Generalized associated polynomials and functions of second kind

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Abstract

Let \( \{ p_v \}_{v \in \mathbb{N}_0}, \ p_v \in \Pi_v \setminus \Pi_{v-1}, \) be a sequence of polynomials, generated by a three-term recurrence relation.

Shifting the recurrence coefficients of the elements of \( \{ p_v \}_{v \in \mathbb{N}_0} \) we get a sequence of so-called associated polynomials, which play an important role in the theory of orthogonal polynomials. We generalize this concept of associating for arbitrary polynomials \( v_n \in \Pi_n. \) Especially, if \( v_n \) is expanded in terms of \( p_v, \ v = 0, \ldots, n, \) their associated polynomials are Clenshaw polynomials which are used in numerical mathematics. As a consequence of it we present some results from the viewpoint of associated polynomials and from the viewpoint of Clenshaw polynomials.

Analogously as for orthogonal polynomials we define functions of second kind for \( v_n. \) We prove some properties of them which depend on the generalized associated polynomials and functions of second kind for orthogonal polynomials.

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1. Introduction

Let \( \{ p_v \}_{v \geq 0} \) be a sequence of polynomials, satisfying the recurrence

\[
p_{v-1}(x) = 0, \quad p_0(x) := 1,
\]

\[
p_{n+2}(x) = (\alpha_{n+1} x - \beta_{n+1}) p_{n+1}(x) - \gamma_{n+1} p_n(x), \quad n \geq -1
\]  

(1.1)
with \( z_n \neq 0 \), \( n \geq 0 \). If \( \gamma_n \neq 0 \) for all \( n \in \mathbb{N}_0 \), then these polynomials are orthogonal with respect to a quasi-definite linear functional \( \mathcal{L} \), defined on the space \( \Pi \) of all algebraic polynomials (cf. [2]). Furthermore, \( \gamma_0 \) can be chosen arbitrarily. For simplicity we choose \( \gamma_0 := \mathcal{L}(p_0^2) \).

Shifting the recurrence coefficients in (1.1) we obtain a sequence \( \{p_n(\tau)\}_{n \in \mathbb{N}_0} \), \( r \in \mathbb{N}_0 \), of polynomials, called \( \tau \)-associated polynomials, satisfying

\[
\begin{align*}
 p_{-1}^{(r)}(x) &= 0, \quad p_0^{(r)}(x) := 1, \\
 p_{n+2}(x) &= (z_{n+r+1} - \beta_{n+r+1})p_{n+1}(x) - \gamma_{n+r+1}p_n(x), \quad n \geq -1.
\end{align*}
\]

(1.2)

For \( r = 0 \) the superscripts can be omitted \( (p_n^{(0)} = p_n) \).

Besides (1.2), they satisfy a ‘dual’ recurrence relation [3,5,11]

\[
\begin{align*}
 p_{-1}^{(k+2)} &= 0, \quad p_0^{(k+1)} = 1, \\
 p_{n-k}^{(k)}(x) &= (z_k x - \beta_k)p_{n-k}^{(k+1)}(x) - \gamma_{k+1}p_{n-1}^{(k+2)}(x), \quad k \in \mathbb{N}_0, \quad n = 0, 1, 2, \ldots .
\end{align*}
\]

(1.3)

In this formula, polynomials with different polynomial degrees and degrees (order) of association are combined. Eqs. (1.1) and (1.3) are used to prove the ‘dual Christoffel–Darboux identity’

\[
\sum_{k=0}^n z_k p_k(x)p_{n-k}^{(k+1)}(y) = \begin{cases} 
 p_{n+1}(x)p_{n+1}(y) - p_{n+1}(x)p_{n+1}(y) \\
 p_{n+1}(x), \quad x \neq y, \\
 p_{n+1}(x), \quad x = y,
\end{cases}
\]

(1.4)

Using (1.4), it is easy to show that

\[
 p_{n-j}^{(j)}(x) = \frac{1}{z_{j-1}}\mathcal{L}(p_j^2)\mathcal{L}([x,y;p_n;p_{n-1}])(x), \quad 1 \leq j \leq n.
\]

(1.5)

Both formulas (1.4), (1.5) are proved in [11, Theorem 1]. Here, we made the assumption that \( \mathcal{L} \) is quasi-definite, i.e. \( \gamma_v \neq 0 \) for all \( v \in \mathbb{N}_0 \), which is assumed to be true from now on. Because

\[
\mathcal{L}(p_m^2) = \frac{z_0}{z_{m-1}}\prod_{v=0}^{m-1} \gamma_v,
\]

