The Monotone Convergence of a Class of Parallel Nonlinear Relaxation Methods for Nonlinear Complementarity Problems

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(Received December 1995; accepted February 1996)

Abstract—We set up a class of parallel nonlinear multisplitting AOR methods by directly multisplitting the nonlinear mapping involved in the nonlinear complementarity problems. The different choices of the relaxation parameters can yield all the known and a lot of new relaxation methods, as well as a lot of new relaxed parallel nonlinear multisplitting methods for solving the nonlinear complementarity problems. The two-sided approximation properties and the influences on the convergence rates from the relaxation parameters about our new methods are shown, and sufficient conditions guaranteeing the methods to converge globally are discussed. Finally, a lot of numerical results show that our new methods are feasible and efficient.

Keywords—Nonlinear complementarity problem, Nonlinear multisplitting, Monotonicity, Global convergence.

1. INTRODUCTION

Let \( \mathbb{R}^n \) denote Euclidean \( n \)-space, \((\cdot, \cdot)\) the usual inner product, and \( \mathbb{R}^n_+ \) the set of \( x \in \mathbb{R}^n \) with \( x \geq 0 \) in the componentwise ordering. Then, given \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the nonlinear complementarity problem (NCP) consists of finding an \( x^* \geq 0 \) such that

\[
F(x^*) \geq 0, \quad (x^*, F(x^*)) = 0.
\] (1.1)

This problem has received a great deal of attention and there has been a lot of research on the existence and uniqueness of its solution, as well as applicable numerical methods for getting its approximated solution by serial computers. Particularly, More [1] studied this nonlinear complementarity problem more systematically in 1974. In his attractive work, a class of nonlinear successive underrelaxation (SUR) methods was introduced. Through the proofs of the monotone convergence of these methods, he deduced an existence result about the solution of this problem for the important Z-function class.

In this paper, motivated by More's result, we set up a class of parallel nonlinear relaxation methods for solving the NCP (1.1) by making use of the nonlinear multisplitting methodology and the successively accelerated overrelaxation technique described in [2-4]. This method is really of strong parallel computational function, and is especially suitable to the SIMD multiprocessor systems. With reasonable choices of the involved relaxation parameters, this method,
besides covering the aforementioned serial successively underrelaxed methods, can yield a lot of novel relaxed methods in the sense of nonlinear multisplitting such as nonlinear multisplitting Gauss-Seidel method, nonlinear multisplitting SOR method, and so on, too. Moreover, its convergence property can be considerably improved. For the Z-function class, as well as the M-function class, we discuss several sufficient conditions which can guarantee the convergence of the nonlinear multisplitting accelerated overrelaxation (AOR) method in the monotonous sense, and investigate different convergence properties of the methods which are caused by different pairs of the relaxation parameters. Finally, we do some numerical tests by a concrete example to show the correctness of our theoretical results.

2. THE NONLINEAR MULTISPLITTING AOR METHOD

Throughout this paper, the $m$th component of a column vector $x \in \mathbb{R}^n$ is denoted by $x_m$ and $e^m$ represents the $m$th unit basis vector of $\mathbb{R}^n$ for $m = 1(1)n$. The natural partial ordering "$\leq$" and "$<$" on $\mathbb{R}^n$ are to be understood componentwise.

For $i = 1, 2, \ldots, \alpha (\alpha \leq n, \text{ an integer})$, let $J_i$ be a nonempty subset of the set $\{1, 2, \ldots, n\}$ satisfying $\cup_{i=1}^\alpha J_i = \{1, 2, \ldots, n\}$, and $E_i = \text{diag}(e^{(i)}_1, e^{(i)}_2, \ldots, e^{(i)}_n) \in \mathcal{L}(\mathbb{R}^n)$ be a nonnegatively diagonal matrix defined by

$$
e^{(i)}_m = \begin{cases} e^{(i)}_m \geq 0, & \text{for } m \in J_i, \\ 0, & \text{for } m \notin J_i, \end{cases} \quad m = 1(1)n,$$

with $\sum_{i=1}^\alpha E_i = I(I \in \mathcal{L}(\mathbb{R}^n) \text{ identity}).$ It is worth mentioning that there may be overlappings among these $J_i (i = 1, 2, \ldots, \alpha)$.

Consider the nonlinear mapping $F : \mathbb{R}^n \to \mathbb{R}^n$. For $i = 1, 2, \ldots, \alpha$, if there exists a mapping $f^{(i)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f^{(i)}(x; x) = F(x)$ for all $x \in \mathbb{R}^n$, then the collection of pairs $(f^{(i)}, E_i) (i = 1, 2, \ldots, \alpha)$ is called a nonlinear multisplitting of $F$.

Of course, there are a lot of meaningful forms of nonlinear multisplittings (see [2]). For example, if we particularly take

$$f^{(i)}_m(x; y) = \begin{cases} F_m \left( \left( I - P^{(i)} \right) x + P^{(i)} y \right), & \text{for } m \in J_i, \\ F_m(x_1, \ldots, x_{m-1}, y_m, x_{m+1}, \ldots, x_n), & \text{for } m \notin J_i, \end{cases} \quad m = 1(1)n, \quad (2.1)$$

for $i = 1, 2, \ldots, \alpha$, where $P^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ is a projection operator given by

$$P^{(i)}(x) = \begin{cases} x_m, & \text{for } m \in J_i, \\ 0, & \text{for } m \notin J_i, \end{cases} \quad m = 1(1)n, \quad (2.2)$$

then (2.1),(2.2), together with the preceding defined weighting matrices $E_i (i = 1, 2, \ldots, \alpha)$, form a special but practical case of nonlinear multisplitting.

Making use of the above concepts, we can now set up the following parallel nonlinear multisplitting AOR method.

**METHOD.** Let $D = \{x \in \mathbb{R}^n_+ \mid F(x) \geq 0\}$ be the feasible set of the NCP (1.1). Given a starting vector $x^0 \in D$, for $p = 0, 1, 2, \ldots$, compute

$$x^{p+1} = \left( x_1^{p+1}, x_2^{p+1}, \ldots, x_n^{p+1} \right)^T$$

through

$$x_m^{p+1} = \sum_{i=1}^\alpha c_m^{(i)} x_m^{p,i}, \quad m = 1(1)n, \quad (2.3)$$
with $x_{m,i}^{p,i}$, the $m$th element of $x^{p,i}$, satisfying
\[ x_{m,i}^{p,i} = \begin{cases} \max \{0, \omega x_{m,i}^{p,i} + (1 - \omega)x^p_m\}, & \text{for } m \in J_i, \\ x^p_m, & \text{for } m \notin J_i, \end{cases} \]  
where $\omega \in (0, \infty)$ is an acceleration factor. For each $i \in \{1, 2, \ldots, \alpha\}$ and each index $m \in J_i$, $\tilde{x}_{m,i}^{p,i} = 0$ is set if
\[ f_m^{(i)} \left( x^p, x_{1,i}^{p,i}, \ldots, x_{m-1,i}^{p,i}, 0, x_{m+1,i}^{p,i}, \ldots, x_{n,i}^{p,i} \right) \geq 0; \]  
otherwise, $\tilde{x}_{m,i}^{p,i}$ is taken as a solution of the nonlinear equation
\[ f_m^{(i)} \left( x^p, x_{1,i}^{p,i}, \ldots, x_{m-1,i}^{p,i}, x_m^{p,i}, x_{m+1,i}^{p,i}, \ldots, x_{n,i}^{p,i} \right) = 0. \]  
Here, for $i = 1, 2, \ldots, \alpha$,
\[ \tilde{x}_{m,i}^{p,i} = \left( \tilde{x}_{1,i}^{p,i}, \tilde{x}_{2,i}^{p,i}, \ldots, \tilde{x}_{n,i}^{p,i} \right)^T, \]  
and the $m$th element $\tilde{x}_{m,i}^{p,i}$ of $\tilde{x}_{m,i}^{p,i}$ is determined by
\[ \tilde{x}_{m,i}^{p,i} = \begin{cases} \max \{0, r\tilde{x}_{m,i}^{p,i} + (1 - r)x^p_m\}, & \text{for } m \in J_i, \\ x^p_m, & \text{for } m \notin J_i, \end{cases} \]  
while $r \in [0, \infty)$ is a relaxation factor.

