New estimates for the div, curl, grad operators and elliptic problems with $L^1$-data in the half-space

Chérif Amrouche$^{a, *}$, Huy Hoang Nguyen$^{b}$

$^a$ Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l’Adour, IPRA, Avenue de l’Université, 64000 Pau, France
$^b$ Departamento de Matemática, IMECC, Universidade Estadual de Campinas, Caixa Postal 6065, Campinas, SP 13083-970, Brazil

**Abstract**

In this Note, we study some properties of the div, curl, grad operators and elliptic problems in the half-space. We consider data in weighted Sobolev spaces and in $L^1$.

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**1. Introduction**

The purpose of this Note is to give some estimates for div, curl, grad operators and elliptic problems with $L^1$-data in the half-space. We know that, if $f \in L^2(\mathbb{R}^n_+)$, there exists $u \in \mathcal{W}^{1,n}(\mathbb{R}^n_+)$ such that $\text{div} u = f$ holds, but does $u \in \mathcal{W}^{1,n}(\mathbb{R}^n_+)$ hold? It does indeed. In Section 2, we give new estimates for $L^1$-vector fields, which improve estimates for the solutions of elliptic systems in $\mathbb{R}^n_+$ with data in $L^1$.

In this Note, we use bold-type characters to denote vector distributions or spaces of vector distributions with $n$ components, and $C > 0$ usually denotes a generic constant (the value of which may change from line to line). For any $q \in \mathbb{N}$, $\mathcal{P}_q$ (respectively, $\mathcal{P}_q^\Delta$) stands for the space of polynomials (respectively harmonic polynomials) of degree $\leq q$. If $q$ is a strictly negative integer, we set by convention $\mathcal{P}_q = \{0\}$. Let $\Omega$ be an open subset in the $n$-dimensional real Euclidean space. A typical point in $\mathbb{R}^n$ is denoted by $x = (x_1, \ldots, x_n)$, and its norm is given by $r = |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. We define the weight function $\rho(x) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent $p'$ is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$.

We now define the weighted Sobolev space $\mathcal{W}^{1,p}_0(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_1} \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$, where $w_1 = 1 + r$ if $p \neq n$ and $w_1 = (1 + r) \ln(2 + r)$ if $p = n$. This space is a reflexive Banach space when endowed with the norm

$$
\|u\|_{\mathcal{W}^{1,p}_0(\Omega)} = \left(\frac{\|u\|_{L^p(\Omega)}}{w_1} + \|\nabla u\|_{L^p(\Omega)}\right)^{\frac{1}{p}}.
$$

We also introduce the space $\mathcal{W}^{2,p}_0(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_2} \in L^p(\Omega), \frac{\nabla u}{w_2} \in L^p(\Omega), \frac{\Delta u}{w_2} \in L^p(\Omega)\}$, where $w_2 = (1 + r)^2$ if $p \notin \left\{\frac{n}{n-2}, n\right\}$, and $w_2 = (1 + r)^2 \ln(2 + r)$ in the remaining case. This space is a reflexive Banach space endowed with its natural
norm given by
\[
\|u\|_{W_0^{2,p}(\Omega)} = \left(\left\|\frac{u}{w_1}\right\|_{L^p(\Omega)}^p + \left\|\frac{\nabla u}{w_1}\right\|_{L^p(\Omega)}^p + \|D^2 u\|_{L^p(\Omega)}^p\right)^{1/p}.
\]

We note that the logarithmic weight only appears if \(p = n\) or \(p = \frac{n}{n-1}\) and all the local properties of \(W_0^{1,p}(\Omega)\) (respectively, \(W_0^{2,p}(\Omega)\)) coincide with those of the corresponding classical Sobolev space \(W^{1,p}(\Omega)\) (respectively, \(W^{2,p}(\Omega)\)). For \(m = 1\) or \(m = 2\), we set \(W_0^{m,p}(\Omega) = D(\Omega)\) and we denote the dual space of \(W_0^{m,p}(\Omega)\) by \(W_0^{-m,p}(\Omega)\), which is the space of distributions. When \(\Omega = \mathbb{R}^n\), we have \(W_0^{m,p}(\mathbb{R}^n) = W_0^{-m,p}(\mathbb{R}^n)\) (see [1] for more details).