(1.6)

we then have \( \mathcal{L}(p_m^2) \neq 0 \). Identity (1.5) gives us an idea how we can define a generalization of associated polynomials for arbitrary polynomials \( v_n \in \Pi_n \) (\( \Pi_n \) denotes the space of all algebraic polynomials of degree \( \leq n \)). This will be done in the next Section 2 where some basic properties of these generalized associated polynomials are given, too. If \( v_n \) is expanded in terms of the \( p_v \), \( v = 0, \ldots, n \), we show in Section 3 that these polynomials are Clenshaw polynomials which are used in various fields of numerical mathematics. Using this connection we present some applications of generalized associated polynomials. A special example is given in Section 4 where we consider divided differences and derivatives of \( v_n \).

Generalized associated polynomials are useful in many mathematical areas. For example, Peherstorfer [6,7] used them (of order 1) to characterize positive quadrature formulas. In Section 3 we show how they can be used to evaluate polynomials and their derivatives with a Clenshaw-like algorithm (cf. [8]).
Furthermore we sketch the main ideas to show how we can invert Vandermonde-like matrices. These few examples may illustrate how generalized associated polynomials can be used in various mathematical fields. On one hand they have some properties which generalize the corresponding ones of orthogonal polynomials. On the other hand they can be represented as special sums in terms of orthogonal polynomials (cf. (2.4), (2.6)) from which we can deduce further properties.

Closely related with associated polynomials are functions of second kind. They can be generalized in a natural manner, too. This is done in Section 5.

2. Generalized associated polynomials

Let $v_n \in \Pi_n$ be expanded in terms of linearly independent polynomials $p_v$, i.e.,

$$v_n = \sum_{v=0}^{n} a_{v,n} p_v.$$  

(2.1)

If all $p_v$, $v = 0, \ldots, n$, satisfy recurrence (1.1) and are orthogonal with respect to $L$ (cf. [2, Chapter I, Theorems 4.1, 4.4]) then, by using (1.4), the first divided difference $[x, y; v_n]$ of $v_n$ can be written as

$$[x, y; v_n] = \sum_{v=0}^{n} a_{v,n} [x, y; p_v]$$

$$= \sum_{v=0}^{n} a_{v,n} \sum_{j=0}^{v-1} a_j p_j(y) p_{v-j-1}(x)$$

$$= \sum_{v=0}^{n} a_{v,n} \sum_{j=1}^{v} a_{j-1} p_{j-1}(y) p_{v-j}(x)$$

$$= \sum_{j=1}^{n} a_{j-1} p_{j-1}(y) \sum_{v=0}^{n-j} a_{v+j,n} p_v(x).$$

(2.2)

If we multiply this identity by $p_{m-1}(y)$, $1 \leq m \leq n$, and apply $L$ with respect to $y$ on it, we get the (normalized) (generalized) associated polynomial $v_{n-m}^{[m]}$ of order $m$,

$$v_{n-m}^{[m]} := \frac{1}{\alpha_{m-1} L(p_{m-1}^2)} L \left( \frac{v_n(\cdot) - v_n(y)}{y} p_{m-1}(y) \right), \quad 1 \leq m \leq n,$$

(2.3)

$$= \sum_{v=0}^{n-m} a_{v+m,n} p_v^{(m)}, \quad 0 \leq m \leq n.$$  

(2.4)

The definition of $v_{n-m}^{[m]}$ depends on $L$ and on the polynomials $p_v$, $v \in \mathbb{N}_0$, which are orthogonal with respect to $L$. By using the orthogonality of $p_{m-1}$ on $\Pi_{n-1}$, $m > n$, with respect to $L$, we see from (2.3) that $v_{n-m}^{[m]} = 0$ if we would have $m > n$. Of course, for an arbitrary $v_n \in \Pi_n$, the corresponding associated polynomials of order $m$ can always be defined as in (2.3). But for (2.4) we need that $v_n$ is given as in (2.1). From (2.4) we see that $v_{n-m}^{[m]} \in \Pi_{n-m}$. 


If the $p_v$ are orthogonal with respect to a measure $\omega$ we often write $\mathcal{L}_\omega$ and $v_{n-m}^{[m]}(\cdot; \omega)$ instead of $\mathcal{L}$ and $v_{n-m}^{[m]}(\cdot)$. For convenience we may assume that $\mathcal{L}$ works on $y$ and that $v_{n-m}^{[m]}$ is constructed with respect to the moment functional with respect to which the $\{p_v\}_{v \in \mathbb{N}_0}$ are orthogonal. Otherwise we mention it explicitly. If it is clear which is the underlying moment functional, we do not label it especially.