For the convenience of our subsequent discussion, we abbreviate this method as NMAOR $(r, \omega)$-method. To illustrate this method more clearly, we especially investigate the linear case of the NCP (1.1) for which
\[ F(x) = Ax - b, \quad A \in L(R^n) \text{ is a P-matrix, } b \in R^n. \]

When $\alpha = 1$, the NMAOR $(r, \omega)$-method automatically reduces to a serial AOR method for solving the linear complementarity problem (LCP), and some special cases of this method have been studied in depth in [5,6], while $1 < \alpha \leq n$, it naturally becomes a parallel matrix multisplitting AOR method for solving the LCP.

In general, when $\alpha = 1$, the NMAOR $(r, \omega)$-method evidently reduces to a serial nonlinear AOR method (simply denoted as NAOR $(r, \omega)$-method) in the sense of nonlinear splitting for solving the NCP (1.1). If we especially choose the relaxation parameters $r$ and $\omega$ in this type of method, we can obtain all the known, as well as several new nonlinear relaxation methods. For example,
- if $r = \omega, \omega \in (0, \infty)$, we get the nonlinear SOR method;
- if $r = 1, \omega \in (0, \infty)$, we get the extrapolated nonlinear Gauss-Seidel method;
- if $r = \omega = 1$, we get the nonlinear Gauss-Seidel method [1,7];
- if $r = 0, \omega \in (0, \infty)$, we get the extrapolated nonlinear Jacobi method;
- if $r = 0, \omega = 1$, we get the nonlinear Jacobi method [1,7].

When $\alpha > 1$, by particularly selecting the relaxation parameters $r$ and $\omega$ in the NMAOR $(r, \omega)$-method, we can also get a novel series of nonlinear multisplitting relaxation methods for solving the NCP (1.1). For example,
- if $r = \omega \in (0, \infty)$, we obtain the nonlinear multisplitting SOR method;
- if $r = 1, \omega \in (0, \infty)$, we obtain the extrapolated nonlinear multisplitting Gauss-Seidel method;
- if $r = \omega = 1$, we obtain the nonlinear multisplitting Gauss-Seidel method;
- if $r = 0, \omega \in (0, \infty)$, we obtain the extrapolated nonlinear multisplitting Jacobi method;
- if $r = 0, \omega = 1$, we obtain the nonlinear multisplitting Jacobi method.
Therefore, the establishment of the NMAOR \((r, \omega)\)-method affords a general theoretical model for us to discuss systematically the nonlinear single and multiple splitting relaxation methods for the nonlinear complementarity problems. Observing that the NMAOR \((r, \omega)\)-method is merely an implicit iteration method due to the implicit nonlinear system of equations (2.6), in practical computations, we will rather implement it approximately by using some efficient numerical programs such as the Newton program, the chord program, the Steffensen program, and so on, than solve it exactly.

3. DEFINITIONS AND PRELIMINARY RESULTS

In the sequel, we will mainly consider the \(Z\)-functions and the \(M\)-functions which were first introduced by Tamir [7].

Let \(F(x)\) be a mapping from \(\mathbb{R}^n\) into \(\mathbb{R}^n\) with components \(F_m(x), m = 1(1)n\). It is called off-diagonally antitone on \(\mathbb{R}^n\) if for all \(x \in \mathbb{R}^n\) and \(m \neq j, m, j = 1(1)n\), the scalar functions \(\varphi_{mj}(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^1\) defined by

\[
\varphi_{mj}(t) = F_m(x + te^j)
\]

are nonincreasing; (strictly) diagonally isotone on \(\mathbb{R}^n\) if for all \(x \in \mathbb{R}^n\) the scalar functions \(\varphi_{mm} : \mathbb{R}^1 \rightarrow \mathbb{R}^1\) defined by

\[
\varphi_{mm}(t) = F_m(x + te^m), \quad m = 1(1)n
\]

are (increasing) nondecreasing; surjectively diagonally isotone on \(\mathbb{R}^n\) if for all \(x \in \mathbb{R}^n\) the scalar functions \(\varphi_{mm}\) are surjective and \(F\) is strictly diagonally isotone on \(\mathbb{R}^n\). The mapping \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is said to be a \(Z\)-function if it is off-diagonally antitone on \(\mathbb{R}^n\) and an \(M\)-function if it is a \(Z\)-function as well as inverse isotone on \(\mathbb{R}^n\), i.e., for any \(x, y \in \mathbb{R}^n\), \(F(x) \leq F(y)\) implies \(x \leq y\).

In fact, the \(Z\)-function and the \(M\)-function are direct nonlinear generalizations of the \(Z\)-matrix and \(M\)-matrix, respectively. Evidently, a continuous, surjective, and inverse isotone mapping \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a homeomorphism from \(\mathbb{R}^n\) onto \(F(\mathbb{R}^n)\).

The mapping \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is called a \(P\)-function if for each \(x \neq y\) in \(\mathbb{R}^n\), there is an index \(m = m(x, y)\) such that

\[
(x_m - y_m)(F_m(x) - F_m(y)) > 0.
\]

For this function, the NCP (1.1) has at most one solution. It is evident that an \(M\)-function is also a \(P\)-function, so there is at most one \(x^* \in R^0_+\) which satisfies (1.1) when \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is an \(M\)-function.

4. CONVERGENCE ANALYSIS OF THE NMAOR \((r, \omega)\)-METHOD

We first set up one theorem which describes the monotone convergence characterization of the NMAOR \((r, \omega)\)-method.

**Theorem 1.** Let \((f^{(i)}, E_i)(i = 1, 2, \ldots, \alpha)\) be a nonlinear multisplitting of \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\). Suppose that for each \(i \in \{1, 2, \ldots, \alpha\}, f^{(i)}(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous, antitone for \(x\) on \(\mathbb{R}^n\), off-diagonally antitone and strictly diagonally isotone for \(y\) on \(\mathbb{R}^n\). If the feasible set \(D\) of the NCP (1.1) is nonempty, then the sequence \(\{x^p\}\) generated by the NMAOR \((r, \omega)\)-method is well defined on \(R^n_+\) and satisfies

(a) \(0 \leq x^{p+1} \leq x^p\);
(b) \(\lim_{p \rightarrow \infty} x^p = x^*\), \(0 \leq x^* \leq x^0\);
(c) \(x^*\) is a solution of the NCP (1.1);
for any solution \( y^* \) of the NCP (1.1), if \( 0 \leq y^* \leq x^0 \), then \( y^* \leq x^* \), provided \( r \in [0, 1] \) and \( \omega \in (0, 1] \). Moreover, if \( F(0) \leq 0 \), then \( F(x^*) = 0 \).

**Proof.** In order to verify (a), we first introduce two number sets

\[
\begin{align*}
Q_i(p) &= \{ m \in J_i \mid f_m^{(i)} \left( x^p; x_1^{p,i}, \ldots, x_{m-1}^{p,i}, x_m^{p,i-1}, 0, x_{m+1}^p, \ldots, x_n^p \right) < 0 \}, \\
\overline{Q}_i(p) &= \{ m \in J_i \mid f_m^{(i)} \left( x^p; x_1^{p,i}, \ldots, x_{m-1}^{p,i}, x_m^{p,i-1}, 0, x_{m+1}^p, \ldots, x_n^p \right) \geq 0 \},
\end{align*}
\]

for \( i \in \{1, 2, \ldots, \alpha\} \) and \( p \in \{0, 1, 2, \ldots\} \). Evidently, there hold

\[
\begin{align*}
Q_i(p) \cap \overline{Q}_i(p) &= \emptyset, \\
Q_i(p) \cup \overline{Q}_i(p) &= J_i,
\end{align*}
\]

for \( i \in \{1, 2, \ldots, \alpha\} \) and \( \forall p \in \{0, 1, 2, \ldots\} \). Now, the proof of (a) can be fulfilled by induction. For this purpose, we suppose that for some \( p \geq 0, m \geq 1 \) and \( i = 1, 2, \ldots, \alpha \):

1. \( 0 \leq x^p \leq x^{p-1} \);
2. \( 0 \leq \bar{x}_j^{p,i} \leq x_j^{p-1,i} \) if \( j \in Q_i(p) \) and \( \bar{x}_j^{p,i} = 0 \) if \( j \in \overline{Q}_i(p) \), \( j \in \{1, 2, \ldots, m-1\} \);
3. \( 0 \leq x_j^{p,i} \leq x_j^{p-1,i} \) if \( j \in \{1, 2, \ldots, m-1\} \);
4. \( 0 \leq x_j^{p+1,i} \leq x_j^p \) if \( j \in \{1, 2, \ldots, m-1\} \);

here we stipulate

\[
\begin{align*}
x^{-1} &= x^0, \\
\bar{x}_j^{-1,i} &= \bar{x}_j^{i-1}, j \in \{1, 2, \ldots, n\}, \\
x_j^{-1,i} &= x_j^0, j \in \{1, 2, \ldots, \alpha\},
\end{align*}
\]

and (2)–(5) are vacuous for \( m = 1 \).

