h) \\ &\ge (\beta+1)f^{\beta}(t)f^{\Delta_{(q,h)}}(t)-\frac{(\beta+1)\beta}{(b-a)^{\beta-1}}g^{\beta-1}(q^2t+[2]_qh)qf(qt+h). \end{aligned}$$

Since f is a non-negative increasing function, we have

$$g(q^2t+[2]_qh)=\int_a^{q^2t+[2]_qh} f(u)\Delta u \le (q^2t+[2]_qh-a)f(q^2t+[2]_qh).$$

Hence,

$$\begin{aligned} H^{\Delta_{(q,h)}}(t) &\geq (\beta+1)f^{\beta}(t)f^{\Delta_{(q,h)}}(t) - \frac{\beta(\beta+1)q(q^2t+[2]_qh-a)^{\beta-1}}{(b-a)^{\beta-1}}f^{\beta-1}(q^2t+[2]_qh)f(qt+h) \\ &\geq (\beta+1)f^{\beta}(t)f^{\Delta_{(q,h)}}(t) - \frac{\beta(\beta+1)q(q^2t+[2]_qh-a)^{\beta-1}}{(b-a)^{\beta-1}}f^{\beta}(q^2t+[2]_qh) \\ &\geq (\beta+1)\{f^{\beta}(t)f^{\Delta_{(q,h)}}(t) - \frac{\beta q(q^2t+[2]_qh-a)^{\beta-1}}{(b-a)^{\beta-1}}f^{\beta}(q^2t+[2]_qh)\} \geq 0. \end{aligned}$$

We conclude that H is an increasing function. Since $H(t) \geq H(a) \geq 0$, it follows that $F^{\Delta_{(q,h)}}(t) = f(t)H(t) \geq 0$. Hence, F is increasing and since $F(t) \geq F(a) = 0$ this concludes the proof of Theorem 3.3. □

The following examples show the Feng Qi type h-integral inequalities and Feng Qi type q-integral inequalities in the sense of forward sum inequalities.

Example 1. Let $q = 1$ in Theorem 3.2. Then, clearly we have

$$f^{\alpha-2}(t)f^{\Delta_h}(t) \geq (\alpha-2)(t+2h-a)^{\alpha-3}f^{\alpha-2}(t+2h),$$

for $t \in [a,b]_{\mathbb{T}_h^{t_0}}$ and $\alpha \geq 3$ implies

$$\int_a^b f^{\alpha}(u)\Delta_h u \geq \left(\int_a^b f(u)\Delta_h u\right)^{\alpha-1}.$$

Let $q = 1$ in Theorem 3.3. Then, we have

$$f^{\beta}(t)f^{\Delta_h}(t) \geq \frac{\beta(t+2h-a)^{\beta-1}}{(b-a)^{\beta-1}}f^{\beta}(t+2h),$$

for $t \in [a,b]_{\mathbb{T}_h^{t_0}}$ and $\beta \geq 3$ implies

$$\int_a^b f^{\beta+2}(u)\Delta_q u \geq \frac{1}{(b-a)^{\beta-1}}\left(\int_a^b f(u)\Delta_q u\right)^{\beta+1}.$$

The above results are found in [6].

Example 2. By letting $h = 0$ in Theorem 3.2, we found that

$$f^{\alpha-2}(t)f^{\Delta_q}(t) \geq (\alpha-2)q(q^2t-a)^{\alpha-3}f^{\alpha-2}(q^2t),$$

for $t \in [a,b]_{\mathbb{T}_q^{t_0}}$ and $\alpha \geq 3$, then

$$\int_a^b f^{\alpha}(u)\Delta_q u \geq \left(\int_a^b f(u)\Delta_q u\right)^{\alpha-1}.$$

By letting $h = 0$ in Theorem 3.3, we have

$$f^{\beta}(t)f^{\Delta_q}(t) \geq \frac{\beta q(q^2t-a)^{\beta-1}}{(b-a)^{\beta-1}}f^{\beta}(q^2t),$$

for $t \in [a,b]_{\mathbb{T}_q^{t_0}}$ and $\beta \geq 3$ implies

$$\int_a^b f^{\beta+2}(u)\Delta_q u \geq \frac{1}{(b-a)^{\beta-1}}\left(\int_a^b f(u)\Delta_q u\right)^{\beta+1}.$$

4. Nabla (q, h)-integral inequalities

In this section, we give the nabla (q,h)-type integral inequalities on $\mathbb{T}_{(q,h)}^{t_0}$. As in the case of delta (q,h)-calculus, we have the following useful lemma.