Choosing $n = k + m$ and $v_{k+m} = p_{k+m}$, i.e., $a_{v,k+m} = \delta_{v,k+m}$, in (2.4) we see

$$p_k^{[m]} = p_k^{(m)}, \quad k, m \in \mathbb{N}_0,$$

which makes it clear why we speak of generalized associated polynomials. Especially for a polynomial $v_n \in \Pi_n$, given as in (2.1), we have by using (2.4)

$$v_{n-m}^{[m]} = \sum_{v=0}^{n-m} a_{v+m,n} p_v^{[m]}, \quad 0 \leq m \leq n. \tag{2.6}$$

A little bit more generally we define for convenience

$$v_{n-m}^{[k]} = \sum_{v=0}^{n-m} a_{v+k,n} p_v^{[k]}, \quad 0 \leq k \leq m \leq n.$$

In particular, $v_{n-m}^{[0]} = v_{n-m}$ denotes the $(n - m)$th partial sum of $v_n$ in (2.1). Because $n$ is fixed and $v_n$ is a given polynomial, there is no confusion in this notation.

Using the generalized recurrence formula

$$p_{v+r}^{(m)} = p_v^{(r+m)} p_r^{(m)} - \gamma_{r+m}^{(r+m+1)} p_{v-1}^{(m)} p_{r-1}^{(m)}, \quad v, r \geq 0 \tag{2.7}$$

(cf. [1, Lemma 2.5 for the monic case]) we get

**Theorem 1.** For $0 \leq s \leq n - m - 1$ we have

$$v_{n-m}^{[m]} = p_s^{(m)} v_{n-m-s}^{[s+m]} - \gamma_{s+m}^{(m)} p_{s-1}^{(m)} v_{n-(m+s+1)}^{[m]} + v_{s-1}^{[m]}, \tag{2.8}$$

where $v_{-1}^{[m]} := 0$.

**Proof.** Using (2.7) we get

$$v_{n-m}^{[m]} = \sum_{v=0}^{n-m} a_{v+m,n} p_v^{(m)}$$

$$= \sum_{v=-s}^{n-m-s} a_{v+m+s,n} p_{v+s}^{(m)}$$
Corollary 2. For \( n \geq m + 2 \) we have

\[
v_{n-m}^{[m]}(x) = a_{m,n} + (\alpha_{m}x - \beta_{m})v_{n-m-1}^{[m+1]}(x) - \gamma_{m+1}v_{n-m-2}^{[m+2]}(x). \tag{2.9}
\]

For \( m = n - s - 1, \ n \geq s + 1 \geq 0 \), which corresponds to (1.2), we get

\[
v_{n+s}^{[n-s-1]}(x) = [a_{n-1,n} + a_{n,n}(\alpha_{n-1}x - \beta_{n-1})]p_{s}^{(n-s-1)}(x)
- \gamma_{n-1}a_{n,n}p_{s-1}^{(n-s-1)}(x) + v_{s-1}^{[n-s-1]}(x) \nonumber
= a_{n,n}p_{s+1}^{(n-s-1)}(x) + a_{n-1,n}p_{s}^{(n-s-1)}(x) + v_{s-1}^{[n-s-1]}(x),
\]

from which we obtain (2.6). Furthermore, from (2.8) we see

\[
p_{s}^{(m)}v_{n-m-s}^{[s+m]} - \gamma_{s+m}p_{s-1}^{(m)}v_{n-(m+s+1)}^{[m+1]} = v_{n-m}^{[m]} - v_{s-1}^{[m]}
= \sum_{v=s}^{n-m} a_{v+m,n}p_{v}^{(m)}
= (v_{n} - v_{s-1})^{[m]}.
\]

Choosing \( m = 0 \) and then replacing \( s \) by \( m + 1 \) in (2.8) we get

\[
v_{n} = \sum_{v=0}^{m} a_{v,n}p_{v} + p_{m+1}v_{n-m-1}^{[m+1]} - \gamma_{m+1}p_{m}v_{n-m-2}^{[m+2]}.
\]

If we compare this equation with (2.9) we see that \( v_{n-m}^{[m]} \) can be obtained from \( v_{n} \) if we replace \( \sum_{v=0}^{m} a_{v,n}p_{v} = v_{m}^{[0]} \) by \( a_{m,n} = v_{0}^{[m]} \), \( p_{m+1} \) by \( p_{1}^{(m)} \) and \( p_{m} \) by \( p_{0}^{(m)} = 1 \), i.e., we increase the order of association by \( m \) and decrease the corresponding polynomial degrees by \( m \).

