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The rationality of Sol-manifolds

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Abstract

Let Γ be the fundamental group of a manifold modeled on 3-dimensional Sol geometry. We prove that Γ has a finite index subgroup G which has a rational growth series with respect to a natural generating set. We do this by enumerating G by a regular language. However, in contrast to most earlier proofs of this sort our regular language is not a language of words in the generating set, but rather reflects a different geometric structure in G.

Keywords: Sol manifold; Three manifold; Group growth; Rational growth; Finite state automaton; Falsification by fellow traveler property

1. Introduction

Let Γ be a group with a finite generating set S. For $g \in \Gamma$, let $\|g\|$ be equal to the length of the shortest word in $S \cup S^{-1}$ representing g, and for $g_1, g_2 \in \Gamma$ set $d(g_1, g_2) = \|g_1^{-1} g_2\|$. This is known as the *word metric* on Γ. The growth of the size of balls in this metric constitutes a central object of study in geometric group theory (see [10, Chapters 6–7] for a survey).

To study the growth of Γ, it is natural to define the *growth series* of Γ to be the power series

$$\mathcal{G}(\Gamma) = \sum_{i=0}^{\infty} c_i z^i,$$

where $c_i = |\{g \in \Gamma\colon \|g\| = i\}|$. In many cases, it turns out that $\mathcal{G}(\Gamma)$ is a rational function. The first non-trivial example of this is in [3], where an exercise outlines a proof that all Coxeter groups

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have rational growth with respect to a Coxeter generating set. Perhaps the most remarkable theorem of this type is in Cannon's paper [5], which proves that all word hyperbolic groups have rational growth with respect to *any* finite generating set ([5] only proves this for fundamental groups of compact hyperbolic manifolds, but it contains all the ideas necessary for the extension to word hyperbolic groups—see [7] for a complete account).

In this paper we study the growth series of the fundamental groups Γ of torus bundles over the circle with Anosov monodromy. In other words, $\Gamma = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$ with $M \in \mathrm{SL}_2(\mathbb{Z})$ a matrix with two distinct real eigenvalues. These are the fundamental groups of 3-manifolds modeled on Sol geometry. Our main theorem is the following:

Theorem 1.1 *(Main theorem). Let Γ be the fundamental group of a 3-dimensional Sol manifold. Then there exists a finite index subgroup G generated by a finite set S so that G has rational growth with respect to S. In other words, Γ is virtually rational.*

This theorem is part of two different streams of research. On the one hand, there have been many papers investigating the growth series for lattices in Thurston's eight 3-dimensional model geometries (see [1,2,5,13,15,16,19]). After Theorem 1.1, the only remaining geometry for which there is not some general theorem is $\widetilde{\mathrm{SL}_2}$, although some progress has been made on this case by Shapiro [16].

On the other hand, there has also been significant research on the growth series of finitely generated solvable groups. Kharlampovich has produced a 3-step solvable group which has an unsolvable word problem [11]. Since all groups with rational growth series have a solvable word problem (the rational growth series allows one to calculate the size of balls in the Cayley graph, which one can then construct using a brute force enumeration), it follows that Kharlampovich's example does not have rational growth with respect to any set of generators.

One can therefore hope for general results only for 1- and 2-step solvable groups. The 1-step solvable groups are the finitely generated abelian groups. Benson has proven that more generally all finitely generated virtually abelian groups have rational growth with respect to any finite set of generators [1]. The 2-step solvable groups are divided into the nilpotent and non-nilpotent groups. A fundamental set of examples of 2-step nilpotent groups are the lattices in 3-dimensional Nil geometry. These correspond to groups of the form $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$ with $M \in \mathrm{SL}_2(\mathbb{Z})$ a matrix with two different non-real eigenvalues, which necessarily must lie on the unit circle. Benson, Shapiro, and Weber [2,15,19] have shown that lattices in 3-dimensional Nil geometry have rational growth with respect to a certain generating set. This has been generalized by Stoll [17], who showed that all 2-step nilpotent groups with infinite cyclic derived subgroup have rational growth with respect to some generating set. He also showed that many such groups (those with "Heisenberg rank at least 2") have transcendental growth with respect to some other generating set. This demonstrates that, in contrast to most natural group properties studied by geometric group theorists, rational growth can depend strongly on the choice of generating set.

The non-nilpotent solvable case is divided into the polycyclic and the non-polycyclic cases. A fundamental set of examples of 2-step solvable non-polycyclic groups are the solvable Baumslag–Solitar groups $BS(1,n)$. Brazil and Collins–Edjvet–Gill have shown that these have rational growth with respect to the standard set of generators [4,6]. A fundamental set of examples of 2-step solvable polycyclic groups are the torsion-free abelian-by-cyclic groups. These are groups of the form $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ with $M \in \mathrm{SL}_n(\mathbb{Z})$. Theorem 1.1 combined with the result of Benson, Shapiro, and Weber referred to in the previous paragraph cover the case $n = 2$, with ours being the "generic" case since our eigenvalues do *not* lie on the unit circle.

The strategy of our proof is as follows. The subgroup G we consider is generated by two elements a and t. We first define a surjective function which associates to a word w in the free group on a and t a pair $(\widetilde{\mathrm{type}}(w), h(w))$ where $\widetilde{\mathrm{type}}(w) \in \mathbb{Z}[z, z^{-1}]$ and $h(w) \in \mathbb{Z}$. The word metric on the free group induces a "size function" on such pairs for which there is a rather simple formula. It turns out that two words w_1 and w_2 determine the same element of G if and only if $h(w_1) = h(w_2)$ and $\widetilde{\mathrm{type}}(w_1) = \widetilde{\mathrm{type}}(w_2)$ modulo a certain principal ideal I of $\mathbb{Z}[z, z^{-1}]$. It is easy to construct a "size-preserving" enumeration of $\mathbb{Z}[z, z^{-1}] \oplus \mathbb{Z}$ by a regular language L. The map

$$\mathbb{Z}\big[z, z^{-1}\big] \oplus \mathbb{Z} \longrightarrow \big(\mathbb{Z}\big[z, z^{-1}\big]/I\big) \oplus \mathbb{Z}$$

induces a "quotient" L/P of L with an induced "size function." We conclude by proving that L/P satisfies a certain negative curvature-like condition (the *falsification by fellow traveler property*). This allows us to enumerate L/P by a regular language, which by well-known results is enough to prove that it (and therefore G) has a rational growth series.

Remark. Though in theory our methods are entirely constructive, in practice the finite state automata we build are so huge that it is impractical to calculate any examples.

History and comments

In his unpublished thesis [9], Grayson claimed to prove Theorem 1.1 whenever the trace of the monodromy is even. However, his proof is insufficient (see the remarks in Section 4 for a more detailed discussion). Our methods are rather different from his methods. He attempts to write down a complicated recurrence relation between balls of different radii. As indicated above, we instead use the theory of finite state automata. In addition, the generating sets we use are slightly different from his generating sets. We do, however, use some of his ideas. In particular, he introduced the notions of *types* and *heights* described in Section 4 (though he did not distinguish between the reduced and unreduced types), and the elegant proof of Theorem 4.2 is due to him.

After this paper was complete, we learned that in an unpublished paper Parry had given a proof of Theorem 1.1, following Grayson's basic outline [14]. Like Grayson, he assumes that the trace of the monodromy is even. However, Parry was able to use a computer to calculate some growth functions explicitly. We reproduce the result of his calculation in Section 6.