Lemma 4.1. *Let $p \geq 1$ be a real number, G be a non-negative increasing function on $\mathbb{T}_{(q,h)}^{t_0}$ and $t \in \mathbb{T}_{(q,h)}^{t_0}$. Then we have*

$$p\,G^{p-1}(q^{-1}(t-h))G^{\nabla_{(q,h)}}(t) \leq [G^p(t)]^{\nabla_{(q,h)}} \leq p\,G^{p-1}(t)G^{\nabla_{(q,h)}}(t). \tag{11}$$

Proof. By (6), we have

$$[G^p(t)]^{\nabla_{q,h}} = \frac{G^p(t) - G^p(q^{-1}(t-h))}{q^{-1}(q't+h)} = \frac{p}{q^{-1}(q't+h)} \int_{G(q^{-1}(t-h))}^{G(t)} u^{p-1} du, \tag{12}$$

where $q' = q - 1$. Since G is a non-negative increasing function, we have

$$\begin{aligned} G^{p-1}(q^{-1}(t-h))[G(t) - G(q^{-1}(t-h))] &\le \int_{G(q^{-1}(t-h))}^{G(t)} u^{p-1} du \\ &\le G^{p-1}(t)[G(t) - G(q^{-1}(t-h))]. \end{aligned}$$

Hence, according to relation (12), we obtain the desired result:

$$p\, G^{p-1}(q^{-1}(t-h)) G^{\nabla_{(q,h)}}(t) \le [G^p(t)]^{\nabla_{(q,h)}} \le p\, G^{p-1}(t) G^{\nabla_{(q,h)}}(t). \quad \square$$

Theorem 4.2. *If $f : [a,b]_{\mathbb{T}_{(q,h)}^{t_0}} \to \mathbb{C}$ is a non-negative increasing function and satisfies*

$$f^{\nabla_{(q,h)}}(t) \ge (\alpha - 2)(t-a)^{\alpha-3},$$

for $t \in [a,b]_{\mathbb{T}_{(q,h)}^{t_0}}$ and $\alpha \ge 3$, then

$$\int_a^b f^\alpha(u) \nabla u \ge \left(\int_a^b f(q^{-1}(u-h)) \nabla u \right)^{\alpha-1}.$$

Proof. For $t \in [a,b]_{\mathbb{T}_{(q,h)}^{t_0}}$, let $g(t) = \int_a^t f(q^{-1}(u-h)) \nabla u$ and

$$F(t) = \int_a^t f^\alpha(qu+h) \nabla u - \left(\int_a^b f(q^{-1}(u-h)) \nabla u \right)^{\alpha-1}.$$

We have

$$F^{\nabla_{(q,h)}}(t) = f^\alpha(t) - [g^{\alpha-1}]^{\nabla_{(q,h)}}(t).$$

Since f and g increase in $[a,b]_{\mathbb{T}_{(q,h)}^{t_0}}$, by virtue of Lemma 4.1, it follows that

$$\begin{aligned} F^{\nabla_{(q,h)}}(t) &\ge f^\alpha(t) - (\alpha-1) g^{\alpha-2}(t) g^{\nabla_{(q,h)}}(t) \\ &= f^\alpha(t) - (\alpha-1) g^{\alpha-2}(t) f(q^{-1}(t-h)) \\ &\ge f^\alpha(t) - (\alpha-1) g^{\alpha-2}(t) f(t) = f(t) G(t) \end{aligned}$$

where $G(t) = f^{\alpha-1}(t) - (\alpha-1) g^{\alpha-2}(t)$.