3. Clenshaw polynomials

The classical way to evaluate \( v_{n} \), given as in (2.1), at a fixed point \( x \) is the Clenshaw algorithm. With \( a_{n+1}^{[1]} := a_{n+2}^{[1]} := 0 \) we define

\[
a_{n-v}^{[1]} := a_{n-v,n} + (\alpha_{n-v}x - \beta_{n-v})a_{n-v+1}^{[1]} - \gamma_{n-v+1}a_{n-v+2}^{[1]} \tag{3.1}
\]
for \( v = 0, \ldots, n \), and obtain \( a_0^{(1)} = v_n(x) \). If all \( a_v = 1 \), \( \beta_v = \gamma_v = 0 \), \( v = 0, \ldots, n-1 \), then we get the Horner algorithm. If we allow \( x \) to be not only fixed but also variable, the \( a_{n-v}^{(1)} \) are polynomials.

**Definition 3.** The polynomials \( a_{n-v}^{(1)} \in \Pi_v \), defined in (3.1), are called the \((n-v)\)th Clenshaw polynomials belonging to \( v_n \). If all \( p_v(x) = x^v \), \( v = 0, \ldots, n \), then the \( a_{n-v}^{(1)} \) are also called Horner polynomials.

Horner polynomials are already known (see e.g., [9,12]). Thus Clenshaw polynomials are a canonical generalization of them. The definition of Clenshaw polynomials requires that the \( p_v \) satisfy a three-term recurrence relation (1.1) but not necessary orthogonality with respect to a moment functional \( \mathcal{L} \). Such a functional is needed for the definition of \( v_{n-m}^{[n]} \) in (2.3).

In the case of monomials, \( p_v(z) = z^v = p_v^{(m)}(z) \) for all \( v, m \in \mathbb{N}_0 \), \( \mathcal{L} \) is given by

\[
\mathcal{L}(p_v p_m) = \frac{1}{2\pi i} \int_{|z|=1} p_v(z) p_m(z) \, dz = \delta_{v,m}.
\]

Especially here we have \( \mathcal{L}(p_m^2) = 1 \) and \( \gamma_m = 0 \) for all \( m \in \mathbb{N}_0 \). Thus \( \mathcal{L} \) cannot be quasi-definite here.

Comparing recurrences (2.9) and (3.1) we see that \( a_{n-v}^{(1)} = v_{n-m}^{[m]} \), \( m = 0, \ldots, n \). From (2.4) it follows

\[
a_{j+1}^{(1)} = \sum_{v=0}^{n-j-1} a_v a_{j+1,v}^{(1)}, \quad j = -1, \ldots, n-1 \tag{3.2}
\]

(for \( j = -1 \) this follows from the property \( a_0^{(1)} = v_n \)) and with (2.2)

\[
[x, y; v_n] = \sum_{j=0}^{n-1} a_{j+1}^{(1)}(x) p_j(y). \tag{3.3}
\]

(Note that the sum on the right is symmetric with respect to \( x \) and \( y \).) In words, the generating function of the Clenshaw polynomials resp. of the generalized associated polynomials are the first divided differences of \( v_n \). Obviously (3.3) generalizes (1.4).

From (3.3) we have

\[
\mathcal{L}([\cdot, y; v_n]) = \sum_{j=0}^{n-1} a_{j+1}^{(1)}(y) \mathcal{L}(p_j(x)) = \sum_{j=0}^{n-1} a_{j+1}^{(1)} \mathcal{L}(p_j(y)) a_0^{[1]} = \mathcal{L}(p_0) a_1^{[1]}
\]

because \( \mathcal{L}(p_j) = 0 \) for \( j > 0 \) and \( \mathcal{L}(p_0) = \mathcal{L}(p_m^2) = \gamma_0 \). Using (2.4) we see again \( v_{n-1}^{[1]} = a_1^{[1]} \).

**Example 4.** The evaluation of a polynomial \( v_n \), where \( n \) is fixed and sufficiently large, can be done by evaluating \( v_n(x) \) directly by the Clenshaw algorithm.

We write \( v_n = v_m + (v_n - v_m) \) and choose \( m \ll n \), e.g., \( m = \lfloor n/2 \rfloor \). Now we get \( v_n(x) \) by evaluating \( v_m \) and \( v_n - v_m = p_{m+1} v_{n-m-1} + \gamma_{m+1} v_{n-m-2} \) at \( x \) (cf. (2.8)). The ‘higher frequency part’ \( v_n - v_m \) can be obtained if we evaluate \( v_{n-m-1}^{[m+1]}(x) =: a_{0,m+1}^{[1]}(x) \) by the Clenshaw algorithm. This yields
The components of this matrix can be obtained by the algorithm which we shall describe in Theorem 6.