1.1. Outline and conventions

In Section 2, we review some preliminary material on Sol manifolds, regular languages, etc. Next, in Section 3 we discuss a technical condition on partitions of regular languages which implies rational growth. This condition, the *falsification by fellow traveler* property, is inspired by but different from the condition of the same name defined by Neumann and Shapiro in [13]. Section 4 is then devoted to the bijection

$$G \longrightarrow \big(\mathbb{Z}\big[z, z^{-1}\big]/I\big) \oplus \mathbb{Z}$$

discussed above. Finally, in Section 5 we construct a sequence of regular languages L_n which enumerate $\mathbb{Z}[z, z^{-1}] \oplus \mathbb{Z}$ in a "size-preserving" manner, and we prove that the partition P_n of L_n induced by the natural map

$$\mathbb{Z}\big[z, z^{-1}\big] \oplus \mathbb{Z} \longrightarrow \big(\mathbb{Z}\big[z, z^{-1}\big]/I\big) \oplus \mathbb{Z}$$

satisfies the falsification by fellow traveler property for sufficiently large n. This proves Theorem 1.1. We conclude by discussing some open questions in Section 6.

We will frequently manipulate *Laurent polynomials* over $\mathbb{Z}$, that is elements of $\mathbb{Z}[z, z^{-1}]$. When we refer to such a polynomial as $\sum_j c_j z^j$, we mean that all but finitely many of the c_j equal 0.

2. Preliminaries

2.1. Sol manifolds

As discussed in [18], 3-dimensional Sol manifolds are 2-dimensional torus bundles over the circle whose monodromy $M \in \mathrm{SL}_2(\mathbb{Z})$ is *Anosov*, that is M has two distinct real eigenvalues. Equivalently, $|\mathrm{trace}(M)| > 2$. Let M be such a matrix, and let a and b be the standard generators for $\mathbb{Z}^2$. Hence, Ma and Mb are well defined. Abusing notation in the obvious way, we say that the *torus bundle group* with monodromy M is the group with the presentation

$$\Gamma = \big\langle a, b, t \mid [a, b] = 1,\ tat^{-1} = Ma,\ tbt^{-1} = Mb\big\rangle.$$

Observe that $G = \langle a, t\rangle$ is a finite index subgroup of Γ. Now, the minimal polynomial of the matrix M is equal to $1 - \mathrm{trace}(M)z + z^2$. Hence $M^k a$ is in the lattice generated by a and Ma. In other words, the group G corresponds to the 2-dimensional torus bundle whose fiber is generated by a and Ma. It is easy to see that G is isomorphic to the torus bundle group with monodromy

$$\begin{pmatrix} 0 & -1 \\ 1 & \mathrm{trace}(M) \end{pmatrix}.$$

We will prove that G has rational growth with respect to the generating set $\{a, t\}$.

2.2. Sized sets and languages

In the course of our proof, we will construct a series of objects whose growth reflects the growth series of G. The "size functions" on these objects come from very different sources. The following formalism provides a language with which to compare these objects:

Definition. A *sized set* is a set X together with a size function $\|\cdot\| : X \to \mathbb{Z}_{\geqslant 0}$.

Set $c_i = |\{x \in X\colon\ \|x\| = i\}|$. We will only consider sized sets with $c_i < \infty$ for all i. There is therefore an associated generating function

$$\mathcal{G}(X) = \sum_{i=0}^{\infty} c_i z^i.$$

Definition. Let X be a sized set and P be a partition of X. In other words, P is a set of pairwise disjoint subsets of X so that

$$\bigcup_{A \in P} A = X.$$

We define X/P to be the sized set whose elements are elements of P and whose size function is

$$\|A\| = \min\{\|x\|\colon x \in A\}.$$

If $x \in X$, then we will denote by $\bar{x}$ the set $A \in X/P$ with $x \in A$. If $x, y \in X$ satisfy $\bar{x} = \bar{y}$, then we will say that x equals y modulo P.

Definition. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be sized sets. A bijection $\psi : X_1 \to X_2$ is a *near-isometry* if there is some constant c so that for all $x \in X_1$ we have $\|x\|_1 = \|\psi(x)\|_2 + c$. If $c = 0$, then a near-isometry is an *isometry*.

Remark. Observe that if X_1 and X_2 are near-isometric with respect to a constant c then $\mathcal{G}(X_1) = z^c\mathcal{G}(X_2)$. In particular, $\mathcal{G}(X_1)$ is a rational function if and only if $\mathcal{G}(X_2)$ is. Our use of near-isometries is purely a matter of convenience—they allow us to have a somewhat simpler definition of the languages L_n we construct in Section 5.

Our primary source of sized sets will be the following:

Definition. Let A be a finite set, which we will call the *alphabet*. A *language* L over A is a subset of A^*, the set of finite sequences of elements of A. Elements of L are called *words*.

Languages can be considered sized sets in the following way. Let L be a language over A. Consider some $\phi : A \to \mathbb{Z}_{>0}$, which we will call the *weighting*. For $a_1 a_2 \cdots a_k \in L$, define

$$\|a_1 a_2 \cdots a_k\| = \sum_{i=1}^{k} \phi(a_i).$$

Example. Let H be a group with a finite set of generators S. Let L be the language of all words in $S \cup S^{-1}$ with weighting 1 for each generator. Finally, let P be the partition which identifies two words if they represent the same element in H. The series $\mathcal{G}(L/P)$ is then the usual growth series for H.

2.3. Regular languages

We quickly review the theory of finite state automata and regular languages. For more details see, e.g., [8, Chapter 1].

Definition. A *finite state automaton* on n strings is a 5-tuple

$$\bigl(A, (V, E), S, F, l\bigr)$$

with A a finite set (called the *alphabet*), (V, E) a finite directed graph (called the *state graph*), $S \in V$ (called the *start state*), $F \subset V$ (called the *final state*), and $l : E \to \prod_{i=1}^{n}(A \cup \{\$\})$ with \$ some symbol disjoint from A (l is called the *transition label*; "\$" is a symbol for the end of a word) satisfying the following condition: if $l(e) = (\dots, \$, \dots)$ with the \$ in the kth place, then $l(f)$ also has a \$ in the kth place for all edges f so that there is a finite (oriented) path

$$e = e_0, e_1, \dots, e_m = f.$$

Definition. Let $Z = (A, (V, E), S, F, l)$ be a finite state automaton on n strings. We define the language $L(Z) \subset \prod_{i=1}^{n} A^*$ to be the following. Consider any element $(w_1, \dots, w_n) \in \prod_{i=1}^{n} A^*$. Assume that the longest word in this tuple has m letters. For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ define w_i^j to be the jth letter of w_i if j is at most the length of w_i and \$ otherwise. Then $(w_1, \dots, w_n) \in L(Z)$ if and only if there is some path

$$S = v_1, e_1, v_2, e_2, \dots, e_m, v_{m+1} \in F$$

so that for $1 \leqslant j \leqslant m$ we have

$$l(e_j) = \left(w_1^j, w_2^j, \dots, w_n^j\right).$$

We say that $L(Z)$ is a *regular language*.

Remark. Observe that we are abusing the word "language" in this definition: only the case $n = 1$ is an actual language.

Remark. One should think of this as a machine able to keep track of a finite amount of information. The vertices of the state graph correspond to the different states in which the machine can be, and the machine moves from the state s_1 to the state s_2 upon reading α if there is an edge e between s_1 and s_2 with $l(e) = \alpha$.