Obviously, (1)–(5) hold for \( p = 0 \) and \( m = 1 \). For \( p \) fixed, we now begin to prove (2)–(5) still hold for \( j = m \).

We first deal with (2). Clearly, when \( m \in \overline{Q}_i(p) \), we see \( \bar{x}_m^{p,i} = 0 \) according to the definition of the NMAOR \((r, \omega)\)-method. When \( m \in Q_i(p) \), by the antitonicity of \( f^{(i)}(x; y) \) for \( x \) and the off-diagonal antitonicity of it for \( y \) on \( R^n \), we have

\[
\begin{align*}
f_m^{(i)} \left( x^p; \bar{x}_1^{p,i}, \ldots, \bar{x}_{m-1}^{p,i}, \bar{x}_m^{p,i}, x_{m+1}^p, \ldots, x_n^p \right) &
\geq f_m^{(i)} \left( x^{p-1}; \bar{x}_1^{p-1,i}, \ldots, \bar{x}_{m-1}^{p-1,i}, \bar{x}_m^{p-1,i}, x_{m+1}^p, \ldots, x_n^p \right) \\
&\geq 0.
\end{align*}
\]

Making use of the continuity of \( f^{(i)}(x; y) \) and the strictly diagonal isotonicity of it for \( y \) on \( R^n \), we can, therefore, conclude that there exists unique \( \bar{x}_m^{p,i} \in [0, \bar{x}_m^{p-1,i}] \) such that

\[
f_m^{(i)} \left( x^p; \bar{x}_1^{p,i}, \ldots, \bar{x}_{m-1}^{p,i}, \bar{x}_m^{p,i}, x_{m+1}^p, \ldots, x_n^p \right) = 0
\]

in this situation. So, (2) is true for \( j = m \).

Note that \( \bar{Q}_i(\cdot) \subseteq \overline{Q}_i(\cdot) \) for \( i = 1, 2, \ldots, \alpha \); (3) can be tested in accordance with the following four cases. First, if \( m \notin J_i \), (2.7) and the induction assumption immediately show

\[
\bar{x}_m^{p,i} = x_m^{p} \leq x_m^{p-1} = \bar{x}_m^{p-1,i}.
\]

Second, if \( m \in Q_i(p) \), (2.7) becomes

\[
\bar{x}_m^{p,i} = r \bar{x}_m^{p,i} + (1 - r) x_m^{p}.
\]
as \( r \in [0, 1] \). Presently, (2) and the induction assumption imply

\[
\bar{x}_m^{p,i} \leq r \bar{x}_m^{p-1,i} + (1 - r)x_m^{p-1}
= \max \{0, r \bar{x}_m^{p-1,i} + (1 - r)x_m^{p-1}\}
= \bar{x}_m^{p-1,i}.
\]

Third, if \( m \in \overline{Q}_i(p - 1) \), from (2.7), again we have

\[
\bar{x}_m^{p,i} = (1 - r)x_m^{p} \leq (1 - r)x_m^{p-1} = \bar{x}_m^{p-1,i},
\]

since we have \( \bar{x}_m^{p,i} = \bar{x}_m^{p-1,i} = 0 \) at this time. At last, if \( m \in \overline{Q}_i(p - 1) \setminus \overline{Q}_i(p - 1) \), by (2.7), we know

\[
\bar{x}_m^{p,i} = (1 - r)x_m^{p} \leq (1 - r)x_m^{p-1},
\]

\[
\bar{x}_m^{p-1,i} = (1 - r)x_m^{p-1} + r \bar{x}_m^{p-1,i} \geq (1 - r)x_m^{p-1},
\]

which clearly show \( \bar{x}_m^{p,i} \leq \bar{x}_m^{p-1,i} \). Till now, we have demonstrated the validity of (3) for \( j = m \).

Equation (4) can be verified analogously to (3), and (5) is direct from (2.3) and (4).

The above discussions show that (2)–(5) hold for all \( m \in \{1, 2, \ldots, n\} \). Therefore,

\[
0 \leq x^{p+1} \leq x^p.
\]

By induction, we know that (a) is true.

(b) is clearly direct from (a).

We now turn to (c). Through the proving process of (a), we know that the sequences \( \{\bar{x}_j^{p,i}\} \) \( (j = 1(1)n, j \in J_i) \), \( \{\bar{x}_j^{p,i}\} \) \( (j = 1(1)n) \) and \( \{x_j^{p,i}\} \) \( (j = 1(1)n) \) have upper bounds and are monotone nonincreasing with respect to \( p \) for \( i = 1, 2, \ldots, \alpha \). Thus, they are convergent as \( p \) tends to infinity. Denote their corresponding limit points as \( \bar{x}_j^p \) \( (j = 1(1)n, j \in J_i) \), \( \bar{x}_j^p \) \( (j = 1(1)n) \) and \( x_j^p \) \( (j = 1(1)n) \), respectively, for \( i = 1, 2, \ldots, \alpha \).

Write

\[
S(m) = \left\{ i \in \{1, 2, \ldots, \alpha\} \mid m \in J_i, \ f_m^{(i)} \left( x^{\ast}; \bar{x}_1^i, \ldots, \bar{x}_{m-1}^i, 0, x_{m+1}^i, \ldots, x_n^i \right) < 0 \right\},
\]

\[
S(m) = \left\{ i \in \{1, 2, \ldots, \alpha\} \mid m \in J_i, \ f_m^{(i)} \left( x^{\ast}; \bar{x}_1^i, \ldots, \bar{x}_{m-1}^i, 0, x_{m+1}^i, \ldots, x_n^i \right) \geq 0 \right\}.
\]

Clearly, there hold

\[
\begin{align*}
S(m) \cap \overline{S}(m) & = \emptyset, \\
S(m) \cup \overline{S}(m) & = \{ i \in \{1, 2, \ldots, \alpha\} \mid m \in J_i \}, \quad m = 1(1)n.
\end{align*}
\]

Taking limits for (2.3)–(2.7), we accordingly have

\[
\begin{align*}
x_m^\ast & = \sum_{i=1}^\alpha e_m^{(i)} \bar{x}_m^i = \sum_{i \in S(m)} e_m^{(i)} \bar{x}_m^i, \\
x_m^\ast & = \sum_{i=1}^\alpha e_m^{(i)} \bar{x}_m^i = \sum_{i \in S(m)} e_m^{(i)} \bar{x}_m^i, \quad m = 1(1)n, \\
x_m^\ast & = \sum_{i=1}^\alpha e_m^{(i)} x_m^i = \sum_{i \in S(m)} e_m^{(i)} x_m^i,
\end{align*}
\]

\[
\begin{cases}
\bar{x}_m^i = 0, & \text{if } i \in S(m), \\
f_m^{(i)} \left( x^{\ast}; \bar{x}_1^i, \ldots, \bar{x}_{m-1}^i, \bar{x}_m^i, x_{m+1}^i, \ldots, x_n^i \right) = 0, & \text{if } i \in S(m),
\end{cases}
\]
For each $m \in \{1, 2, \ldots, n\}$, define
\[
\bar{x}_m = \begin{cases} 
\frac{r \bar{x}_m^i + (1 - r)x_m^*}{r}, & \text{for } m \in J_i, \\
x_m^*, & \text{for } m \notin J_i,
\end{cases}
\]
\begin{equation}
\bar{x}_m^i = \begin{cases} 
\omega \bar{x}_m^i + (1 - \omega)x_m^*, & \text{for } m \in J_i, \\
x_m^*, & \text{for } m \notin J_i,
\end{cases}
\end{equation}

As
\[
\sum_{i=1}^{\alpha} e_m^{(i)} = \sum_{i \in S(m) \cup S'(m)} e_m^{(i)} = 1,
\]

it is obvious that
\[
\begin{cases} 
\bar{x}_m^{o(m)} \leq x_m^*, & m \in J_o(m), \\
x_m^* \leq \bar{x}_m^{a(m)}, & m \in J_a(m), 
\end{cases} \quad m = 1(1)n. \tag{4.4}
\]

We now show by induction that the following facts are correct.