Now we compute \( p_{m+1}(x)v_{n-m-1}(x) - \gamma_{m+1}p_m(x)v_{n-m-2}(x) \). Adding the evaluated ‘lower frequency part’ \( v_m(x) \) to this result we get \( v_{m+1}(x) \).

If \( m = \lfloor n/2 \rfloor \) only need to evaluate polynomials of degree \( \leq \max(m+1, n-m-1) \leq m+1 \). If \( m \) or \( n-m \) is too ‘large’ we can repeat this proceeding with \( v_m \) or \( v_{m+1}^{[m+1]} \) and \( v_{m+2}^{[m+2]} \) instead of \( v_n \). Hence a ‘divide and conquer’ strategy for polynomial evaluation can be derived.

**Example 5 (Inversion of Vandermonde-like matrices).** Let \( x_k, k = 0, \ldots, n-1 \), be (real or complex) nodes with \( x_v \neq x_{\mu} \) for \( v \neq \mu \) and \( v_n := \prod_{\nu=0}^{n-1}(\cdot - x_{\nu}) \). From (3.3) we have

\[
\frac{1}{v_n'(x_v)} \sum_{j=0}^{n-1} a_j a_{j+1}^{[1]}(x_v) p_j(x_v) = \delta_{v,\mu}, \quad 0 \leq \mu, v \leq n-1.
\]

With \( V^{-1}_{n-1} := (p_{\nu}(x_{\mu}))_{\nu,\mu=0,\ldots,n-1} \) we obtain

\[
V^{-1}_{n-1} = \left( \frac{a_j a_{j+1}^{[1]}(x_v)}{v_n'(x_v)} \right)_{j,v=0,\ldots,n-1}.
\]

The components of this matrix can be obtained by the algorithm which we shall describe in Theorem 6.

### 4. Divided differences

Now we consider higher divided differences of \( v_n \). From (3.3) we obtain

\[
[x_0, x_1, \ldots; v_n] = \sum_{j=0}^{n-2} a_j [x_0, x_1; a_{j+1}^{[1]}] p_j
\]

because \( a_n^{[1]} \in \Pi_0 \) is a constant. Inductively, we get

\[
[x_0, x_1, \ldots, x_{k-1}, \ldots; v_n] = \sum_{j=0}^{n-k} a_j [x_0, x_1, \ldots, x_{k-1}; a_{j+1}^{[1]}] p_j, \quad k = 1, \ldots, n, \tag{4.1}
\]

where \( x_0, \ldots, x_{k-1} \in \mathbb{C} \) or \( \mathbb{R} \). For \( x_0 = \cdots = x_{k-1} = x \) we have

\[
[x, \ldots, x, y; v_n] = \sum_{j=0}^{n-k} a_j [x, \ldots, x; a_{j+1}^{[1]}] p_j(y), \quad k = 1, \ldots, n, \tag{4.2}
\]

\( k \)-times

\( k \)-times
and especially

\[
\frac{1}{k} \frac{d^k}{dx^k} v_n(x) = \sum_{j=0}^{n-k} a_j \frac{d^{k-1}}{dx^{k-1}} a_{j+1}^{[1]}(x) p_j(x), \quad k = 1, \ldots, n.
\]

From this formula one can see that derivatives can be obtained by the Clenshaw algorithm.

**Theorem 6** (Skrzipek [8]). Let us define

\[
\tilde{a}_{k+j}^{[k]} := \begin{cases} 
  a_j^{[0]} := a_{j,n}, & k = 0, \\
  a_j^{[k]} a_{k+j}^{[k]}, & k = 1, \ldots, n, \\
  j = n - k, \ldots, 0
\end{cases}
\]

(note that \(a_{k+j}^{[k]}, \tilde{a}_{k+j}^{[k]}\) depends on \(x\)). Restarting Clenshaw’s algorithm with coefficients \(\{\tilde{a}_{k+j}^{[k]}\}_{j=n-k,\ldots,0}\) for \(k = 0, \ldots, n - 1\), we get sequences \(\{a_{k+j}^{[k+1]}\}_{j=n-k,\ldots,0}, k = 0, \ldots, n\), with

\[
a_{k+j}^{[k+1]} = \frac{1}{k!} \frac{d^k}{dx^k} v_n(x), \quad k = 0, \ldots, n.
\]