The following theorem demonstrates the flexibility of regular languages:

Theorem 2.1. [8, Proposition 1.1.4, Theorem 1.2.8, Corollary 1.4.7] *The class of regular languages is closed under all first order predicates* (*i.e.* $\cup$, $\cap$, $\neg$, $\forall$, *and* $\exists$) *and under concatenation. In addition, if L is a regular language on n strings then the following language is regular*:

$$\operatorname{rev}(L) = \left\{(a_{1,1} \cdots a_{1,m_1}, \dots, a_{n,1} \cdots a_{n,m_n}) \colon (a_{1,m_1} \cdots a_{1,1}, \dots, a_{n,m_n} \cdots a_{n,1}) \in L\right\}.$$

We will also need the following theorem:

Theorem 2.2. *Let $Z = (A, (V, E), S, F, l)$ be a finite state automaton on one string and let $\phi : A \to \mathbb{Z}_{>0}$ be a weighting. Then the generating function $\mathcal{G}(L(Z))$ with the language $L(Z)$ weighted by ϕ is a rational function.*

Proof. It is a standard fact (see, for instance, [7, Theorem 9.1]) that $\mathcal{G}(L(Z))$ is rational if ϕ is the constant function 1. To deduce the general case from this, replace each edge e in (V, E) by a path of length $\phi(l(e))$ with each edge in the path labeled by $l(e)$. $\square$

Remark. When we refer to a regular language without specifying how many strings it has, we are referring to a regular language on one string.

3. Partitioning regular languages

Fix a regular language L with weighting ϕ. Consider a partition P of L. By Theorem 2.2, we know that L has a rational generating function. In this section we give a sufficient condition for L/P (see Section 2.2 for the definition of L/P) to have a rational generating function. Our condition, the *falsification by fellow traveler* property, allows us to construct a regular sublanguage of L containing exactly one word of minimal size from each set in P. It is inspired by the property of the same name in [13]. We begin with two preliminary definitions.

Definition. We say that L/P has a *regular cross section* if there is some regular sublanguage $L' \subset L$ so that for all $A \in P$ there is a unique $x \in L'$ with $x \in A$. If in addition all such x satisfy

$$\|x\| = \min\{\|x'\|\colon x' \in A\}$$

then we say that L' is a *regular minimal cross section* of L/P.

Definition. We say that a regular language $R \subset L \times L$ is an *acceptor* for a partition P of L if

$$(w, w') \in R \quad \Longrightarrow \quad \overline{w} = \overline{w}' \quad \text{and} \quad (w', w) \in R.$$

Our condition is the following:

Definition. We say that a partition P with an acceptor R has the *falsification by fellow traveler* property if there is some constant K and some regular sublanguage L' of L containing at least one minimal size representative of each set in P so that if $w \in L'$ is not a minimal size representative in L/P then there is some word $w' \in L$ so that the following are true.

- $(w, w') \in R$ (and, in particular, $\overline{w} = \overline{w}'$).
- $\|w'\| < \|w\|$.
- For any j, let s and s' be the initial segments of w and w' of length j. Then $|\|s\| - \|s'\|| \leqslant K$ (the words w and w' are said to *K-fellow travel*).

We also require that if $w, w' \in L'$ are both minimal size representatives of the same element of L/P then $(w, w') \in R$.

Our main theorem about such partitions is the following:

Theorem 3.1. *Let P be a partition of a weighted regular language L with an acceptor R. Assume that P has the falsification by fellow traveler property. Then L/P has a regular minimal cross section.*

Theorems 3.1 and 2.2 imply the following:

Corollary 3.2. *Let P be a partition of a weighted regular language L with an acceptor R so that P has the falsification by fellow traveler property. Then L/P has rational growth.*

Before proving Theorem 3.1, we need a lemma.

Lemma 3.3. *Let P be a partition of a regular language L with a weighting ϕ and an acceptor R, and let K be a natural number. Then the following language is regular*:

$$L_K = \{(w_1, w_2) \in L \times L\colon\ (w_1, w_2) \in R,\ \|w_1\| > \|w_2\|,\ \textit{and } w_1 \textit{ and } w_2\ K\textit{-fellow travel}\}.$$

Proof. Observe that L_K is the intersection of R and the language

$$L'_K = \{(w_1, w_2) \in (A^*)^2\colon\ \|w_1\| > \|w_2\| \text{ and } w_1 \text{ and } w_2\ K\text{-fellow travel}\}.$$

By Theorem 2.1 it is therefore enough to show that L'_K is regular. We construct an automaton accepting L'_K as follows. For simplicity, we will extend ϕ to $A \cup \{\$\}$ by setting $\phi(\$) = 0$. Our automaton has $2K + 1$ states labeled $-K, \dots, K$ plus a failure state. The label on a numbered state represents the difference between the portions of w_1 and w_2 read thus far. We begin in state 0. Now assume that we are in a state i and read a from w_1 and b from w_2. If $|i + \phi(a) - \phi(b)| > K$, then w_1 and w_2 have ceased to K-fellow travel, so we go to the failure state. Otherwise, we go to the state $i + \phi(a) - \phi(b)$. We succeed and accept (w_1, w_2) if we end in a state with a positive label, and we fail otherwise. □

We now prove Theorem 3.1.

Proof of Theorem 3.1. Let L' be the regular sublanguage of L and K be the constant given by the definition of the falsification by fellow traveler property. By Lemma 3.3

$$L_K = \{(w_1, w_2) \in L \times L\colon\ (w_1, w_2) \in R,\ \|w_1\| > \|w_2\|,\ \text{and } w_1 \text{ and } w_2\ K\text{-fellow travel}\}$$

is a regular language. Hence by Theorem 2.1

$$L'' = \{w \in L'\colon\ \text{there does not exist any } w' \in L \text{ so that } (w, w') \in L_K\}$$

is a regular language. This language is composed of minimal size representatives in L/P. It contains at least one representative of each element. By [8, Remark, p. 57], the language

$$S = \{(w_1, w_2) \in L'' \times L''\colon\ w_1 \text{ is short-lex less than } w_2\}$$

is regular (see [8, p. 56] for the definition of the short-lex ordering. For our purposes its only important property is that it is a total ordering on the set of words). We conclude from Theorem 2.1 that

$$L''' = \{w \in L''\colon\ \text{for all } w' \in L'' \text{ we have } (w', w) \notin S \cap R\}$$

is regular. By the definition of the falsification by fellow traveler property, L''' contains a *unique* representative of minimal length for each element of L/P; i.e. it is a regular minimal cross section of L/P. □

4. Types and heights

Fix a torus bundle group Γ with monodromy M. Recall that we are examining the finite index subgroup $G = \langle a, t\rangle$.

4.1. Definitions

Consider some $g \in G$. Since $G \subset \Gamma = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$, we can regard g as a pair (x, h) with $x \in \mathbb{Z}^2$ and $h \in \mathbb{Z}$. We will call h the *height* of g (denoted $h(g)$) and x the *type* of g (denoted $\operatorname{type}(g)$).

Denote by F_T the free group on a set T. Consider $w \in F_{\{a,t\}}$ which maps to $\overline{w} \in G$. Set $h(w) = h(\overline{w})$ and $\operatorname{type}(w) = \operatorname{type}(\overline{w})$ (we will refer to these as the *height* and *type* of w). We wish to determine the relationship between w and $\operatorname{type}(w)$. Let N be the normal subgroup of $F_{\{a,t\}}$ generated by a. The exact sequence

$$1 \longrightarrow N \longrightarrow F_{\{a,t\}} \longrightarrow F_{\{t\}} \longrightarrow 1$$

splits, so we have $F_{\{a,t\}} = N \rtimes F_{\{t\}}$. This fits into the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d] & N \rtimes F_{\{t\}} \arrow[r] \arrow[d] & F_{\{t\}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \rtimes_M \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

The map $N \to \mathbb{Z}^2$ factors through the abelianization N^{ab} of N. Now, it is well known (see, e.g., [12, Exercise 3.2.3–4]) that N is the free group on the generating set

$$\{t^k a t^{-k} \colon k \in \mathbb{Z}\}.$$

The map

$$t^k a t^{-k} \longmapsto z^k$$

therefore defines an isomorphism from N^{ab} to the group $\mathbb{Z}[z, z^{-1}]$ of Laurent polynomials. Summing up, we have factored the map

$$\operatorname{type} \colon F_{\{a,t\}} \longrightarrow \mathbb{Z}^2$$

as a composition

$$F_{\{a,t\}} \longrightarrow N \longrightarrow N^{\mathrm{ab}} = \mathbb{Z}[z, z^{-1}] \longrightarrow \mathbb{Z}^2.$$

Denote by $\widetilde{\operatorname{type}}(w)$ the image of w in $\mathbb{Z}[z, z^{-1}]$; we will call this the *unreduced type* of w.