2. The \(\text{div}, \text{grad}, \text{curl}\) operators and elliptic systems

First of all, we set \(B_{2^+}^a = \{x \in \mathbb{R}^n_+; |x| < a\}\) with \(a > 0\). In this section, we consider the case \(n \geq 2\). We introduce the following theorem.

Theorem 2.1. Let \(f \in L^n(\mathbb{R}^n_+)\). Then there exists \(u \in W_0^{1,n}(\mathbb{R}^n_+) \cap L^{\infty}(\mathbb{R}^n_+)\) such that \(\text{div} u = f\). Moreover, we have the following estimate:
\[
\|u\|_{L^{\infty}(\mathbb{R}^n_+)} + \|u\|_{W_0^{1,n}(\mathbb{R}^n_+)} \leq C\|f\|_{L^n(\mathbb{R}^n_+)}. \tag{1}
\]

Proof. Let \((f_k)_{k \in \mathbb{N}} \subset D(\mathbb{R}^n_+)\) converge towards \(f \in L^n(\mathbb{R}^n_+)\) and \(B_{2^+}^a \subset \mathbb{R}^n_+\) such that \(\text{supp} f_k \subset B_{2^+}^a\). We set \(g(x) = r_k f(r_k x)\). Thanks to Theorem 3 of [2], there exists \(v \in W_0^{1,n}(B_{2^+}^a) \cap L^{\infty}(B_{2^+}^a)\) satisfying \(\text{div} v = g\) in \(B_{2^+}^a\) and the following estimate:
\[
\|v\|_{L^{\infty}(B_{2^+}^a)} + \|\nabla v\|_{L^n(B_{2^+}^a)} \leq C\|g\|_{L^n(B_{2^+}^a)}.
\]

We now set \(u_k(x) = v\left(\frac{x}{r_k}\right)\) with \(x \in B_{2^+}^a\). It is easy to show that \(u_k \in W_0^{1,n}(B_{2^+}^a) \cap L^{\infty}(B_{2^+}^a)\) satisfies \(\text{div} u_k = f_k\), and we have the following estimate:
\[
\|u_k\|_{L^{\infty}(B_{2^+}^a)} + \|\nabla u_k\|_{L^n(B_{2^+}^a)} \leq C\|f_k\|_{L^n(B_{2^+}^a)} \leq C\|f_k\|_{L^n(B_{2^+}^a)}.
\]

The conclusion of Theorem 2.1 is equivalent to the estimate
\[
\forall u \in L^{n/(n-1)}(\mathbb{R}^n_+), \quad \|u\|_{L^{n/(n-1)}(\mathbb{R}^n_+)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^n_+)}^n + W_0^{-1,n/(n-1)}(\mathbb{R}^n_+). \tag{2}
\]

Indeed, let us consider the following unbounded operator:
\[
A = -\nabla : L^{n/(n-1)}(\mathbb{R}^n_+) \to L^1(\mathbb{R}^n_+) + W_0^{-1,n/(n-1)}(\mathbb{R}^n_+).
\]

Then the domain \(D(A)\) is dense in \(L^{n/(n-1)}(\mathbb{R}^n_+)\) and \(A\) is closed. Because the adjoint operator
\[
A^* = \text{div} : W_0^{1,n}(\mathbb{R}^n_+) \cap L^{\infty}(\mathbb{R}^n_+) \to L^n(\mathbb{R}^n_+)
\]
is surjective, we deduce the estimate (2).

Thanks to (2), the range \(A\) is closed in \(L^1(\mathbb{R}^n_+) + W_0^{-1,n/(n-1)}(\mathbb{R}^n_+)\). Then \((\text{Ker} A^*)^\perp = \text{Im} A\), and as a consequence we obtain the following version of De Rham’s Theorem: for any \(f\) belonging to the space \(L^1(\mathbb{R}^n_+) + W_0^{-1,n/(n-1)}(\mathbb{R}^n_+)\) and satisfying the following compatibility condition:
\[
\forall v \in W_0^{1,n}(\mathbb{R}^n_+) \cap L^{\infty}(\mathbb{R}^n_+), \quad \text{with div } v = 0, (f, v) = 0,
\]
there exists a unique \(\pi \in L^{n/(n-1)}(\mathbb{R}^n_+)\) such that \(f = \nabla \pi\).