(i) $F_m(x^*) \geq 0$, $x_m^* F_m(x^*) = 0$, $m = 1(1)n$;
(ii) $\bar{x}_m^i = x_m^*$, $m \in J_i$, $i = 1, 2, \ldots, \alpha$;
(iii) $x_m^i = \bar{x}_m^i = x_m^*$, $m = 1(1)n$, $i = 1, 2, \ldots, \alpha$.

When $m = 1$, if $S(1) = \emptyset$, then we have
\[
\bar{x}_1^i = 0, \quad i \in \bar{S}(1)
\]
from (4.2), and $x_1^* = 0$ from (4.1). Hence, it holds that
\[
x_1^i = \bar{x}_1^i = x_1^* = 0
\]
by (4.3). Additionally,
\[
F_1(x^*) = f_1^{(i)}(x^*, x_1^i, \ldots, x_n^*), \quad i \in \bar{S}(1)
\]
\[
= f_1^{(i)}(x^*; 0, x_2^*, \ldots, x_n^*)
\]
\[
\geq 0.
\]

On the other hand, if $S(1) \neq \emptyset$, then $\bar{x}_1^i > 0 (i \in S(1))$, and they satisfy
\[
f_1^{(i)}(x^*; \bar{x}_1^i, x_2^*, \ldots, x_n^*) = 0, \quad i \in S(1).
\]

Therefore, by (4.4) and the strictly diagonal isotonicity of $f^{(i)}(x; y)(i = 1, 2, \ldots, \alpha)$ for $y$ on $R^n$, we have
\[
F_1(x^*) = f_1^{(s(1))}(x^*; x_1^*, \ldots, x_n^*)
\]
\[
\leq f_1^{(s(1))}(x^*; \bar{x}_1^*, x_2^*, \ldots, x_n^*)
\]
\[
= 0,
\]
and
\[
F_1(x^*) = f_1^{(s(1))}(x^*; x_1^*, \ldots, x_n^*)
\]
\[
\geq f_1^{(s(1))}(x^*; \bar{x}_1^*, x_2^*, \ldots, x_n^*)
\]
\[
= 0.
\]

So, $F_1(x^*) = 0$, which demonstrates that (i) holds for $m = 1$. 

Since
\[ f_1^{(s_1(i))} (x^*; x_1^*, \ldots, x_n^*) = 0 = f_1^{(s_1(i))} (x^*; x_1^*, x_2^*, \ldots, x_n^*), \]
\[ f_1^{(s_0(i))} (x^*; x_1^*, x_2^*, \ldots, x_n^*) = 0 = f_1^{(s_0(i))} (x^*; x_1^*, x_2^*, \ldots, x_n^*), \]
the strictly diagonal isotonicity of \( f^{(i)}(x; y)(i = 1, 2, \ldots, \alpha) \) for \( y \) on \( \mathbb{R}^n \) implies
\[ \tilde{x}_1^{s_0(1)} = \tilde{x}_1^{s_1(1)} = x_1^*. \]

Hence,
\[ \tilde{x}_i^i = x_i^*, \quad i \in S(1). \]

So, (ii) holds for \( m = 1 \). (iii) clearly holds for \( m = 1 \) by directly using (4.3).

Suppose that for all \( m \leq j - 1 \), (i)-(iii) are valid. When \( m = j \), if \( S(j) = \emptyset \), then we have
\[ \tilde{x}_j^i = 0, \quad i \in S(j) \]
from (4.2) and \( x_j^* \) from (4.1). Hence, it holds that
\[ x_j^i = \tilde{x}_j^i = x_j^* = 0 \]
by (4.3). Additionally,
\[ F_j(x^*) = f_j^{(i)} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \quad i \in S(j) \]
\[ = f_j^{(i)} (x^*; x_1^*, \ldots, x_{j-1}^*, 0, x_{j+1}^*, \ldots, x_n^*) \]
\[ \geq 0. \]

On the other hand, if \( S(j) \neq \emptyset \), then \( \tilde{x}_j^i > 0(i \in S(j)) \), and they satisfy
\[ f_j^{(i)} (x^*; x_1^*, \ldots, x_{j-1}^*, \tilde{x}_j^0, x_{j+1}^*, \ldots, x_n^*) = 0, \quad i \in S(j). \]

Therefore, by (4.4) and the strictly diagonal isotonicity of \( f^{(i)}(x; y)(i = 1, 2, \ldots, \alpha) \) for \( y \) on \( \mathbb{R}^n \) we have
\[ F_j(x^*) = f_j^{(s_1(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*) \]
\[ \leq f_j^{(s_1(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, \tilde{x}_j^{s_1(j)}, x_{j+1}^*, \ldots, x_n^*) \]
\[ = f_j^{(s_1(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^{s_1(j)}, x_{j+1}^*, \ldots, x_n^*) \]
\[ = 0, \]
and
\[ F_j(x^*) = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*) \]
\[ \geq f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, \tilde{x}_j^{s_0(j)}, x_{j+1}^*, \ldots, x_n^*) \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, \tilde{x}_j^{s_0(j)}, \tilde{x}_j^{s_0(j)}, x_{j+1}^*, \ldots, x_n^*) \]
\[ = 0. \]

So, \( F_j(x^*) = 0 \), which indicates that (i) validates for \( m = j \). Because of
\[ 0 = f_j^{(s_1(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, \tilde{x}_j^{s_1(j)}, x_{j+1}^*, \ldots, x_n^*) \]
\[ = f_j^{(s_1(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
\[ 0 = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, \tilde{x}_j^{s_0(j)}, x_{j+1}^*, \ldots, x_n^*), \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
\[ = f_j^{(s_0(j))} (x^*; x_1^*, \ldots, x_{j-1}^*, x_j^*, x_{j+1}^*, \ldots, x_n^*), \]
the strictly diagonal isotonicity of \( f^{(i)}(x; y)(i = 1, 2, \ldots, \alpha) \) for \( y \) on \( \mathbb{R}^n \) again yields
\[
\bar{x}^0(i) = \bar{x}^1(j) = x^*_j.
\]
Hence,
\[
\bar{x}^i_j = x^*_j, \quad i \in S(j).
\]
So, (ii) is also true for \( m = j \). Likewise, the correctness of (iii) can be directly concluded for \( m = j \) from (4.3), too. By induction, we know that \( x^* \) is a solution of the NCP (1.1).

The proof of (d) can also be completed by induction. Notice that \( y^* \leq x^0 \). Suppose that for some \( p \geq 0 \) and \( m \geq 1 \):

1. \( y^* \leq x^p \);
2. \( y^*_j \leq \bar{x}^p_{ij}, \quad j \in \{1, 2, \ldots, m-1\}, \quad j \in J_i; \)
3. \( y^*_j \leq \min\{\bar{x}^p_{ji}, x^p_{ji}\}, \quad j \in \{1, 2, \ldots, m-1\}; \)
4. \( y^*_j \leq x^p_{j+1}, \quad j \in \{1, 2, \ldots, m-1\} \)

hold for \( i = 1, 2, \ldots, \alpha \), where we stipulate that (2)–(4) are vacuous for \( m = 1 \). Obviously, (1)–(4) hold for \( p = 0 \) and \( m = 1 \). For \( p \) fixed, we now start to prove that (2)–(4) also hold for \( j = m \).