Inductively, we have for fixed \(x\) (cf. (4.2))

\[
a_{k+j}^{[k]} = [x, \ldots, x; a_{k+j}^{[1]}] \frac{1}{k} \frac{d^k}{dx^k} a_{j+1}^{[1]}(x), \quad k = 0, \ldots, n
\]

and

\[
[x, \ldots, x; v_n] = \sum_{j=0}^{n-k} \tilde{a}_{k+j}^{[k]} p_j(x), \quad k = 0, \ldots, n.
\]

Analogously we get from (3.3)

\[
[x, x_0, \ldots, x_{k-1}; v_n] = \sum_{j=0}^{n-1} a_j^{[1]}(x)[x_0, \ldots, x_{k-1}; p_j] = \sum_{j=k-1}^{n-1} a_j^{[1]}(x)[x_0, \ldots, x_{k-1}; p_j] = \sum_{j=0}^{n-k} a_{j+k-1}^{[1]}(x)[x_0, \ldots, x_{k-1}; p_{j+k-1}].
\]
Replacing $j$ by $n - k - j$ in the sum and using (4.1) we see the identity
\[
\mathcal{L} \left( x_0, \ldots, x_{k-1}; v_n \right) = \sum_{j=0}^{n-k} \alpha_j \left[ x_0, \ldots, x_{k-1}; v_{n-j-1} \right] p_j
\]
which shows a duality relation between the $\{ p_j \}_{j=0}^n$ resp. $\{ v_j \}_{j=0}^n$ and the corresponding Fourier-coefficients for $\left[ \cdot; x_0, \ldots, x_{k-1}; v_n \right]$. Furthermore, from (3.3) we derive
\[
\frac{d^p}{dx^p} \frac{d^q}{dy^q} \left[ x, y; v_n \right] = \sum_{j=0}^{n-1} \alpha_j \frac{d^p}{dx^p} a_{j+1}^{[1]} (x) \frac{d^q}{dy^q} p_j (y).
\]
For $x = y$ the left side of (4.3) becomes $(d^{p+q+1}/dx^{p+q+1}) \delta_{n,0}$. By (3.2) we may assume that $a_{j+1}^{[1]} = v_{n-j-1}^{[j]}$ is a polynomial expanded in terms of a system of orthogonal polynomials. As described before, the derivatives of this polynomial can be evaluated by the Clenshaw algorithm too. The same holds for
\[
\frac{d^p}{dx^p} \left( \sum_{j=0}^{n} \delta_{j,p} \delta_{j,q} p_j \right) = 0 \quad \text{for} \quad 0 \leq p < m_1, \quad 0 \leq q < m_2.
\]
Following some ideas given in [12] we see how (4.3) can be used for inverting Vandermonde-like matrices with some multiple nodes.

5. Functions of the second kind

From (1.5) we have
\[
p_{n-1}^{(1)} = \frac{1}{\gamma_0} \mathcal{L} \left( x, y; p_n \right),
\]
where $\mathcal{L}$ works on $y$. Now we choose $\mathcal{L}_\omega (f) := \mathcal{L} (f) := \int_\mathbb{R} f(t) \, dw(t)$, $\gamma_0 := \mathcal{L}_\omega (p_0^2)$, where $\omega$ denotes the measure with respect to which the elements of $\{ p_n \}_{n \in \mathbb{N}_0}$ are orthogonal. Then $q_n$, $q_n (z) := \mathcal{L}_\omega \left( \frac{p_n (y)}{z-y} \right), \quad z \notin \text{supp}(\omega), \quad n \in \mathbb{N}_0,$
is the $n$th function of second kind which belongs to $p_n$. Obviously we have [4], [10, pp. 53, 54]
\[
q_n = q_0 p_n - \gamma_0 p_{n-1}^{(1)}, \quad n \geq 0.
\]
Defining the function of second kind of $v_n$ as
\[
w_n (z) := \mathcal{L}_\omega \left( \frac{v_n (y)}{z-y} \right), \quad z \notin \text{supp}(\omega),
\]
Theorem 7. The value \( w_n(z) \), \( z \notin \text{supp}(\omega) \), can be obtained from \( q_0(z) \) and evaluating \( v_n(z) \) by the Clenshaw algorithm (3.1). Similarly one can proceed for functions of second kind and order \( m \). With

\[
q_n^{[m-1]} := \mathcal{L}_\omega \left( \frac{p_m(y) p_{m-1}(y)}{\cdot - y} \right), \quad 0 \leq n, \quad 1 \leq m.
\]

(eespecially we have \( q_0^{[m-1]} = q_m^{[n]} \) and \( q_0^{[m-1]} = q_m^{[0]} \)) we define in \( \mathbb{C}\setminus\text{supp}(\omega) \)

\[
w_n^{[m-1]} := \mathcal{L}_\omega \left( \frac{v_n(y) p_{m-1}(y)}{\cdot - y} \right), \quad 0 \leq n, \quad 1 \leq m.
\]