The following theorem, given without proof, is proved by Bourgain and Brézis [3] (see also [4]) in the case where \(\mathbb{R}^n_+\) is replaced by a bounded smooth domain \(\Omega\).
Theorem 2.2. (i) Let $\varphi \in W_0^{1,n}(\mathbb{R}_+^n)$. Then there exists $\psi = (\psi_1, \ldots, \psi_n) \in W_0^{1,n}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}_+^n)$ such that $\psi_n = 0$ on $\Gamma = \mathbb{R}^{n-1} \times \{0\}$, satisfying
\[ \varphi = \psi + \nabla \eta, \]
and the following estimate holds:
\[ \|\psi\|_{L^\infty(\mathbb{R}_+^n)} + \|\psi\|_{W_0^{1,n}(\mathbb{R}_+^n)} + \|\eta\|_{W_0^{2,n}(\mathbb{R}_+^n)} \leq C \|\varphi\|_{W_0^{1,n}(\mathbb{R}_+^n)}. \] (4)

(ii) If $\varphi \in W_0^{1,n}(\mathbb{R}_+^n)$, then the same result holds with $\psi \in W_0^{1,n}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ and $\eta \in W_0^{2,n}(\mathbb{R}_+^n)$ and the corresponding estimate.

Define the space $X(\mathbb{R}_+^n) = \{v \in L^1(\mathbb{R}_+^n); \text{div } v \in W_0^{-2,1/(n-1)}(\mathbb{R}_+^n)\}$. Let $f \in X(\mathbb{R}_+^n)$. Thanks to Theorem 2.2, the linear operator $F : \varphi \mapsto \int_{\mathbb{R}_+^n} f \cdot \varphi \, dx$ satisfies
\[ \forall \varphi \in \mathcal{D}(\mathbb{R}_+^n), \quad |(F, \varphi)| \leq C \|f\|_{X(\mathbb{R}_+^n)}. \]

As $\mathcal{D}(\mathbb{R}_+^n)$ is dense in $W_0^{1,n}(\mathbb{R}_+^n)$, and by applying Hahn–Banach Theorem, we can uniquely extend $F$ by an element $\tilde{F} \in W_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ satisfying $\|\tilde{F}\|_{W_0^{-1,n/(n-1)}(\mathbb{R}_+^n)} \leq C \|f\|_{X(\mathbb{R}_+^n)}$. Besides, the linear operator $f \mapsto \tilde{F}$ from $X(\mathbb{R}_+^n)$ into $W_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ is continuous and injective. Therefore, $X(\mathbb{R}_+^n)$ can be identified to a subspace of $W_0^{-1,n/(n-1)}(\mathbb{R}_+^n)$ with continuous and dense embedding. Thanks to [5], we can then deduce that, for any $f \in X(\mathbb{R}_+^n)$, the problem
\[ -\Delta u = f \quad \text{in } \mathbb{R}_+^n \quad \text{and} \quad u = 0 \quad \text{on } \Gamma, \] (5)
has a unique solution $u \in W_0^{1,n/(n-1)}(\mathbb{R}_+^n)$ satisfying the following estimate: $\|u\|_{W_0^{1,n/(n-1)}(\mathbb{R}_+^n)} \leq C \|f\|_{X(\mathbb{R}_+^n)}$.

The first statement of the following corollary is the equivalent of Theorem 2.1 for the curl operator, and the second statement gives a Helmholtz decomposition.

Corollary 2.3. (i) Let $f \in L^1(\mathbb{R}_+^n)$ such that $\text{div } f = 0$ in $\mathbb{R}_+^n$ and $f_2 = 0$ on $\Gamma$. Then there exists $\psi \in W_0^{1,3}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ such that $f = \text{curl } \psi$ and we have the following estimate:
\[ \|\psi\|_{W_0^{1,3}(\mathbb{R}_+^n)} + \|\psi\|_{L^\infty(\mathbb{R}_+^n)} \leq C \|f\|_{L^1(\mathbb{R}_+^n)}. \] (6)

(ii) Let $f \in L^3(\mathbb{R}_+^n)$. Then there exist $\varphi \in W_0^{1,3}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ and $\pi \in W_0^{1,3}(\mathbb{R}_+^n)$ unique up to an additive constant and satisfying
\[ f = \text{curl } \varphi + \nabla \pi. \] (7)
Moreover, we have the following estimate:
\[ \|\varphi\|_{W_0^{1,3}(\mathbb{R}_+^n)} + \|\varphi\|_{L^\infty(\mathbb{R}_+^n)} + \|\nabla \pi\|_{L^3(\mathbb{R}_+^n)} \leq C \|f\|_{L^3(\mathbb{R}_+^n)}. \] (8)

In the following proposition, we improve the estimate given by Corollary 1.4 of Van Schaftingen [6].