If \( y^*_m = 0 \), then (2)–(4) are trivial. Otherwise, we must have \( F_m(y^*) = 0 \) since \( y^* \) is a solution of the NCP (1.1). At this time, by applying the antitonicity of \( f^{(i)}(x; y) \) for \( x \) on \( \mathbb{R}^n \), the off-diagonal antitonicity and the strictly diagonal isotonicity of \( f^{(i)}(x; y) \) for \( y \) on \( \mathbb{R}^n \), we can get
\[
\begin{align*}
\bar{f}^{(i)}(x^p_{j1}, \ldots, \bar{x}^p_{m-1}, x^*_m, x^p_{m+1}, \ldots, x^p_n) &< \bar{f}^{(i)}(x^p_{j1}, \ldots, \bar{x}^p_{m-1}, y^*_m, x^p_{m+1}, \ldots, x^p_n) \\
&\leq \bar{f}^{(i)}(y^*_j; y^*) \\
&= F_m(y^*) \\
&= 0,
\end{align*}
\]

or in other words, \( m \in Q_i(p) \). Hence, from the definition of the NMAOR \((r, \omega)\)-method, we see that
\[
f^{(i)}_m\left(x^p_{j1}, \ldots, \bar{x}^p_{m-1}, \bar{x}^p_{m}, x^p_{m+1}, \ldots, x^p_n\right) = 0.
\]

Similarly, we can obtain
\[
\begin{align*}
f^{(i)}_m\left(y^*_1, \ldots, y^*_m, \bar{x}^p_{m+1}, \ldots, x^p_n\right) &\geq f^{(i)}_m\left(x^p_{j1}, \bar{x}^p_{m+1}, \ldots, \bar{x}^p_{m-1}, \bar{x}^p_{m}, x^p_{m+1}, \ldots, x^p_n\right) \\
&= 0 \\
&= f^{(i)}_m\left(y^*_1, \ldots, y^*_m, y^*_m, y^*_m, \ldots, y^*_n\right).
\end{align*}
\]

The strictly diagonal isotonicity of \( f^{(i)}(x; y) \) for \( y \) on \( \mathbb{R}^n \) then again shows
\[
\bar{x}^p_{mi} \geq y^*_m.
\]

Presently, for \( j = m \), (3), (4) can be easily deduced from (2.4), (2.7), and (2.3), respectively.

The above discussions show that (2)–(4) hold for all \( m \in \{1, 2, \ldots, n\} \). Therefore, \( y^* \leq x^{p+1} \). Taking limits on both sides of this inequality, we obtain \( y^* \leq x^* \) at once.

Finally, if \( F(0) \leq 0 \), but \( F_m(x^*) > 0 \) for some \( m \in \{1, 2, \ldots, n\} \), then \( x^*_m = 0 \), and therefore, the off-diagonal antitonicity of \( f^{(i)}(x; y) \) with respect to \( y \) on \( \mathbb{R}^n \), as well as the antitonicity of it with respect to \( x \) on \( \mathbb{R}^n \), implies that
\[
F_m(x^*) = f^{(i)}_m(x^*_j; x^*_j) \leq f^{(i)}_m(0; 0) = F_m(0) \leq 0,
\]

which is a contradiction. So, \( F(x^*) = 0 \) provided \( F(0) \leq 0 \).
The following theorem characterizes the dependence of the NMAOR \((r, \omega)\)-method upon the choice of the acceleration factor \(\omega \in (0, 1]\).

**Theorem 2.** Suppose that the conditions of Theorem 1 are satisfied. Let \(\omega, \omega', r \in [0, 1]\) be given for which
\[
0 < \omega < \omega' 
\]
in all cases with \(x^0 \in D\) as starting points. Let \(\{x^p\}\) and \(\{x'^p\}\) be sequences yielded by the NMAOR \((r, \omega)\)-method with \((r, \omega)\) and \((r, \omega')\) being the iteration parameters, respectively. By Theorem 1, these sequences are well defined, monotonously nonincreasingly convergent to the maximum solution \(x^*\) of the NCP (1.1) in \(\{x \in R^n_+ | 0 \leq x \leq x^0\}\). Moreover, the relations
\[
x^p \geq x'^p \geq x^*, \quad p = 0, 1, 2, \ldots
\]
hold.

**Proof.** Suppose that for some \(p \geq 0\) and \(m \geq 1:\)
\[
\begin{align*}
(i) \quad & x^p \geq x'^p; \\
(ii) \quad & x^p_j \geq x'^p_j, \quad j \in \{1, 2, \ldots, m - 1\} \text{ and } j \in J_i; \\
(iii) \quad & x^p_j \geq x'^p_j, \quad j \in \{1, 2, \ldots, m - 1\}; \\
(iv) \quad & x^p_j \geq x'^p_j, \quad j \in \{1, 2, \ldots, m - 1\}; \\
(v) \quad & x^{p+1}_j \geq x'^{p+1}_j, \quad j \in \{1, 2, \ldots, m - 1\}
\end{align*}
\]
hold for \(i = 1, 2, \ldots, \alpha\). Here we stipulate that (ii)-(v) are vacuous for \(m = 1\). (i)-(v) are clearly valid for \(p = 0\) and \(m = 1\). Observing that for \(j = m\), (iii)-(v) can be trivially deduced from (ii) by directly applying (2.4), (2.7), and (2.3), so we are only required to test the correctness of (ii) for \(j = m\). For this purpose, let us again introduce two number sets
\[
Q_i(p) = \left\{ m \in J_i \mid f_m^{(i)} \left( x^p; x^p_1, \ldots, x^p_m, 0, x^p_{m+1}, \ldots, x^p_n \right) < 0 \right\}, \\
\overline{Q}_i(p) = \left\{ m \in J_i \mid f_m^{(i)} \left( x^p; x^p_1, \ldots, x^p_m, 0, x^p_{m+1}, \ldots, x^p_n \right) \geq 0 \right\}
\]
for \(\forall i \in \{1, 2, \ldots, \alpha\}\) and \(\forall p \in \{0, 1, 2, \ldots\}\) corresponding to the iterative sequence \(\{x'^p\}\). Clearly, there also hold
\[
\begin{align*}
\left\{ Q_i(p) \cap \overline{Q}_i(p) = \emptyset, \quad i = 1, 2, \ldots, \alpha; \quad p = 0, 1, 2, \ldots \right\}, \\
\left\{ Q_i(p) \cup \overline{Q}_i(p) = J_i, \quad i = 1, 2, \ldots, \alpha; \quad p = 0, 1, 2, \ldots \right\}
\end{align*}
\]
Moreover, we can assert that the relations
\[
\left\{ Q_i(p) \subseteq \overline{Q}_i(p), \quad i = 1, 2, \ldots, \alpha; \quad p = 0, 1, 2, \ldots \right\}
\]
hold. In fact, let \(m \in Q'_i(p)\). Then, by the antitonicity of \(f_m^{(i)}(x; y)\) for \(x\) on \(R^n\) and the off-diagonal antitonicity of it for \(y\) on \(R^n\), we have
\[
f_m^{(i)} \left( x^p; x^p_1, \ldots, x^p_m, 0, x^p_{m+1}, \ldots, x^p_n \right) \\
\leq f_m^{(i)} \left( x'^p; x'^p_1, \ldots, x'^p_m, 0, x'^p_{m+1}, \ldots, x'^p_n \right) \\
< 0,
\]
which shows \(m \in Q_i(p)\), too. Therefore, \(Q'_i(p) \subseteq Q_i(p)\). Similarly, we can verify \(\overline{Q}_i(p) \subseteq \overline{Q}_i(p)\).
Presently, if \( m \in Q_i(p) \), in light of the definition of the NMAOR \((r, w)\)-method, we know

\[
\tilde{x}^{p,i}_m = x^{p,i}_m = 0;
\]