More concretely, the splitting $F_{\{a,t\}} = N \rtimes F_{\{t\}}$ shows that every word $w \in F_{\{a,t\}}$ can be expressed as a product

$$w = \left(\prod_{i=1}^{n} t^{k_i} a^{l_i} t^{-k_i}\right) t^h$$

with $h, k_i \in \mathbb{Z}$ and $l_i \in \{\pm 1\}$. Observe that $h(w) = h$. Also, $\widetilde{\text{type}}(w)$ equals the Laurent polynomial

$$\sum_{i=1}^{n} l_i z^{k_i} \in \mathbb{Z}\bigl[z, z^{-1}\bigr].$$

Since (continuing our systemic confusion of a with the vector $(1,0) \in \mathbb{Z}^2$)

$$\text{type}\bigl(t^{k_i} a^{l_i} t^{-k_i}\bigr) = l_i M^{k_i} a$$

we have the following lemma:

Lemma 4.1. *All $w \in F_{\{a,t\}}$ satisfy* $\text{type}(w) = [\widetilde{\text{type}}(w)(M)] \cdot a$.

4.2. Appearance of types

We now determine the length of the shortest word with a specified unreduced type and height. We begin with some terminology. Consider an unreduced type

$$t(z) = \sum_i c_i z^i \in \mathbb{Z}\bigl[z, z^{-1}\bigr]$$

with $c_i \in \mathbb{Z}$ and a height $h \in \mathbb{Z}$. The Laurent polynomial $t(z)$ can be divided into three different pieces (depending on h). There are two cases. If $h \geqslant 0$, we define

$$\mathcal{T}_h(t) := \sum_{i=-\infty}^{-1} c_i z^i,$$

$$\mathcal{C}_h(t) := \sum_{i=0}^{h} c_i z^i,$$

$$\mathcal{H}_h(t) := \sum_{i=h+1}^{\infty} c_i z^i,$$

$$\overline{\mathcal{T}}_h := \max\bigl\{|i|\colon\ i = 0 \text{ or } i < 0,\ c_i \neq 0\bigr\},$$

$$\overline{\mathcal{H}}_h := \max\{i - h\colon\ i = h \text{ or } i > h,\ c_i \neq 0\}.$$

If $h \leqslant 0$, we define

$$\mathcal{T}_h(t) := \sum_{i=-\infty}^{h-1} c_i z^i,$$

$$\mathcal{C}_h(t) := \sum_{i=h}^{0} c_i z^i,$$

$$\mathcal{H}_h(t) := \sum_{i=1}^{\infty} c_i z^i,$$

$$\overline{\mathcal{T}}_h := \max\{|i| - |h|\colon\ i = h \text{ or } i < h,\ c_i \neq 0\},$$

$$\overline{\mathcal{H}}_h := \max\{i\colon\ i = 0 \text{ or } i > 0,\ c_i \neq 0\}.$$

We will refer to $\mathcal{T}_h(t)$ as the *tail*, $\mathcal{C}_h(t)$ as the *center*, and $\mathcal{H}_h(t)$ as the *head*. Also, we will call $\overline{\mathcal{T}}_h(t)$ the *length of the tail* and $\overline{\mathcal{H}}_h(t)$ the *length of the head*. Observe that

$$t = \mathcal{T}_h(t) + \mathcal{C}_h(t) + \mathcal{H}_h(t).$$

Our theorem is the following:

Theorem 4.2. *Let h be a height and let*

$$t(z) = \sum_i c_i z^i \in \mathbb{Z}[z, z^{-1}]$$

be an unreduced type. Then the shortest word with this unreduced type and height has length

$$2\overline{\mathcal{T}}_h(t) + 2\overline{\mathcal{H}}_h(t) + |h| + \sum_i |c_i|.$$

Proof. We begin by describing an algorithm for determining the unreduced type and height of a word w in $\{a^{\pm 1}, t^{\pm 1}\}$. The algorithm keeps track of two pieces of data, the *partial height* $H \in \mathbb{Z}$ and the *partial unreduced type* $T \in \mathbb{Z}[z, z^{-1}]$. Both are initialized to 0. We read w from left to right. If we read the letter t^l with $l = \pm 1$, we add l to H. If we read the letter a^l with $l = \pm 1$, we add lz^H to T. After reading all of w, it is clear that $H = h(w)$ and that $T = \widetilde{\mathrm{type}}(w)$.

Now consider any word w with the desired height and unreduced type. Observe that each $a^{\pm 1}$ in w contributes exactly one term of the form $\pm z^i$. Hence w must contain at least $\sum_i |c_i|$ letters of the form $a^{\pm 1}$. To prove that w is at least as long as the theorem indicates, it is therefore enough to show that w contains at least $2\overline{\mathcal{T}}_h(t) + 2\overline{\mathcal{H}}_h(t) + |h|$ letters of the form $t^{\pm 1}$. We first consider the case $h \geqslant 0$. In this case, either $\mathcal{T}_h(t) = 0$ or $\mathcal{T}_h(t)$ must contain a non-zero term of degree $-\overline{\mathcal{T}}_h(t)$. This implies that during our algorithm the partial height H must at some point equal $-|\mathcal{T}_h(t)|$. Similarly, either $\mathcal{H}_h(t) = 0$ or $\mathcal{H}_h(t)$ must contain a non-zero term of degree $h + \overline{\mathcal{H}}_h(t)$. This implies that during our algorithm the partial height H must at some point equal $h + \overline{\mathcal{H}}_h(t)$. Since $h(w) = h$, our algorithm must end with $H = h$. Summing up, the partial height H (which changes by ± 1 each time a letter of the form $t^{\pm 1}$ is read) starts at 0, ends at h, at some point equals $-\overline{\mathcal{T}}_h(t)$, and at some other point equals $h + \overline{\mathcal{H}}_h(t)$. Clearly at least $2\overline{\mathcal{T}}_h(t) + 2\overline{\mathcal{H}}_h(t) + |h|$ letters of the form $t^{\pm 1}$ are necessary, as desired. The case of $h \leqslant 0$ is proven in a similar fashion, with the roles of $\mathcal{T}_h(t)$ and $\mathcal{H}_h(t)$ reversed.

This proves that the indicated expression is a lower bound on the length of a word with the desired unreduced type and height. We now prove that this lower bound is realized. Like in the proof of the lower bound, the proofs in the cases $h \geqslant 0$ and $h \leqslant 0$ are similar; we will only consider $h \geqslant 0$. In this case, the following word has the desired length, unreduced type, and height:

$$t^{-\overline{\mathcal{T}}_h(t)}\left(\prod_{i=-\overline{\mathcal{T}}_h(t)}^{-1} a^{c_i}t\right)a^{c_0}\left(\prod_{i=1}^{h+\overline{\mathcal{H}}_h(t)} ta^{c_i}\right)t^{-\overline{\mathcal{H}}_h(t)}. \qquad \square$$

Remark. After proving a version of Theorem 4.2, Grayson attempts to set up a complicated system of recurrence relations between various subsets of the group. He expresses the growth function as a power series whose coefficients are themselves power series. He demonstrates that there is a sort of linear recurrence relation between these (power series) coefficients. He then claims that this is enough to prove that the growth series is rational. However, absent a proof that (say) the first coefficient is in fact a rational function this is insufficient.