Proposition 2.4. Let $f \in L^1(\mathbb{R}_+^n)$ such that $\text{div } f = 0$. Then we have the following estimate:
\[ \forall \varphi \in W_0^{1,3}(\mathbb{R}_+^n), \quad |(F, \varphi)| \leq C \|f\|_{L^1(\mathbb{R}_+^n)} \|\text{curl } \varphi\|_{L^3(\mathbb{R}_+^n)}. \] (9)

Proof. First, note that, from the hypothesis, we deduce that $f \in W_0^{-1,3/2}(\mathbb{R}_+^n)$. Let $\varphi \in W_0^{1,3}(\mathbb{R}_+^n)$. Then we have $\text{curl } \varphi \in L^3(\mathbb{R}_+^n)$. Thanks to Corollary 2.3, there exists $\psi \in W_0^{1,3}(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ such that $\text{curl } \psi = \text{curl } \varphi$ with the following estimate:
\[ \|\psi\|_{W_0^{1,3}(\mathbb{R}_+^n)} + \|\psi\|_{L^\infty(\mathbb{R}_+^n)} \leq C \|\text{curl } \varphi\|_{L^3(\mathbb{R}_+^n)}. \] (10)

Besides, there exists $\eta \in W_0^{2,3}(\mathbb{R}_+^n)$ such that $\varphi = \psi + \nabla \eta$ in $\mathbb{R}_+^n$. Then we have
\[ (f, \varphi)_{W_0^{-1,3/2}(\mathbb{R}_+^n) \times W_0^{1,3}(\mathbb{R}_+^n)} = \int_{\mathbb{R}_+^n} f \cdot \psi + (f, \nabla \eta)_{W_0^{-1,3/2}(\mathbb{R}_+^n) \times W_0^{1,3}(\mathbb{R}_+^n)} = \int_{\mathbb{R}_+^n} f \cdot \psi. \]
Therefore, the estimate (9) is deduced from the estimate (10). $\square$
Here is a variant of Theorem 2.2.

**Theorem 2.5.** Let \( \varphi \in W^{1,3}_0(\mathbb{R}^3_+) \). Then there exist \( \psi \in W^{1,3}_0(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+) \) and \( \eta \in W^{2,3}_0(\mathbb{R}^3_+) \) such that

\[
\varphi = \psi + \text{curl} \, \eta \quad \text{with} \quad \text{div} \, \Delta^2 \eta = 0 \quad \text{in} \quad \mathbb{R}^3_+.
\]

**Proof.** From the hypothesis, we have \( \text{div} \, \varphi \in L^1(\mathbb{R}^3_+) \). Thanks to Theorem 2.1, there exists \( \psi \in W^{1,3}_0(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+) \) such that \( \text{div} \, \psi = \text{div} \, \varphi \) and we have the following estimate:

\[
\| \psi \|_{W^{1,3}_0(\mathbb{R}^3_+)} + \| \psi \|_{L^\infty(\mathbb{R}^3_+)} + \| \eta \|_{W^{2,3}_0(\mathbb{R}^3_+)} \leq C \| \varphi \|_{W^{1,3}_0(\mathbb{R}^3_+)}.\]

We set \( \varphi = \psi - \eta \); then \( \varphi \in W^{1,3}_0(\mathbb{R}^3_+) \) and \( \text{div} \, \varphi = 0 \). We extend \( \varphi \) to \( \mathbb{R}^3 \) as follows:

\[
\tilde{\varphi}(x', x_3) = \begin{cases} 
\varphi(x', x_3) & \text{if } x_3 > 0, \\
\varphi(x', -x_3) & \text{if } x_3 < 0.
\end{cases}
\]

Then we have \( \varphi \in W^{1,3}_0(\mathbb{R}^3 \cup \mathbb{R}^3_+) \) and \( \text{div} \, \varphi = 0 \). It is easy to show that \( \text{curl} \, \varphi = \text{curl} \, \tilde{\varphi} \). Therefore, we deduce that \( \Delta^2 \varphi = 0 \), and hence \( \varphi = \text{curl} \, \tilde{\varphi} \). Thus, we have the following estimate:

\[
\| \varphi \|_{L^1(\mathbb{R}^3_+)} \leq C \| \varphi \|_{L^1(\mathbb{R}^3 \cup \mathbb{R}^3_+)}.
\]

We introduce the following proposition.