On the other hand, we have

$$G^{\nabla_{(q,h)}}(t) = [f^{\alpha-1}]^{\nabla_{(q,h)}}(t) - (\alpha-1)[g^{\alpha-2}]^{\nabla_{(q,h)}}(t).$$

Applying Lemma 4.1 again, we get

$$\begin{aligned} G^{\nabla_{(q,h)}}(t) &\ge (\alpha-1) f^{\alpha-2}(q^{-1}(t-h)) f^{\nabla_{(q,h)}}(t) - (\alpha-1)(\alpha-2) g^{\alpha-3}(t) g^{\nabla_{(q,h)}}(t) \\ &= (\alpha-1) f^{\alpha-2}(q^{-1}(t-h)) f^{\nabla_{(q,h)}}(t) - (\alpha-1)(\alpha-2) g^{\alpha-3}(t) f(q^{-1}(t-h)) \\ &= (\alpha-1) f(q^{-1}(t-h)) \{ f^{\alpha-3}(q^{-1}(t-h)) f^{\nabla_{(q,h)}}(t) - (\alpha-2) g^{\alpha-3}(t) \}. \end{aligned}$$

Since f is a non-negative increasing function, we have

$$g(t) = \int_a^t f(q^{-1}(u-h)) \nabla u \le (t-a) f(q^{-1}(t-h)).$$

Hence,

$$G^{\nabla_{(q,h)}}(t) = (\alpha-1) f^{\alpha-2}(q^{-1}(t-h)) \{ f^{\nabla_{(q,h)}}(t) - (\alpha-2)(t-a)^{\alpha-3} \}.$$

We conclude that G is an increasing function. Since $G(t) \ge G(a) \ge 0$, it follows that $F^{\nabla_{(q,h)}}(t) = f(t)G(t) \ge 0$. Hence, F is increasing and since $F(t) \ge F(a) = 0$ this concludes the proof of Theorem 4.2. $\square$

Theorem 4.3. *Let $f : [a,b]_{\mathbb{T}_{(q,h)}^{t_0}} \to \mathbb{C}$ is a non-negative increasing function satisfying*

$$f^{\nabla_{(q,h)}}(t) \ge \beta \left(\frac{t-a}{b-a} \right)^{\beta-1},$$

for $t \in [a, b]_{\mathbb{T}^{t_0}_{(q,h)}}$ *and* $\beta \geq 3$. *Then*

$$\int_a^b f^{\beta+2}(u)\nabla u \geq \frac{1}{(b-a)^{\beta-1}} \left(\int_a^b f(q^{-1}(u-h))\nabla u\right)^{\beta+1}.$$

Proof. For each $t \in [a, b]_{\mathbb{T}^{t_0}_{(q,h)}}$, let $g(t) = \int_a^t f(q^{-1}(u-h))\nabla u$ and

$$F(t) = \int_a^t f^{\beta+2}(u)\nabla u - \frac{1}{(b-a)^{\beta-1}} \left(\int_a^t f(q^{-1}(u-h))\nabla u\right)^{\beta+1}.$$

Then, we have

$$F^{\nabla_{(q,h)}}(t) = f^{\beta+2}(t) - \frac{1}{(b-a)^{\beta-1}}[g^{\beta+1}(t)]^{\nabla_{(q,h)}}.$$

Since f and g increase in $[a, b]_{\mathbb{T}^{t_0}_{(q,h)}}$, by virtue of Lemma 4.1, we have

$$\begin{aligned} F^{\nabla_{(q,h)}}(t) &\geq f^{\beta+2}(t) - \frac{(\beta+1)}{(b-a)^{\beta-1}} g^{\beta}(t) g^{\nabla_{(q,h)}}(t) \\ &= f^{\beta+2}(t) - \frac{(\beta+1)}{(b-a)^{\beta-1}} g^{\beta}(t) f(q^{-1}(t-h)) \\ &\geq f^{\beta+2}(t) - \frac{(\beta+1)}{(b-a)^{\beta-1}} g^{\beta}(t) f(t) = f(t)H(t) \end{aligned}$$

where $H(t) = f^{\beta+1}(t) - \frac{(\beta+1)}{(b-a)^{\beta-1}} g^{\beta}(t)$.