Let $T = \mathrm{trace}(M)$. Since M is Anosov, it has two distinct real eigenvalues. Let λ and λ' be the eigenvalues with eigenvectors v and v'. Let $\alpha, \alpha' \in \mathbb{R}$ be such that

$$(1,0) = \alpha v + \alpha' v'.$$

Theorem 4.3. *Let w_1 and w_2 be words in $\{a^{\pm 1}, t^{\pm 1}\}$. Then w_1 and w_2 represent the same element of G if and only if $h(w_1) = h(w_2)$ and $1 - Tz + z^2$ divides the Laurent polynomial $\widetilde{\mathrm{type}}(w_1) - \widetilde{\mathrm{type}}(w_2)$.*

Proof. Let

$$\widetilde{\mathrm{type}}(w_1) = \sum_i c_i z^i,$$

$$\widetilde{\mathrm{type}}(w_2) = \sum_i c'_i z^i.$$

Observe that with respect to the basis $\{v, v'\}$ Lemma 4.1 says that we have

$$\mathrm{type}(w_1) = \left(\alpha \sum_i c_i \lambda^i, \alpha' \sum_i c_i \lambda'^i\right),$$

$$\mathrm{type}(w_2) = \left(\alpha \sum_i c'_i \lambda^i, \alpha' \sum_i c'_i \lambda'^i\right).$$

Since M is a 2×2 matrix with irrational eigenvalues, λ and λ' have the same minimal polynomial as M; i.e. $1 - Tz + z^2$, whence the theorem. $\square$

Consider the set

$$X = \mathbb{Z}\left[z, z^{-1}\right] \times \mathbb{Z}.$$

Define a size function on X by setting

$$\left\| \left(\sum_i c_i z^i, h \right) \right\| := 2\overline{\mathcal{T}}_h\left(\sum_i c_i z^i \right) + 2\overline{\mathcal{H}}_h\left(\sum_i c_i z^i \right) + |h| + 1 + \sum_i |c_i|.$$

Define a partition P on X by the following equivalence relation:

$$(t_1, h_1) \sim (t_2, h_2) \quad \Longleftrightarrow \quad \left[h_1 = h_2 \text{ and } 1 - Tz + z^2 \text{ divides } t_1 - t_2\right].$$

We can now state the following important corollary to the above calculations:

Corollary 4.4. *G is near-isometric to X/P (with constant $c = 1$).*

Remark. The extra 1 in the definition of the size function on X simplifies the language L we construct in Section 5, as it forces *every* monomial in the center of a Laurent polynomial to contribute something to the size, even if it equals 0.

5. The language

By Corollary 3.2, to prove Theorem 1.1 it is enough to produce a regular language L with a partition P' satisfying the falsification by fellow traveler property so that L/P' is isometric to X/P. We first prove a number of finiteness results about X. Next, we will define a series of languages L_n and a series of corresponding partitions P_n. Finally, we will prove that for n sufficiently large L_n/P_n is isometric to X/P and satisfies the falsification by fellow traveler property.

5.1. *Finiteness lemmas*

The coefficients of the Laurent polynomials associated to elements of X are unbounded. To apply the theory of finite state automata to X/P, we will first prove a lemma which bounds the coefficients of the Laurent polynomials associated to elements of minimal size in a single subset in P. We will then prove two other lemmas which bound the information we need to keep track of while comparing elements of X modulo P.

Lemma 5.1. *Let $x = (t, h) \in X$ be so that*

$$\|x\| = \min\left\{\|x'\|\colon\ x = x' \text{ modulo } P\right\}.$$

Then the coefficients c_i of t satisfy $|c_i| < 5|T|$.

Proof. By the definition of P, we can for each i add or subtract $z^{i-1} - Tz^i + z^{i+1}$ from t without changing $\bar{x}$. Now, if $|c_i| \geqslant 5|T|$, add or subtract $5z^{i-1} - 5Tz^i + 5z^{i+1}$ in such a way as to decrease $|c_i|$. Examining the formula for the size of an element of X, we see that we have subtracted $5|T|$ from the size of x and added at most $2 + 2 + 5 + 5 = 14$. Since $|T| \geqslant 3$, we conclude that x was not of minimal size, a contradiction. □

Lemma 5.2. *For every positive integer A, there exists some positive integer B_A so that the following holds. For $i = 1, 2$ let $f_i = \sum_j c_{i,j} z^j$ with $|c_{i,j}| \leqslant A$. Assume that $1 - Tz + z^2$ divides $f_1 - f_2$. Then the coefficients of $(f_1 - f_2)/(1 - Tz + z^2)$ are bounded by B_A.*

Proof. Since $|T| \geqslant 3$, the largest coefficient which is left when we expand out $(1 - Tz + z^2)g(z)$ is at least as large as the largest coefficient of g. Hence we may set $B_A = 2A$. □

Lemma 5.3. *For all positive integers A and B, there exists some positive integer $C_{A,B}$ so that the following holds. For $i = 1, 2$ let $f_i = \sum_j c_{i,j} z^j$ with $|c_{i,j}| \leqslant A$. Assume that*

$$f_1 - f_2 = \left(\left(1 - Tz + z^2\right) \sum_j d_j z^j\right) + (e_1 z + e_2)$$

with $|d_j| \leqslant B$. Then $|e_1|, |e_2| \leqslant C_{A,B}$.

Proof. Observe that the coefficients of $(1 - Tz + z^2) \sum_j d_j z^j$ are bounded by $B(|T| + 2)$. Hence the coefficients of

$$e_1 z + e_2 = f_1 - f_2 - \left(\left(1 - Tz + z^2\right) \sum_j d_j z^j\right)$$

are bounded by $C_{A,B} := 2A + B(|T| + 2)$. □

5.2. The language

Fix a natural number $n \geqslant 1$. Let

$$A_n = \{-n, \ldots, n\} \times \{-1, 1, 2\}$$

be an alphabet with weighting

$$\phi(c, k) = |c| + |k|.$$

Consider the language L_n on A_n whose words are of the following form:

$$(\cdot, 2) \cdots (\cdot, 2)(\cdot, \pm 1) \cdots (\cdot, \pm 1)(\cdot, 2) \cdots (\cdot, 2).$$

We require words w in L_n to satisfy the following conditions.

(1) w must contain at least one letter of the form $(\cdot, \pm 1)$.
(2) The second entries in all the middle terms of w must be identical.
(3) If the common second entry in all the middle terms of w is -1, then there must be at least two such middle terms.
(4) If the first or last letters of w equal $(c, 2)$, then $c \neq 0$.

We also define the language L'_n to consist of all such words w satisfying conditions (1)–(3) but not necessarily (4). Both L_n and L'_n are clearly regular. Define a map $\psi' : L'_n \to X$ by

$$\psi'\left(\prod_{i=1}^{n_1}(c_i, 2)\prod_{i=0}^{n_2}\left(c'_i, 1\right)\prod_{i=1}^{n_3}\left(c''_i, 2\right)\right) = \left(\sum_{i=1}^{n_1} c_i z^{i-n_1-1} + \sum_{i=0}^{n_2} c'_i z^i + \sum_{i=1}^{n_3} c''_i z^{n_2+i}, n_2\right)$$

and

$$\psi'\left(\prod_{i=1}^{n_1}(c_i, 2)\prod_{i=0}^{n_2}\left(c'_i, -1\right)\prod_{i=1}^{n_3}\left(c''_i, 2\right)\right) = \left(\sum_{i=1}^{n_1} c_i z^{i-n_1-n_2-1} + \sum_{i=0}^{n_2} c'_i z^{i-n_2} + \sum_{i=1}^{n_3} c''_i z^i, -n_2\right).$$