**Proposition 2.6.** Let \( f \in L^1(\mathbb{R}^3_+) \) such that \( \text{div} \, f = 0 \) in \( \mathbb{R}^3_+ \). Then there exists a unique \( \varphi \in L^{3/2}(\mathbb{R}^3_+) \) such that \( \text{curl} \, \varphi = f \), \( \text{div} \, \varphi = 0 \) in \( \mathbb{R}^3_+ \) and \( \varphi_3 = 0 \) on \( \Gamma' \), satisfying the following estimate:

\[
\| \varphi \|_{L^{3/2}(\mathbb{R}^3_+)} \leq C \| f \|_{L^1(\mathbb{R}^3_+)}.\]

**Proof.** It is easy to try to prove the existence of \( \varphi \). We know that \( f \in W^{-1,3/2}(\mathbb{R}^3_+) \). Then, there exists a unique \( \varphi \in W^{1,3/2}(\mathbb{R}^3_+) \) such that \( \text{curl} \, \varphi = f \), \( \text{div} \, \varphi = 0 \) in \( \mathbb{R}^3_+ \) and \( \varphi_3 = 0 \) on \( \Gamma' \). The function \( \varphi = \text{curl} \, \varphi \) is the required function. \( \Box \)

A variant of this result can be obtained. If \( f \in L^1(\mathbb{R}^3_+) \) such that \( \text{div} \, f = 0 \) in \( \mathbb{R}^3_+ \) and \( f_3 = 0 \) on \( \Gamma' \), then we can prove the existence of a unique \( \varphi \in L^{3/2}(\mathbb{R}^3_+) \) such that \( \text{curl} \, \varphi = f \) with \( \text{div} \, \varphi = 0 \) in \( \mathbb{R}^3_+ \) and \( \varphi_3 = 0 \) on \( \Gamma' \).

More generally, we can prove the following result.

**Theorem 2.7.** (i) Let \( f \in L^1(\mathbb{R}^3_+) + W^{-1,3/2}(\mathbb{R}^3_+) \) such that \( \text{div} \, f = 0 \). Then there exists a unique \( \varphi \in L^{3/2}(\mathbb{R}^3_+) \) such that \( \text{curl} \, \varphi = f \) with \( \text{div} \, \varphi = 0 \) in \( \mathbb{R}^3_+ \) and \( \varphi_3 = 0 \) on \( \Gamma' \), satisfying the following estimate:

\[
\| \varphi \|_{L^{3/2}(\mathbb{R}^3_+)} \leq C \| f \|_{L^1(\mathbb{R}^3_+) + W^{-1,3/2}(\mathbb{R}^3_+)}.\]

(ii) Let \( f \in X(\mathbb{R}^3_+) \). Then there exists a unique \( \varphi \in L^{3/2}(\mathbb{R}^3_+) \) such that \( \text{curl} \, \varphi = 0 \) with \( \varphi_3 = 0 \) on \( \Gamma' \) and a unique \( \pi \in L^{3/2}(\mathbb{R}^3_+) \) satisfying \( \varphi = \text{curl} \, \varphi + \nabla \pi \) with the corresponding estimate.
Combining the above results with [8,5], we can solve the following elliptical systems.

**Theorem 2.8.** (i) Let \( g' \in L^1(\Gamma) \) and \( g_n \in W_0^{-1+\frac{2}{n},\frac{1}{n}}(\Gamma) \) satisfy the compatibility condition \( \int_{\Gamma} g' = 0 \) and \( \langle g_n, 1 \rangle = 0 \). If \( \text{div}'g' \in W_0^{-2+\frac{1}{n},\frac{n}{n}}(\Gamma) \), then the system

\[
- \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n_+ \quad \text{and} \quad u = g \quad \text{on} \quad \Gamma
\]

has a unique very weak solution \( u \in L^{n/(n-1)}(\mathbb{R}^n_+) \).