On the other hand, we have

$$H^{\nabla_{(q,h)}}(t) = [f^{\beta+1}(t)]^{\nabla_{(q,h)}} - \frac{(\beta+1)}{(b-a)^{\beta-1}}[g^{\beta}]^{\nabla_{(q,h)}}(t).$$

Applying Lemma 4.1 again, we get

$$\begin{aligned} H^{\nabla_{(q,h)}}(t) &\geq (\beta+1) f^{\beta}(q^{-1}(t-h)) f^{\nabla_{(q,h)}}(t) - \frac{(\beta+1)\beta}{(b-a)^{\beta-1}} g^{\beta-1}(t) g^{\nabla_{(q,h)}}(t) \\ &\geq (\beta+1) f(q^{-1}(t-h)) \left\{ f^{\beta-1}(q^{-1}(t-h)) f^{\nabla_{(q,h)}}(t) - \frac{\beta}{(b-a)^{\beta-1}} g^{\beta-1}(t) \right\}. \end{aligned}$$

Since f is a non-negative increasing function, we have

$$g(t) = \int_a^t f(q^{-1}(u-h))\nabla u \leq (t-a) f(q^{-1}(t-h)).$$

Hence,

$$\begin{aligned} H^{\nabla_{(q,h)}}(t) &\geq (\beta+1) f(q^{-1}(t-h)) \left\{ f^{\beta-1}(q^{-1}(t-h)) f^{\nabla_{(q,h)}}(t) - \frac{\beta(t-a)^{\beta-1}}{(b-a)^{\beta-1}} f^{\beta-1}(q^{-1}(t-h)) \right\} \\ &= (\beta+1) f^{\beta}(q^{-1}(t-h)) \left\{ f^{\nabla_{(q,h)}}(t) - \frac{\beta(t-a)^{\beta-1}}{(b-a)^{\beta-1}} \right\}. \end{aligned}$$

We conclude that H is an increasing function. Since $H(t) \geq H(a) \geq 0$, it follows that $F^{\nabla_{(q,h)}}(t) = f(t)H(t) \geq 0$. Hence, F is increasing and since $F(t) \geq F(a) = 0$ this concludes the proof of Theorem 4.3. □

The following examples show the Feng Qi type h-integral inequalities and q-integral inequalities in the sense of backward sum inequalities.

Example 3. Let $q = 1$ in Theorem 4.2. Then, we have

$$f^{\nabla_h}(t) \geq (\alpha-2)(t-a)^{\alpha-3},$$

for $t \in [a, b]_{\mathbb{T}^{t_0}_h}$ and $\alpha \geq 3$, implies that

$$\int_a^b f^{\alpha}(u)\nabla_h u \geq \left(\int_a^b f(u-h)\nabla_h u\right)^{\alpha-1}.$$

Let $q = 1$ in Theorem 4.3. Then, we have

$$f^{\nabla_h}(t) \geq \beta \left(\frac{t-a}{b-a}\right)^{\beta-1},$$

for $t \in [a, b]_{\mathbb{T}_h^{t_0}}$ and $\beta \geq 3$, implies that

$$\int_a^b f^{\beta+2}(u)\nabla_h u \geq \frac{1}{(b-a)^{\beta-1}} \left(\int_a^b f(u-h)\nabla_h u\right)^{\beta+1}.$$

Example 4. Let $h = 0$ in Theorem 4.2. Then, we have

$$f^{\nabla_q}(t) \geq (\alpha-2)(t-a)^{\alpha-3},$$

for $t \in [a, b]_{\mathbb{T}_q^{t_0}}$ and $\alpha \geq 3$, implies that

$$\int_a^b f^{\alpha}(u)\nabla_q u \geq \left(\int_a^b f(q^{-1}(u-h))\nabla_q u\right)^{\alpha-1}.$$

Let $h = 0$ in Theorem 4.3. Then

$$f^{\nabla_q}(t) \geq \beta \left(\frac{t-a}{b-a}\right)^{\beta-1},$$

for $t \in [a, b]_{\mathbb{T}_q^{t_0}}$ and $\beta \geq 3$, implies that

$$\int_a^b f^{\beta+2}(u)\nabla_q u \geq \frac{1}{(b-a)^{\beta-1}} \left(\int_a^b f(q^{-1}u)\nabla_q u\right)^{\beta+1}.$$

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