Let ψ be the restriction of ψ' to $L_n \subset L'_n$. Observe that ψ' induces a partition P'_n of L'_n and ψ induces a partition P_n of L_n. The map ψ is clearly a size-preserving map from L_n to X, and the fact that we require that if the sign of the center terms is negative then there must be at least two center terms forces it to be an injection (this condition prevents trouble from occurring when $h = 0$). Lemma 5.1 implies the following:

Theorem 5.4. *For $n \geqslant 5|T|$ the induced map $\overline{\psi} : L_n/P_n \to X/P$ is an isometry.*

We now observe that the tripartite division of words in L'_n reflects the tail–center–head division of the corresponding Laurent polynomials. If $w \in L'_n$ with $\psi'(w) = (t, h)$, we define

$$\mathcal{T}(w) = \mathcal{T}_h(t),$$
$$\mathcal{C}(w) = \mathcal{C}_h(t),$$
$$\mathcal{H}(w) = \mathcal{H}_h(t).$$

We will refer to these as the *tail*, the *center*, and the *head* of w. We also define $\overline{\mathcal{T}}(w)$ and $\overline{\mathcal{H}}(w)$ to equal the number of letters of the form $(c, 2)$ at the beginning and end of w. We remark that if w begins or ends with $(0, 2)$, then $\overline{\mathcal{T}}(w) \neq \overline{\mathcal{T}}_h(t)$ or $\overline{\mathcal{H}}(w) \neq \overline{\mathcal{H}}_h(t)$.

5.3. The acceptor

Fix positive integers n and i. Define a language

$$R_{n,i} = \left\{(w_1, w_2) \in L_n \times L_n \colon\ \overline{w}_1 = \overline{w}_2 \text{ in } L_n/P_n \text{ and } \left|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2)\right|, \left|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2)\right| \leqslant i\right\}.$$

This section is devoted to proving the following theorem:

Theorem 5.5. *$R_{n,i}$ is a regular language.*

This has the following immediate corollary:

Corollary 5.6. *$R_{n,i}$ is an acceptor for the partition P_n of the language L_n.*

To prove Theorem 5.5, we need the following lemma:

Lemma 5.7. *Define*

$$R'_n = \{(w_1, w_2) \in L'_n \times L'_n\colon\ \overline{w}_1 = \overline{w}_2 \text{ in } L'_n/P'_n \\ \text{and } \overline{\mathcal{T}}(w_1) = \overline{\mathcal{T}}(w_2),\ \overline{\mathcal{H}}(w_1) = \overline{\mathcal{H}}(w_2)\}.$$

Then R'_n is a regular language.

Proof. By Theorem 2.1, it is enough to construct an automaton accepting $\operatorname{rev}(R'_n)$, or (to put it in another way) to construct an automaton which reads w_1 and w_2 from right to left. Let $B = B_n$ be the constant from Lemma 5.2 and $C = C_{n,B_n}$ be the constant from Lemma 5.3. Our strategy will be to imitate the usual polynomial long division algorithm to divide the difference between the Laurent polynomials associated to w_1 and w_2 by $1 - Tz + z^2$. We also will make sure that w_1 and w_2 "line up" properly; that is that they have heads, centers, and tails of the same length.

Our automaton has a failure state plus the following set of states:

$$\{(r, l)\colon\ r = c_1 z + c_2 \text{ with } c_i \in \mathbb{Z} \text{ so that } |c_i| \leqslant C \text{ and} \\ l \in \{\mathcal{H}, \mathcal{T}, \mathcal{C}_1, \mathcal{C}_{-1}, \mathcal{C}_{-1,1}\}\}.$$

The second entry in a state keeps track of where we are in w_1 and w_2 (the label $\mathcal{C}_1$ means that the center portion consists of terms of the form $(\cdot, 1)$, the label $\mathcal{C}_{-1,1}$ means that the center portion consists of terms of the form $(\cdot, -1)$ and we have only read one term of that form, and the label $\mathcal{C}_{-1}$ means that the center portion consists of terms of the form $(\cdot, -1)$ and we have read at least two terms of that form). The first entry keeps track of the remainder obtained by dividing the difference of the portion read so far by $1 - Tz + z^2$. Recalling that our automaton reads w_1 and w_2 from right to left, we begin in the state $(0, \mathcal{H})$. Assume that $\psi'(w_i) = (t_i, h_i)$ with

$$t_i = \sum_{j=-N_1}^{N_2} c_{i,j} z^j.$$

Assume now that we are in the state (r, l) after reading k letters. This means that there exists some Laurent polynomial q (whose value does not matter—all that matters for determining the transitions are the values of r and l) so that

$$\sum_{i=N_2-k+1}^{N_2} (c_{1,i} - c_{2,i}) z^i = z^{N_2-k+1}\bigl(q \cdot \bigl(1 - Tz + z^2\bigr) + r\bigr).$$

If we do not read entries of the form (c_{1,N_2-k}, e) from w_1 and (c_{2,N_2-k}, e) from w_2 (in other words, if at this point w_1 and w_2 cease to "line up"), then we fail. Otherwise, the difference between the portions read so far is

$$\sum_{i=N_2-k}^{N_2} (c_{1,i} - c_{2,i}) z^i = z^{N_2-k}\bigl(zq \cdot \bigl(1 - Tz + z^2\bigr) + \bigl(zr + (c_{1,N_2-k} - c_{2,N_2-k})\bigr)\bigr).$$

Note that $zr + (c_{1,N_2-k} - c_{2,N_2-k})$ is a quadratic polynomial. Divide it by $1 - Tz + z^2$ to get

$$zr + (c_{1,N_2-k} - c_{2,N_2-k}) = q'\bigl(1 - Tz + z^2\bigr) + r',$$

where r' is a linear function. If the coefficients of r' are not bounded by C, then by Lemma 5.3 it is impossible for w_1 and w_2 to define equal elements of L'_n modulo P'_n, and we fail. Otherwise, we make the following transition: If $l = \mathcal{H}$ or $l = \mathcal{T}$ and $e = 2$, then we transition to (r', l). If $l = \mathcal{H}$ and $e = 1$, then we transition to $(r', \mathcal{C}_1)$. If $l = \mathcal{H}$ and $e = -1$, then we transition to $(r', \mathcal{C}_{-1,1})$. If $l = \mathcal{C}_1$ and $e = 1$, we transition to $(r', \mathcal{C}_1)$. If $l = \mathcal{C}_1$ or $l = \mathcal{C}_{-1}$ and $e = 2$, we transition to $(r', \mathcal{T})$. If $l = \mathcal{C}_{-1,1}$ or $l = \mathcal{C}_{-1}$ and $e = -1$, we transition to $(r', \mathcal{C}_{-1})$. If we are not in one of these situations, we fail.

Assume now that we manage to successfully read all of w_1 and w_2 and end in a state (r, l). This implies, in particular, that the heads, centers, and tails of w_1 and w_2 are of the same length. Also, it is clear from the algorithm that r is the remainder of the difference of the Laurent polynomials associated to w_1 and w_2 divided by $1 - Tz + z^2$. We succeed if we end in one of the following three states: $(0, \mathcal{T})$, $(0, \mathcal{C}_1)$, or $(0, \mathcal{C}_{-1})$. The restriction on l is required to guarantee that both w_1 and w_2 contain centers of the appropriate form. □

We now prove Theorem 5.5.