(ii) Let \( f \in L^1(\mathbb{R}^n_+) \), \( g' \in L^1(\Gamma) \) and \( g_n \in W_0^{-1+\frac{2}{n},\frac{1}{n}}(\Gamma) \) satisfy the compatibility condition \( \int_{\mathbb{R}^n_+} f' + \int_{\Gamma} g' = 0 \) and \( \int_{\mathbb{R}^n_+} f_n + \langle g_n, 1 \rangle = 0 \). If

\[
[f, g'] = \sup_{\xi \in W_0^{0,n}(\mathbb{R}^n_+), \xi \neq 0} \frac{\left| \int_{\mathbb{R}^n_+} f \cdot \nabla \xi + \int_{\Gamma} g' \cdot \nabla \xi \right|}{\| \xi \|_{W_0^{2,n}(\mathbb{R}^n_+)}} < \infty,
\]

then the system

\[
- \Delta u = f \quad \text{in} \quad \mathbb{R}^n_+ \quad \text{and} \quad \frac{\partial u}{\partial \nu_n} = g \quad \text{on} \quad \Gamma
\]

has a unique solution \( u \in W_0^{1,n/(n-1)}(\mathbb{R}^n_+) \).

(iii) Let \( f \in L^1(\mathbb{R}^n_+) \) such that \( \text{div}f \in [W_0^{2,n}(\mathbb{R}^n_+) \cap W_0^{-1,n}(\mathbb{R}^n_+)'] \) and \( \int_{\mathbb{R}^n_+} f_n = 0 \). Then \( f \in W_0^{-1,n/(n-1)}(\mathbb{R}^n_+) \), and the system

\[
- \Delta u = f \quad \text{in} \quad \mathbb{R}^n_+; \quad u' = 0 \quad \text{and} \quad \frac{\partial u_n}{\partial \nu_n} = 0 \quad \text{on} \quad \Gamma
\]

has a unique solution \( u \in W_0^{1,n/(n-1)}(\mathbb{R}^n_+) \).

(iv) Let \( f \in L^1(\mathbb{R}^n_+) \) such that \( \int_{\mathbb{R}^n_+} f' = 0 \). If

\[
[f] = \sup_{\xi \in D(\mathbb{R}^n_+), \frac{\partial \xi}{\partial \nu_n} = 0 \text{on} \Gamma} \frac{\left| \int_{\mathbb{R}^n_+} f \cdot \nabla \xi \right|}{\| \xi \|_{W_0^{2,n}(\mathbb{R}^n_+)}} < \infty
\]

holds, then the system

\[
- \Delta u = f \quad \text{in} \quad \mathbb{R}^n_+; \quad u_n = 0 \quad \text{and} \quad \frac{\partial u'}{\partial \nu_n} = 0 \quad \text{on} \quad \Gamma
\]

has a unique solution \( u \in W_0^{1,n/(n-1)}(\mathbb{R}^n_+) \).

**Proof.** We will prove only the first point (i).

**Step 1:** The proof is started by showing that \( g' \in W_0^{-1+\frac{1}{n},\frac{2}{n}}(\Gamma) \). Let \( \mu \in D(\Gamma) \) and \( \varphi \in W_0^{1,n}(\mathbb{R}^n_+) \) such that \( \varphi = \mu \) on \( \Gamma \). Thanks to Theorem 2.2, there exist \( \psi \in W_0^{1,n}(\mathbb{R}^n_+) \cap \Omega_0^{\infty}(\mathbb{R}^n_+) \) and \( \eta \in W_0^{2,n}(\mathbb{R}^n_+) \) such that \( \psi_n = 0 \) on \( \Gamma \), satisfying \( \varphi = \psi + \nabla \eta \) and the estimate

\[
\| \psi \|_{L_0^{\infty}(\mathbb{R}^n_+)} + \| \psi \|_{W_0^{1,n}(\mathbb{R}^n_+)} + \| \eta \|_{W_0^{2,n}(\mathbb{R}^n_+)} \leq C \| \varphi \|_{W_0^{1,n}(\mathbb{R}^n_+)},
\]