If \( v_n \) is given as in (2.1) we have outside of \( \text{supp}(\omega) \)

\[
w_n^{[m-1]}(z) = \sum_{r=0}^{n} a_{v,n} q_r^{[m-1]}(z), \quad 0 \leq n, \quad 1 \leq m.
\]

(5.2)

**Theorem 7.** The value \( w_n^{[m-1]}(z) \), \( 0 \leq n, \quad 1 \leq m \), can be obtained from the evaluation of \( p_m \) and \( v_n \) at \( z \) by the Clenshaw algorithm and from the evaluation of \( q_0(z) \). In particular, \( w_n^{[m-1]} \) can be extended for \( z \in \text{supp}(\omega) \) iff \( q_0 \) is defined there. (Usually \( q_0 \) is defined in \( \text{supp}(\omega) \) as the Cauchy principal value if this exists.)

**Proof.** From (2.3) and (1.6) we see

\[
w_n^{[m-1]} = v_n q_0^{[m-1]} - \gamma_{m-1} \mathcal{L}_\omega (p_m^2) q_{n-m}^{[m]}
= a_0^{[1]} q_{m-1} - \gamma_m a_0^{[1]} \prod_{r=0}^{m-1} \gamma_r
\]

(5.3)

which means that \( w_n^{[m-1]}(z) \) can be obtained from \( q_{m-1}(z) \) and from the evaluation of \( v_n(z) \) from which we get \( a_0^{[1]} \), \( a_0^{[1]} \cdot q_{m-1}(z) \) satisfies recurrence (1.1) but with initial values \( q_{-1}(z) = z_0 \) and \( q_0(z) = \mathcal{L}_\omega (1/(z - y)) \). Thus the Clenshaw algorithm can be used to evaluate \( q_{m-1}(z) \), too. More explicitly, \( q_{m-1}(z) \) can be evaluated by using (5.1) from which we get

\[
w_n^{[m-1]} = a_0^{[1]} \left[ q_0 p_m - \gamma_0 a_0^{[1]} p_{m-1} \right] - \gamma_m a_0^{[1]} \prod_{r=0}^{m-1} \gamma_r
= a_0^{[1]} q_0 p_m - \gamma_0 \left[ a_0^{[1]} \gamma_0 p_{m-1} - a_0^{[1]} \prod_{r=0}^{m-1} \gamma_r \right].
\]

Evaluating \( p_m(z) \) by the Clenshaw algorithm yields \( p_{m-1}^{(1)}(z) \), too. Thus we only need to evaluate \( q_0 \), \( p_m \), and \( v_n \) at \( z \) to obtain \( w_n^{[m-1]}(z) \). □
In particular, for \( v_n = p_n, \) i.e. \( a_{v,n} = \delta_{v,n}, \) \( v = 0, \ldots, n, \) we have from (5.3)
\[
q_n^{[m-1]} = p_n q_{m-1} - \gamma_{m-1} L_\omega(p_{m-1}^{(m)}) p_{n-m}^{(m)} \quad n \geq 0, \ m \geq 1. \tag{5.4}
\]

**Theorem 8.** The functions of second kind \( u_n^{[m-1]} \) satisfy the recurrence relations
\[
w_n^{[m-1]} = v_n^{[s]} q_s^{[m-1]} - \gamma_s v_n^{[s+1]} q_{s-1}^{[m-1]} + w_s^{[m-1]}, \quad m \leq s \leq n - 1, \tag{5.5}
\]
\[
w_k^{[m-1]} = v_n^{[k+m]} q_n^{[m-1]} - \gamma_{k+m} v_n^{[k+m+1]} q_{n-k+m+1}^{[m-1]} + w_k^{[m-1]}, \quad 0 \leq k \leq n - m - 1, \tag{5.6}
\]
where the integers \( s, k \) can be chosen arbitrarily in the given regions.