if \( m \in Q_i(p) \setminus Q'_i(p) \), the definition of the method gives us

\[
\tilde{x}^{p,i}_m \geq 0 = x^{p,i}_m
\]

again; and if \( m \in Q'_i(p) \), that is to say,

\[
\ldots, \tilde{x}_{m-1}, \tilde{x}_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_n
\]

\[
= \sum_{m-1}^{n} x^{p,i}_m
\]

\[
= f_m^{(i)}(x^{p,i}_1, \ldots, \tilde{x}^{p,i}_{m-1}, \tilde{x}^{p,i}_m, \tilde{x}^{p,i}_{m+1}, \ldots, x^{p,i}_n)
\]

according to the antitonicity of \( f^{(i)}(x; y) \) for \( x \) on \( \mathbb{R}^n \) and the off-diagonal antitonicity of it for \( y \) on \( \mathbb{R}^n \), there hold

\[
\ldots, \ldots = \sum_{m-1}^{n} x^{p,i}_m
\]

\[
f_m^{(i)}(x^{p,i}_1, \ldots, \tilde{x}^{p,i}_{m-1}, \tilde{x}^{p,i}_m, \tilde{x}^{p,i}_{m+1}, \ldots, x^{p,i}_n)
\]

\[
\geq f_m^{(i)}(x^{p,i}_1, \ldots, \tilde{x}^{p,i}_{m-1}, \tilde{x}^{p,i}_m, \tilde{x}^{p,i}_{m+1}, \ldots, x^{p,i}_n)
\]

and therefore,

\[
\tilde{x}^{p,i}_m \geq x^{p,i}_m
\]

is implied by the strictly diagonal isotonicity of \( f^{(i)}(x; y) \) for \( y \) on \( \mathbb{R}^n \).

In conclusion, we have demonstrated that \((ii)\) is true for \( j = m \).

The above discussion indicates that \((ii)-(v)\) hold for all \( m \in \{1, 2, \ldots, n\} \). Therefore, \( x^{p+1} \geq x^{p+1} \). By induction, we have completed the proof of this theorem.

Besides the conditions of Theorem 1, if we also assume that \( F \) is surjective and inverse isotone, we can prove the global convergence of the NMAOR \((r, \omega)\)-method. This fact is precisely stated in the following theorem.

**Theorem 3.** Let \( (f^{(i)}, E_i)(i = 1, 2, \ldots, \alpha) \) be a nonlinear multisplitting of a surjective, inverse isotone function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \). Suppose that for each \( i \in \{1, 2, \ldots, \alpha\} \), \( f^{(i)}(x; y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous, antitone for \( x \) on \( \mathbb{R}^n \), off-diagonally antitone and surjectively diagonally isotone for \( y \) on \( \mathbb{R}^n \). Then, any sequence generated by the NMAOR \((r, \omega)\)-method starting from any \( x^0 \in \mathbb{R}^n \) converges to the unique solution \( x^* \in \mathbb{R}^n \) of the NCP (1.1) provided \((r, \omega) \in [0, 1] \times (0, 1] \).

**Proof.** We first demonstrate that under the conditions of the theorem, \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is an \( M \)-function. To prove this, we only need to verify that \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a \( Z \)-function. As a matter of fact, for \( \forall m, j = (1)n \) with \( m \neq j \), \( \forall x \in \mathbb{R}^n \), and \( \forall t, s \in \mathbb{R}^1 \), if \( t \geq s \), then for some \( i \in \{1, 2, \ldots, \alpha\} \), we have

\[
\psi_{mj}(t) := F_m (x + te^j) = f_m^{(i)}(x + te^j; x + te^j) \leq f_m^{(i)}(x + se^j; x + se^j) = F_m (x + se^j) = \psi_{mj}(s),
\]

which show that \( F \) is a \( Z \)-function. In the above deduction, the antitonicity of \( f^{(i)}(x; y) \) for \( x \) on \( \mathbb{R}^n \) and the off-diagonal antitonicity of \( f^{(i)}(x; y) \) for \( y \) on \( \mathbb{R}^n \) have been considered. Thereafter, we know that the NCP (1.1) has at most one solution.

For a given \( x^0 \in \mathbb{R}^n_+ \), we define

\[
b_m = \max \{F_m (x^0), 0\}, \quad m = 1(1)n,
\]
y_0 = F^{-1}(b).

Then, by the inverse isotonicity of $F$ on $\mathbb{R}^+$, we have

$$F(y_0) \geq 0, \quad y_0 \geq x_0 \geq 0.$$  

Making use of Theorem 1, we immediately see that there is at least one $x^* \in \mathbb{R}_+$ which satisfies (1.1). Hence, the NCP (1.1) has unique solution $x^*$ on $\mathbb{R}_+$. Let $\{x^p\}$ and $\{y^p\}$ be sequences generated by the NMAOR $(r, \omega)$-method starting from $x_0$ and $y_0$, respectively, with the same parameters $r$ and $\omega$. By the surjectively diagonal isotonicity of $f^{(i)}(x; y)$ with respect to $y$ on $\mathbb{R}^n$ for each $i = 1, 2, \ldots, \alpha$, we know that $\{\bar{x}_{n}^{(i)}\}$ and $\{\bar{y}_{n}^{(i)}\}$ ($m = 1(1)n$, $m \in J_i$) are uniquely existing for $i = 1, 2, \ldots, \alpha$ and $p = 0, 1, 2, \ldots$, which indicates that the corresponding NMAOR $(r, \omega)$-method is well defined. From Theorem 1, we get

$$0 \leq y_{p+1} \leq y_p, \quad p = 0, 1, 2, \ldots,$$  

and

$$\lim_{p \to \infty} y_p = x^*.$$  

Now, suppose that for some $p \geq 0$ and $m \geq 1$,

$$(1') \quad 0 \leq x^p \leq y^p;$$

$$(2') \quad 0 \leq x_{j}^{p,i} \leq \bar{x}_{j}^{p,i}, \quad j \in \{1, 2, \ldots, m - 1\} \text{ and } j \in J_i;$$

$$(3') \quad 0 \leq x_{j}^{p,i} \leq \bar{y}_{j}^{p,i}, \quad j \in \{1, 2, \ldots, m - 1\};$$

$$(4') \quad 0 \leq x_{j}^{p,i} \leq y_{j}^{p,i}, \quad j \in \{1, 2, \ldots, m - 1\};$$

$$(5') \quad 0 \leq x_{j}^{p+1} \leq \bar{y}_{j}^{p+1}, \quad j \in \{1, 2, \ldots, m - 1\}$$

hold for $i = 1, 2, \ldots, \alpha$. Here we stipulate that (2')–(5') are vacuous for $m = 1$, so (1')–(5') are clearly valid for $p = 0$ and $m = 1$.

For $p$ fixed, we want now to prove that (2')–(5') is true for $j = m$, too. Notice that for $j = m$, (3')–(5') can be directly obtained from (2') by the definitions of the sequences, so we only need to verify the validity of (2') for $j = m$. For this purpose, let us denote analogously to Theorem 1 by $Q_i^f(p)$, $\overline{Q}_i^f(p)$ and $Q_i^r(p)$, $\overline{Q}_i^r(p)$ the number sets corresponding to the sequences $\{x^p\}$ and $\{y^p\}$, respectively. Similarly, we can demonstrate that there hold the inclusion relations

$$\begin{cases} 
Q_i^f(p) \subseteq Q_i^r(p), \\
\overline{Q}_i^f(p) \subseteq \overline{Q}_i^r(p), \quad i = 1, 2, \ldots, \alpha; \quad p = 0, 1, 2, \ldots. 
\end{cases}$$

If $m \in Q_i^f(p)$, according to the definition of the NMAOR $(r, \omega)$-method, we know

$$\bar{x}_m^{p,i} = \bar{y}_m^{p,i} = 0;$$

if $m \in Q_i^r(p) \setminus Q_i^f(p)$, the definition of the method shows us

$$\bar{y}_m^{p,i} \geq 0 = \bar{x}_m^{p,i}$$

again; and if $m \in Q_i^r(p)$, that is to say,

$$f^{(i)}_m \left( x^p; \bar{x}_m^{p,i}, \ldots, \bar{x}_{m-1}^{p,i}, \bar{x}_m^{p,i}, x_{m+1}^p, \ldots, x_n^p \right) = 0$$

$$= f^{(i)}_m \left( y^p; \bar{y}_m^{p,i}, \ldots, \bar{y}_{m-1}^{p,i}, \bar{y}_m^{p,i}, y_{m+1}^p, \ldots, y_n^p \right),$$