Then

\[
\langle g, \mu \rangle_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} = \int_{\Gamma} g' \psi - \langle \text{div}'g', \eta \rangle_{W_0^{-2+\frac{1}{n},\frac{n}{n}}(\Gamma) \times W_0^{-1+\frac{2}{n},\frac{1}{n}}(\Gamma)} + \langle g_n, \varphi_n \rangle_{W_0^{-1+\frac{1}{n},\frac{n}{n}}(\Gamma) \times W_0^{-\frac{1}{n},\frac{n}{n}}(\Gamma)}
\]

and

\[
\| g \|_{\mathcal{D}'(\Gamma) \times \mathcal{D}(\Gamma)} \leq \| g' \|_{L_0^1(\Gamma)} \| \psi \|_{L_0^{\infty}(\mathbb{R}^n_+)} + \| \text{div}'g' \|_{W_0^{-2+\frac{1}{n},\frac{n}{n}}(\Gamma)} \| \eta \|_{W_0^{2,n}(\mathbb{R}^n_+)} + \| g_n \|_{W_0^{-1+\frac{1}{n},\frac{n}{n}}(\Gamma)} \| \varphi_n \|_{W_0^{-\frac{1}{n},\frac{n}{n}}(\Gamma)}.
\]

Thanks to the density of \( D(\Gamma) \) in \( W_0^{-1+\frac{1}{n},\frac{n}{n}}(\Gamma) \) and (15), we can deduce that \( g' \in W_0^{-1+\frac{1}{n},\frac{n}{n}}(\Gamma) \).

**Step 2:** System (13) is equivalent to the following one. Find \( u \) belonging \( L^{n/(n-1)}(\mathbb{R}^n_+) \) such that, for all \( v \in W_0^{2,n}(\mathbb{R}^n_+) \cap \mathcal{W}^{1,n}_-(\mathbb{R}^n_+) \),

\[
\int_{\mathbb{R}^n_+} u \cdot \Delta v = - \left( g \cdot \frac{\partial v}{\partial \nu} \right)_{W_0^{-1+\frac{1}{n},\frac{n}{n}}(\Gamma) \times W_0^{-1+\frac{1}{n},\frac{n}{n}}(\Gamma)}.
\]
But, for all $F \in L^n(\mathbb{R}^n)$, there exists $v \in W^{2,n}_0(\mathbb{R}^n) \cap \overset{\circ}{W}^{1,n}_0(\mathbb{R}^n)$, unique up to an element of $x_n\mathbb{R}^n$, such that $-\Delta v = F$ in $\mathbb{R}^n$, $v = 0$ on $\Gamma$ and the following estimate holds:

$$\|v\|_{W^{2,n}_0(\mathbb{R}^n)} \le C\|F\|_{L^n(\mathbb{R}^n)}.$$ 

Then, we have, for all $a \in \mathbb{R}^n$,

$$\left\| \left( g, \frac{\partial v}{\partial x_n} \right)_{W^{0,-1+\frac{n}{n-1}}_0(\Gamma) \times W^{1,-1}^n(\Gamma)} \right\| \le C\|g\|_{W^{0,-1+\frac{n}{n-1}+\frac{n}{n}}(\Gamma)} \|v + ax_n\|_{W^{2,n}_0(\mathbb{R}^n)}.$$ 

Consequently, taking the infimum, we have

$$\left\| \left( g, \frac{\partial v}{\partial x_n} \right)_{W^{0,-1+\frac{n}{n-1}}_0(\Gamma) \times W^{1,-1}^n(\Gamma)} \right\| \le C\|g\|_{W^{0,-1+\frac{n}{n-1}+\frac{n}{n}}(\Gamma)} \|F\|_{L^n(\mathbb{R}^n)}.$$ 

Then, the linear operator

$$T : F \rightarrow \left( g, \frac{\partial v}{\partial x_n} \right)_{W^{0,-1+\frac{n}{n-1}}_0(\Gamma) \times W^{1,-1}^n(\Gamma)}$$

is continuous on $L^n(\mathbb{R}^n)$ and, thanks to the Riesz representation theorem, there exists a unique $u \in L^{n/(n-1)}(\mathbb{R}^n)$ such that $T(F) = \int_{\mathbb{R}^n} u \cdot F$, i.e., $u$ is the solution of (13).  

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References