Proof of Theorem 5.5. Set $Q_r = \prod_{j=1}^{r}(0, 2)$. Observe that

$$\begin{aligned} R_{n,i} = \bigl\{&(w_1, w_2)\colon\ w_1, w_2 \in L_n \text{ and there exist } r, s \in \mathbb{Z} \text{ so that } 0 \leqslant r, s \leqslant i \\ &\text{and either } (Q_r w_1 Q_s, w_2) \in R'_n,\ (Q_r w_1, w_2 Q_s) \in R'_n, \\ &(w_1 Q_s, Q_r w_2) \in R'_n, \text{ or } (w_1, Q_r w_2 Q_s) \in R'_n\bigr\}. \end{aligned}$$

Since the integers r and s which appear in this expression are bounded, it can be expressed using first order predicates and concatenation. Theorem 2.1 therefore implies that $R_{n,i}$ is a regular language. □

5.4. The falsification by fellow traveler property and proof of the main theorem

In this section we will complete the proof of Theorem 1.1. We will need the following definition:

Definition. Consider $w_1, w_2 \in L_n$ with $\psi(w_i) = (\sum_j c_{i,j} z^j, h_i)$. The *divergence* of w_1 and w_2 is the maximal absolute value of

$$\sum_{j=-\infty}^{k} |c_{1,j}| - |c_{2,j}|$$

as k varies.

The key step in our proof will be the following theorem:

Theorem 5.8. *There exist constants K, L, and N so that $N \geqslant 5|T|$ and the following are true.*

(1) *If $w_1 \in L_{5|T|}$ is not a minimal size representative modulo $P_{5|T|}$, then there exists some $w_2 \in L_N$ so that*:
   - $\overline{w}_1 = \overline{w}_2$ *and* $\|w_2\| < \|w_1\|$.
   - $|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2)| \leqslant L$ *and* $|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2)| \leqslant L$.
   - *The divergence of w_1 and w_2 is bounded by K.*

(2) *If $w_1, w_2 \in L_{5|T|}$ are two different minimal size representatives of the same element modulo $P_{5|T|}$, then $|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2)| \leqslant L$ and $|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2)| \leqslant L$.*

Before proving Theorem 5.8, we will use it to prove Theorem 1.1.

Proof of Theorem 1.1. By Corollary 4.4 and Theorem 5.4, it is enough to show that L_N/P_N has a rational growth series for large N. Let K, L, and N be the constants from Theorem 5.8. We will prove that P_N is a partition of L_N with acceptor $R_{N,L}$ satisfying the falsification by fellow traveler condition with respect to the constant $K + (N + 6)L$. By Theorem 3.2, this will imply that L_N/P_N has a rational growth series, as desired.

We begin by observing that by Lemma 5.1, $L_{5|T|}$ contains minimal size elements from each set in P_N. Now, let $w_1 \in L_{5|T|}$ not be a minimal size representative modulo $P_{5|T|}$. Consider the $w_2 \in L_N$ given by the first conclusion of Theorem 5.8. It is clear that $(w_1, w_2) \in R_{N,L}$ and that $\|w_2\| < \|w_1\|$. We must prove that w_1 and w_2 $(K + (N + 6)L)$-fellow travel.

We will assume that $\overline{\mathcal{T}}(w_2) \leqslant \overline{\mathcal{T}}(w_1)$; the other case is similar. Consider length j initial segments v_1 and v_2 of w_1 and w_2. Let v_2' be the initial segment of w_2 of length $j - (\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2))$. Since the divergence of w_1 and w_2 is bounded by K, we know that

$$\big|\|v_1\| - \|v_2'\|\big| \leqslant K + 4L.$$

The $4L$ term comes from the fact that each term in the initial segment of length $\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2) \leqslant L$ of v_1 contributes an extra 2 to the difference, and in addition either v_1 or v_2' may contain at most L terms from the head which are absent from the other, each possibly contributing 2 more to the difference. The remaining portion of v_2 has length at most L and each term contributes at most $2 + N$ to the size of v_2. Hence we conclude that

$$\big|\|v_1\| - \|v_2\|\big| \leqslant K + 4L + L(2 + N) = K + (N + 6)L$$

as desired.

Now, by the second conclusion of Theorem 5.8, if w_1 and w_2 are two minimal size representatives of the same element of $L_{5|T|}$ modulo $P_{5|T|}$, then $(w_1, w_2) \in R_{N,L}$. This completes the proof of the falsification by fellow traveler property, and hence of the theorem. □

We now prove Theorem 5.8.

Proof of Theorem 5.8. Let $B := B_{5|T|}$ be the constant from Lemma 5.2. We will prove that the following choices of L, K, and N suffice:

$$L = \big(|T| + 2\big)B,$$
$$K = \big(|T| + 2\big)(3B + 4) + 8L + 1,$$
$$N = 5|T| + \big(|T| + 2\big)B.$$

We begin by proving the first conclusion of the theorem. Let $w_1 \in L_{5|T|}$ not be a minimal size representative modulo $P_{5|T|}$. By Lemma 5.1, there exists some $w_2 \in L_{5|T|}$ so that $\overline{w}_1 = \overline{w}_2$ and $\|w_2\| < \|w_1\|$. Let $t_1, t_2 \in \mathbb{Z}[z, z^{-1}]$ and $h \in \mathbb{Z}$ be so that $(t_i, h) = \psi(w_i)$. Expand the t_i as $t_i = \sum_j c_{i,j} z^j$. By Lemma 5.2, there exists some Laurent polynomial $q = \sum_j d_j z^j$ with $|d_j| \leqslant B$ so that $t_2 = t_1 + (1 - Tz + z^2)q$.

Our goal will be to modify t_2 and q to produce new Laurent polynomials t_2' and q' with $t_2' = t_1 + (1 - Tz + z^2)q'$ so that (expanding q' and t_2' as $q' = \sum_j d_j' z^j$ and $t_2' = \sum_j c_{2,j}' z^j$ and setting $w_2' = \psi^{-1}(t_2', h)$) the following conditions are satisfied:

(1) $|d_j'| \leqslant B$ and $\|w_2'\| < \|w_1\|$.
(2) $\overline{\mathcal{T}}(w_2') - \overline{\mathcal{T}}(w_1) \leqslant L$ and $\overline{\mathcal{H}}(w_2') - \overline{\mathcal{H}}(w_1) \leqslant L$.
(3) $\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2') \leqslant L$ and $\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2') \leqslant L$.
(4) For all k we have $\sum_{j=-\infty}^{k} (|c_{2,j}'| - |c_{1,j}|) \leqslant K$.
(5) For all k we have $\sum_{j=-\infty}^{k} (|c_{1,j}| - |c_{2,j}'|) \leqslant K$.

The first part of condition (1) implies that $w_2' \in L_N$, and the rest of the conditions imply the first conclusion of the theorem. Our modification will take several steps; to prevent a proliferation of new notation we will continue to refer to the modified polynomials, words, and coefficients as t_2, q, w_2, d_j, and $c_{2,j}$. The modifications are the following:

Claim 1. *We can modify w_2 so that conditions* (1) *and* (2) *are satisfied.*

Proof. We will indicate how to achieve $\overline{\mathcal{T}}(w_2) \leqslant \overline{\mathcal{T}}(w_1) + L$; the other modification is similar. Assume that $\overline{\mathcal{T}}(w_2) > \overline{\mathcal{T}}(w_1) + (|T| + 2)B$. We will show how to find a w_2' so that condition (1) is satisfied and so that $\|w_2'\| < \|w_2\|$; repeating this process will eventually yield the desired conclusion. The idea of our construction is that since each element of the tail of w_2 contributes something to $\|w_2\|$, if the tail is sufficiently long then we can remove the first few terms from it and shrink $\|w_2\|$. Let M be the smallest integer with $d_M \neq 0$. Hence

$$t_2 = t_1 + \big(1 - Tz + z^2\big) \sum_{j=M}^{\infty} d_j z^i.$$

Set $M' = M + (|T| + 2)B$ and

$$t_2' = t_1 + \left(1 - Tz + z^2\right) \sum_{j=M'}^{\infty} d_j z^i.$$

Expand this as $t_2' = \sum_j c_{2,j}' z^j$ and set $w_2' = \psi^{-1}(t_2', h)$.