\(\text{if } m \in Q_i^r(p) \setminus Q_i^f(p), \text{ the definition of the method shows us } \)
in light of the antitonicity of $f^{(i)}(x; y)$ for $x$ on $\mathbb{R}^n$ and the off-diagonal antitonicity of it for $y$ on $\mathbb{R}^n$, there hold

\[
 f^{(i)}_m \left( x^p, \bar{\xi}_m^p, \ldots, \bar{\xi}_{m-1}^p, \bar{\xi}_m^p, x^p_{m+1}, \ldots, x^p_n \right) = f^{(i)}_m \left( y^p, \bar{\xi}_m^p, \ldots, \bar{\xi}_{m-1}^p, \bar{\xi}_m^p, y^p_{m+1}, \ldots, y^p_n \right)
\]

\[\leq f^{(i)}_m \left( x^p, \bar{\xi}_m^p, \ldots, \bar{\xi}_{m-1}^p, \bar{\xi}_m^p, x^p_{m+1}, \ldots, x^p_n \right),\]

and thereafter,

\[
\bar{\xi}_m^p \leq \bar{\xi}_m^p
\]

is implied by the strictly diagonal isotonicity of $f^{(i)}(x; y)$ for $y$ on $\mathbb{R}^n$.

In conclusion, we have proven the correctness of (2') for $j = m$.

The previous investigations show that (2')–(5') hold for all $m \in \{1, 2, \ldots, n\}$. Therefore, $0 \leq x^{p+1} \leq y^{p+1}$. By induction, we have proved

\[
0 \leq x^p \leq y^p, \quad p = 0, 1, 2, \ldots \tag{4.7}
\]

Let $\{x^{p_k}\}$ be any subsequence of $\{x^p\}$. If it is convergent, we can verify that we have

\[
\lim_{k \to \infty} x^{p_k} = x^* \tag{4.10}
\]

by a similar demonstration to the proof of Theorem 1(c). Furthermore, we can assert that there is no divergent subsequence of $\{x^p\}$. In fact, assume that $\{x^{p_{k'}}\}$ is a divergent subsequence of $\{x^p\}$. Then there exists $\varepsilon_0 > 0$ such that for all $K > 0$ there holds

\[
|x^{p_{k'}} - x^*| \geq \varepsilon_0 \epsilon
\]

provided $k' \geq K$, where $\epsilon = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$; that is to say, there holds either

\[
x^{p_{k'}} \geq x^* + \varepsilon_0 \epsilon, \quad \forall k' \geq K, \tag{4.8}
\]

or

\[
x^{p_{k'}} \leq x^* - \varepsilon_0 \epsilon, \quad \forall k' \geq K. \tag{4.9}
\]

If (4.8) holds, noticing (4.7), we see that

\[
x^* + \varepsilon_0 \epsilon \leq x^{p_{k'}}.
\]

Making use of (4.6), we can get

\[
x^* + \varepsilon_0 \epsilon \leq x^*,
\]

which is a contradiction.

If (4.9) holds, we easily know that $\{x^{p_{k'}}\}$ is a bounded sequence, and it must include a convergent subsequence $\{x^{p_{k''}}\}$ which is obviously a subsequence of $\{x^p\}$ satisfying

\[
0 \leq x^{p_{k''}} \leq x^* - \varepsilon_0 \epsilon. \tag{4.10}
\]

Remembering that this $\{x^{p_{k''}}\}$ still converges to $x^*$, we can obtain from (4.10) that

\[
x^* \leq x^* - \varepsilon_0 \epsilon,
\]

which is also a contradiction.

Therefore, any subsequence of $\{x^p\}$ converges to $x^*$, which is equivalent to the convergence of $\{x^p\}$ to $x^*$. Up to now, the proof of this theorem is thoroughly completed.
5. SOME REMARKS

In this section, we make several remarks about the NMAOR \((r, \omega)\)-method as well as its convergence conclusions established in the last section.

**Remark 1.** There are examples which show that each assumption in Theorem 3 is indispensable for ensuring the validity of its conclusion. Because of the length of this paper, we will not list them here.

**Remark 2.** For the special nonlinear multisplitting defined by (2.1), (2.2), if we replace the condition '\(f(i)(x; y) (i = 1, 2, \ldots, \alpha)\) are antitone for \(x\) on \(\mathbb{R}^n\) and off-diagonal antitone for \(y\) on \(\mathbb{R}^n\)' by '\(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is off-diagonally antitone on \(\mathbb{R}^n\)', all the conclusions of Theorems 1 and 2 still hold.

**Remark 3.** For the above mentioned special nonlinear multisplitting, assumption '\(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\), a surjective \(M\)-function,' is sufficient for the validity of the conclusion of Theorem 3.

**Remark 4.** All the previous conclusions still hold if for each \(i \in \{1, 2, \ldots, \alpha\}\), the strictness requirement on the diagonal isotonicity of \(f(i)(x; y)\) with respect to \(y\) on \(\mathbb{R}^n\) is removed provided the NMAOR \((r, \omega)\)-method is slightly modified in the case that \(z_{m,i}^p\) is obtained by solving the nonlinear equation (2.6). In the modified method, \(z_{m,i}^p\) is defined to be

\[
\tilde{z}_{m,i}^p = \min \{x_m | x_m \in \mathcal{X}_{m,i}^p\};
\]

in that case, \(\mathcal{X}_{m,i}^p\) is the set of solutions \(x_m\) of the nonlinear equation

\[
f_m^p\left(x_p, z_{m,i}^p, \ldots, z_{m-1,i}^p, x_{m-1}, x_{m+1}, \ldots, x_n\right) = 0.
\]

Note that the solution of this equation may be not unique at this time. The proofs of the conclusions corresponding to the modified method can be fulfilled in the same ways as those to the NMAOR \((r, \omega)\)-method, and so they are omitted here.

6. NUMERICAL RESULTS

In this section, we will imitatively implement the parallel nonlinear multisplitting AOR method, i.e., the NMAOR \((r, \omega)\)-method, by solving the nonlinear complementarity problem

\[
F(x) = Ax + \varphi(x) + b \geq 0, \quad x \geq 0, \quad \text{and} \quad (x, F(x)) = 0,
\]

where

\[
A = \begin{pmatrix}
a_{11} & -1 & & & \\
-1 & a_{22} & -1 & & \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & -1 & a_{nn} \\
\end{pmatrix}, \quad \varphi(x) = \begin{pmatrix}
c_1x_1^3 \\
c_2x_2^3 \\
\vdots \\
c_{n-1}x_{n-1}^3 \\
c_nx_n^3 \\
\end{pmatrix}, \quad b = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_{n-1} \\
b_n \\
\end{pmatrix}.
\]

and

\[
a_{mm} = 2 + \frac{1}{10m^2(n-m+1)^2},
\]

\[
c_m = \frac{1}{50m(n-m+1)},
\]

\[
b_m = \begin{cases}
\frac{8}{5m^2n(n-m+1)}, & \text{for } m \text{ odd}, \\
\frac{2}{m^2n(n-m+1)}, & \text{for } m \text{ even},
\end{cases}
\]

\(m = 1, 2, \ldots, n.\)
We take $\alpha = 2$. The nonlinear multisplitting is given by (2.1), (2.2) with

$$J_1 = \{1, 2, \ldots, m_1\}, \quad J_2 = \{m_2, m_2 + 1, \ldots, n\},$$

and by

$$e_m^{(1)} = \begin{cases} 1.0, & \text{for } 1 \leq m \leq m_2 - 1, \\ 0.5, & \text{for } m_2 \leq m \leq m_1, \\ 0, & \text{for } m_1 + 1 \leq m \leq n, \end{cases}$$

$$e_m^{(2)} = \begin{cases} 0, & \text{for } 1 \leq m \leq m_2 - 1, \\ 0.5, & \text{for } m_2 \leq m \leq m_1, \\ 1.0, & \text{for } m_1 + 1 \leq m \leq n, \end{cases}$$

where $m_1$ and $m_2$ are positive integers which reflect the overlapping between the nonlinear splittings, and are chosen, respectively, according to the following three cases:

(a) $m_1 = \text{Int}(n/2)$, $m_2 = \text{Int}(n/2) + 1$;
(b) $m_1 = \text{Int}(2n/3)$, $m_2 = \text{Int}(n/3)$;
(c) $m_1 = \text{Int}(4n/5)$, $m_2 = \text{Int}(n/5)$.