**Proof.** From (5.2) and (2.8) we have
\[
w_n^{[m-1]} = v_n^{[s]} q_s^{[m-1]} - \gamma_s v_n^{[s+1]} q_{s-1}^{[m-1]} + w_s^{[m-1]},
\]
\[
- \gamma_{m-1} L_\omega(p_{m-1}^{(m)}) q_{m-1}^{[m-1]} + \gamma_{m+k} p_{m-1}^{(m)} q_{n-k-m+1}^{[m-1]} + w_k^{[m-1]}, \quad 0 \leq k \leq n - m - 1,
\]
where \( 0 \leq s \leq n - 1, \ 0 \leq k \leq n - m - 1. \) Choosing \( k = s - m, m \leq s \leq n - 1, \) we get
\[
w_n^{[m-1]} = v_n^{[s]} p_s q_{m-1} - \gamma_{m-1} L_\omega(p_{m-1}^{(m)}) p_{s-m}^{(m)} - \gamma_{m-1} L_\omega(p_{m-1}^{(m)}) q_{s-m-1}^{[m-1]}
\]
\[
+ v_{s-1} q_{m-1}^{[s]} - \gamma_{s} v_{n-s}^{[s+1]} p_{s-1}^{(m)} q_{m-1}^{[m-1]} - \gamma_{m-1} L_\omega(p_{m-1}^{(m)}) p_{s-m}^{(m)} q_{s-m-1}^{[m-1]}
\]
\[
+ v_{n-s} q_{s}^{[m-1]} - \gamma_{s} v_{n-s}^{[s+1]} q_{s-1}^{[m-1]} + w_{s-1}^{[m-1]},
\]
where we have used (5.4). Thus (5.5) is proved; (5.6) is obtained if we choose \( s = k + m. \)

For \( s = n - 1 \) resp. \( k = n - m - 1 \) in (5.5) resp. (5.6) we have
\[
w_n^{[m-1]}(z) = (\alpha_{n-1} z - \beta_{n-1}) a_{n,n} + a_{n-1,n} q_{n-1}^{[m-1]}(z)
\]
\[
- \gamma_{n-1} a_{n,n} q_{n-2}^{[m-1]}(z) + w_{n-2}^{[m-1]}(z)
\]
\[
= a_{n,n}([\alpha_{n-1} z - \beta_{n-1}] q_{n-1}^{[m-1]}(z) - \gamma_{n-1} q_{n-2}^{[m-1]}(z)]
\]
\[
+ a_{n-1,n} q_{n-1}^{[m-1]}(z) + w_{n-2}^{[m-1]}(z)
\]
\[
= a_{n,n} q_{n-1}^{[m-1]}(z) + a_{n-1,n} q_{n-1}^{[m-1]}(z) + w_{n-2}^{[m-1]}(z)
\]
from which we get (5.2) again.

For \( s = m \geq 1 \) resp. \( k = 0 \) in (5.5) resp. (5.6) we have
\[
w_n^{[m-1]}(z) = v_{n-m}^{[m]}(z) q_{m-1}^{[m-1]}(z) - \gamma_{m} v_{n-m-1}^{[m]}(z) d_{m-1}^{[m-1]}(z) + w_{m-1}^{[m-1]}(z),
\]
where \( w_{m-1}^{[m-1]} = v_{m-1} q_{m-1} \) which follows from (5.3).

At the end of this article, we notice some special values for \( w_n^{[m-1]} \). If \( x \) is a zero of \( v_n \) then we have
\[
w_n^{[m-1]}(x) = L_\omega \left( \frac{v_n(x) - v_n(y)}{x - y} p_{m-1}^{(m)}(y) \right) = v_{n-m}^{[m]}(x).
\]
If $x$ is a zero of $p_{m-1}$ we get

$$w_n^{[m-1]}(x) = L_{\nu \omega} \left( v_n(y) \frac{p_{m-1}(y) - p_{m-1}(x)}{x - y} \right)$$

$$= - \sum_{\nu=0}^{n} a_{\nu,n} L_{\nu \omega} \left( \frac{p_{m-1}(x) - p_{m-1}(y)}{x - y} p_{\nu}(y) \right)$$

$$= - \sum_{\nu=0}^{m-2} a_{\nu,n} L_{\nu \omega} (p_{\nu}^2) p_{m-v-2}^{(v+1)}(x) - \sum_{\nu=m-1}^{n} a_{\nu,n} L_{\nu \omega} ([x, y; p_{m-1}] p_{\nu}(y))$$

$$= - \sum_{\nu=0}^{m-2} a_{\nu,n} L_{\nu \omega} (p_{\nu}^2) p_{m-v-2}^{(v+1)}(x)$$

$$= - \sum_{\nu=0}^{m-2} a_{m-v-2} L_{\nu \omega} (p_{m-v-2}^2) a_{m-v-2, n} p_{\nu}^{(m-v-1)}(x)$$

by (1.5). Here, we have used that $[x, \cdot; p_{m-1}] \in \Pi_{m-2}$ and that the integral in the second sum on the right side vanishes for $\nu \geq m - 1$ because of the orthogonality of the $p_{\nu} \in \Pi_{\nu}$.

References