Here, we use $\text{Int}(\bullet)$ to denote the integer part of the corresponding positive number.

In our computations, all iterations are started with an initial vector $x^0 \in D$ having all components equal to one, and terminated once the current iterations $x^p$ obey

$$\text{Res}^p := (Ax^p + \gamma(x^p) + b)^T x^p \leq 10^{-5}.$$ 

The solution of the nonlinear equation (2.6) is approximately obtained by one step of the Newton program.

### Table 1. NMAOR (0.94,0.98)-method (Case a).

<table>
<thead>
<tr>
<th>$p$</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
<th>28</th>
<th>32</th>
<th>36</th>
<th>39</th>
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<tbody>
<tr>
<td>$100 \times x^p$</td>
<td>14.7529</td>
<td>6.8730</td>
<td>2.5123</td>
<td>0.1258</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
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<tr>
<td>$100 \times x^p$</td>
<td>29.0854</td>
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<td>2.3479</td>
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<td>1.4414</td>
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<td>35.5212</td>
<td>18.9575</td>
<td>9.8888</td>
<td>4.9276</td>
<td>2.6087</td>
<td>1.5855</td>
<td>1.336</td>
<td>0.9308</td>
<td>0.8566</td>
</tr>
<tr>
<td>$100 \times x^p$</td>
<td>36.3144</td>
<td>19.7105</td>
<td>10.6408</td>
<td>5.6796</td>
<td>3.1232</td>
<td>1.9600</td>
<td>1.4702</td>
<td>1.2429</td>
<td>1.1597</td>
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<tr>
<td>$100 \times x^p$</td>
<td>25.9474</td>
<td>14.0604</td>
<td>7.5757</td>
<td>4.0287</td>
<td>2.1754</td>
<td>1.3351</td>
<td>0.9602</td>
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<td>0.7319</td>
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<td>$100 \times x^p$</td>
<td>13.5498</td>
<td>7.5542</td>
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<td>2.4953</td>
<td>1.5589</td>
<td>1.1336</td>
<td>0.9438</td>
<td>0.8592</td>
<td>0.8283</td>
</tr>
<tr>
<td>$100 \times \text{Res}^p$</td>
<td>9.3464</td>
<td>2.7202</td>
<td>0.7715</td>
<td>0.2077</td>
<td>0.0515</td>
<td>0.0148</td>
<td>0.0050</td>
<td>0.0019</td>
<td>0.0010</td>
</tr>
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</table>

### Table 2. NMAOR (0.94,0.98)-method (Case b).

<table>
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<th>$p$</th>
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<th>8</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$100 \times x^p$</td>
<td>28.9070</td>
<td>14.6107</td>
<td>5.7513</td>
<td>1.3642</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$100 \times x^p$</td>
<td>55.1980</td>
<td>27.9967</td>
<td>13.0852</td>
<td>5.7788</td>
<td>2.5345</td>
<td>1.6689</td>
<td>1.3738</td>
<td>1.2712</td>
</tr>
<tr>
<td>$100 \times x^p$</td>
<td>65.3900</td>
<td>31.3969</td>
<td>14.6486</td>
<td>6.5370</td>
<td>2.7629</td>
<td>1.4584</td>
<td>1.0078</td>
<td>0.8511</td>
</tr>
<tr>
<td>$100 \times x^p$</td>
<td>62.0839</td>
<td>29.1286</td>
<td>13.8994</td>
<td>6.5556</td>
<td>3.0815</td>
<td>1.7630</td>
<td>1.3044</td>
<td>1.1449</td>
</tr>
<tr>
<td>$100 \times x^p$</td>
<td>46.0868</td>
<td>21.3026</td>
<td>10.1306</td>
<td>4.7667</td>
<td>2.1983</td>
<td>1.1951</td>
<td>0.8452</td>
<td>0.7234</td>
</tr>
<tr>
<td>$100 \times x^p$</td>
<td>23.8121</td>
<td>11.2332</td>
<td>5.5842</td>
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<td>$100 \times \text{Res}^p$</td>
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<td>1.6181</td>
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<td>0.0565</td>
<td>0.0113</td>
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<td>0.0099</td>
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</table>
We have done numerous numerical experiments through choosing different $n$, the dimension of the problem, and different pairs of the relaxation parameters $r$ and $\omega$. All our numerical results are satisfactory and closely coincide with the theory. Because of the length of the paper, we just list the iteration data for $n = 6$ corresponding to the three cases of the nonlinear multisplittings and some pairs $(r, \omega)$ of the relaxation parameters in the following tables. For convenience, we just list several decimal parts of our numerical results which are sufficient for us to illustrate the monotonicity of our new method.

**Table 3. NMAOR (0.94,0.98)-method (Case c).**

<table>
<thead>
<tr>
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<th>33</th>
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</thead>
<tbody>
<tr>
<td>100 $x_1^p$</td>
<td>34.9479</td>
<td>18.6078</td>
<td>8.0077</td>
<td>2.6301</td>
<td>0.1397</td>
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<td>0.0000</td>
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<tr>
<td>100 $x_2^p$</td>
<td>55.9163</td>
<td>29.7298</td>
<td>14.4046</td>
<td>6.6694</td>
<td>2.8942</td>
<td>1.7610</td>
<td>1.0457</td>
<td>1.2823</td>
<td>1.2670</td>
</tr>
<tr>
<td>100 $x_3^p$</td>
<td>64.5539</td>
<td>32.5382</td>
<td>15.7668</td>
<td>7.3554</td>
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<td>0.8447</td>
</tr>
<tr>
<td>100 $x_4^p$</td>
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<td>29.8834</td>
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<td>7.2354</td>
<td>3.4845</td>
<td>1.9054</td>
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<td>1.1622</td>
<td>1.1384</td>
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<tr>
<td>100 $x_5^p$</td>
<td>45.2929</td>
<td>21.8345</td>
<td>10.7718</td>
<td>5.2599</td>
<td>2.5015</td>
<td>1.3037</td>
<td>0.8830</td>
<td>0.7366</td>
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<tr>
<td>100 $x_6^p$</td>
<td>23.3953</td>
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<td>0.8308</td>
<td>0.8216</td>
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<tr>
<td>100 $\text{Res}^p$</td>
<td>31.1333</td>
<td>8.3082</td>
<td>1.9671</td>
<td>0.4304</td>
<td>0.0785</td>
<td>0.0147</td>
<td>0.0037</td>
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**Table 4. NMAOR (0.90,0.96)-method (Case d).**

<table>
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<th>28</th>
<th>32</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 $x_1^p$</td>
<td>35.4120</td>
<td>19.6117</td>
<td>8.9722</td>
<td>3.3434</td>
<td>0.4358</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>100 $x_2^p$</td>
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<td>31.4984</td>
<td>15.9535</td>
<td>7.7787</td>
<td>3.5359</td>
<td>1.9732</td>
<td>1.4963</td>
<td>1.3204</td>
<td>1.2659</td>
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<tr>
<td>100 $x_3^p$</td>
<td>66.1721</td>
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<td>1.1994</td>
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<td>0.8438</td>
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<tr>
<td>100 $x_4^p$</td>
<td>63.0093</td>
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<td>8.4365</td>
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<td>2.2566</td>
<td>1.5042</td>
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</tr>
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<td>12.1133</td>
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<td>1.5810</td>
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**REFERENCES**