The only non-trivial fact we must prove is $\|w_2'\| < \|w_2\|$. Observe first that by construction we have

$$\overline{\mathcal{T}}(w_2) - \overline{\mathcal{T}}\left(w_2'\right) \geqslant \left(|T| + 2\right)B.$$

Also,

$$\left|c_{2,j}'\right| = \begin{cases} 0 & \text{for } j < M', \\ |c_{2,j} - d_{j-2} + Td_{j-1}| \leqslant |c_{2,j}| + (|T| + 1)B & \text{for } j = M', \\ |c_{2,j} - d_{j-2}| \leqslant |c_{2,j}| + B & \text{for } j = M' + 1, \\ |c_{2,j}| & \text{for } j > M' + 1. \end{cases}$$

Finally, we may have lengthened the head of w_2 by 1; i.e.

$$\overline{\mathcal{H}}(w_2) - \overline{\mathcal{H}}\left(w_2'\right) \geqslant -1.$$

Summing up,

$$\begin{aligned} \|w_2\| - \left\|w_2'\right\| &= 2\left(\overline{\mathcal{T}}(w_2) - \overline{\mathcal{T}}\left(w_2'\right)\right) + 2\left(\overline{\mathcal{H}}(w_2) - \overline{\mathcal{H}}\left(w_2'\right)\right) + \sum_j |c_{2,j}| - \left|c_{2,j}'\right| \\ &\geqslant 2\left(|T| + 2\right)B - 2 - \left(|T| + 1\right)B - B = \left(|T| + 2\right)B - 2 > 0 \end{aligned}$$

as desired. □

Claim 2. *We can modify the w_2 produced in Claim* 1 *so that conditions* (1)–(3) *are satisfied.*

Proof. Assume that $\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2) > L$. The idea of our construction of w_2' is that since each term of the tail of w_1 contributes something to $\|w_1\|$ and w_2 has a much shorter tail than w_1 we can use only the initial portion of the quotient q to shorten the tail of w_1 by enough to shrink $\|w_1\|$. Let M be the smallest integer with $c_{1,M} \neq 0$. Set $M' = M + (|T| + 2)B$ and let

$$t_2' = t_1 + \left(1 - Tz + z^2\right) \sum_{j=-\infty}^{M'-1} d_j z^i.$$

Expand this as $t_2' = \sum_j c_{2,j}' z^j$ and set $w_2' = \psi^{-1}(t_2', h)$.

Observe that $c_{2,j}' = 0$ for $j < M'$. Informally, we have "chopped off" the first $(|T| + 2)B$ terms from the tail of w_1. However, we may have been too successful: possibly $c_{2,M'}' = 0$, indicating that we have shortened the tail more than we intended. If this is the case, add or subtract

$z^{M'}(1 - Tz + z^2)$ from t_2' in such a way as to insure that we still have $w_2' \in L_N$. There is therefore some integer $E \in \{-1, 0, 1\}$ so that

$$\left|c'_{2,j}\right| = \begin{cases} 0 & \text{if } j < M', \\ |c_{1,j} + d_{j-2} - Td_{j-1} + E| & \text{if } j = M', \\ |c_{1,j} + d_{j-2} - ET| & \text{if } j = M'+1, \\ |c_{1,j} + E| & \text{if } j = M'+2, \\ |c_{1,j}| & \text{if } j > M'+2 \end{cases}$$
$$\leqslant \begin{cases} 0 & \text{if } j < M', \\ |c_{1,j}| + (|T|+1)B + 1 & \text{if } j = M', \\ |c_{1,j}| + B + |T| & \text{if } j = M'+1, \\ |c_{1,j}| + 1 & \text{if } j = M'+2, \\ |c_{1,j}| & \text{if } j > M'+2. \end{cases}$$

Also,

$$\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}\big(w_2'\big) = \big(|T|+2\big)B = L.$$

Finally, we may have changed the length of the head of w_1 by 1; i.e.

$$\left|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}\big(w_2'\big)\right| \leqslant 1.$$

These facts imply that w_2' satisfies conditions (2) and (3). To show that w_2' also satisfies condition (1), we must show that $\|w_2'\| < \|w_1\|$. This follows from the following calculation:

$$\begin{aligned} \|w_1\| - \left\|w_2'\right\| &= 2\big(\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}\big(w_2'\big)\big) + 2\big(\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}\big(w_2'\big)\big) + \sum_j |c_{1,j}| - \left|c'_{2,j}\right| \\ &\geqslant 2\big(|T|+2\big)B - 2 - \big(\big(|T|+1\big)B + 1\big) - \big(B + |T|\big) - 1 \\ &= \big(|T|+2\big)B - |T| - 4 = |T|(B-1) + (2B-4) > 0. \end{aligned}$$

The final inequality follows from the fact that $B \geqslant 2$. In a similar way, one can show that if $\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2) > L$ then one can modify w_2 in an appropriate way. □

Claim 3. *We can modify the w_2 produced in Claim* 2 *so that conditions* (1)–(4) *are satisfied.*

Proof. Assume that for some k we have

$$\sum_{j=-\infty}^{k} \big(|c_{2,j}| - |c_{1,j}|\big) > \big(|T|+2\big)B + 8L + 1 + 2\big(|T|+2\big).$$

We will show that we can find some w_2'' satisfying conditions (1)–(3) so that $\|w_2''\| < \|w_2\|$; repeating this process will eventually yield the desired conclusion. This construction will be a two-step process. The idea of the first part of our construction is that since the initial portion of w_2 is so much larger than the corresponding portion of w_1 we can remove the initial portion from q to get a w_2' which begins like w_1 and ends like w_2 and is smaller than w_2. Set

$$t_2' = t_1 + \left(1 - Tz + z^2\right) \sum_{j=k+1}^{\infty} d_j z^j.$$

Expand this as $t_2' = \sum_j c_{2,j}' z^j$ and set $w_2' = \psi^{-1}(t_2', h)$. Observe that

$$\left|c_{2,j}'\right| = \begin{cases} |c_{1,j}| & \text{if } j \leqslant k, \\ |c_{2,j} - d_{j-2} + T d_{j-1}| \leqslant |c_{2,j}| + (|T|+1)B & \text{if } j = k+1, \\ |c_{2,j} - d_{j-2}| \leqslant |c_{2,j}| + B & \text{if } j = k+2, \\ |c_{2,j}| & \text{if } j > k+2. \end{cases}$$

Now, in a manner similar to that in Claim 2, we may have inadvertently shortened the tail or head of w_2 so much that w_2' no longer satisfies condition (3). To fix this, create a new Laurent polynomial t_2'' by adding $(E_1 z^{M_1} + E_2 z^{M_2})(1 - Tz + z^2)$ with $E_1, E_2 \in \{-1, 0, 1\}$ and $M_1, M_2 \in \mathbb{Z}$ to t_2' in such a way as to assure that (setting $w_2'' = \psi^{-1}(t_2'', h)$) we have $w_2'' \in L_N$ and

$$\left|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}\left(w_2''\right)\right|, \left|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}\left(w_2''\right)\right| \leqslant L.$$

Observe that the divergence of w_2' and w_2'' is bounded by $2(|T|+2)$. This implies that

$$\begin{aligned}
\|w_2\| - \left\|w_2''\right\| &\geqslant 2\left(\overline{\mathcal{T}}(w_2) - \overline{\mathcal{T}}\left(w_2''\right)\right) + 2\left(\overline{\mathcal{H}}(w_2) - \overline{\mathcal{H}}\left(w_2''\right)\right) \\
&\quad + \left(\sum_j \left(|c_{2,j}| - \left|c_{2,j}'\right|\right) - 2\left(|T|+2\right)\right) \\
&\geqslant 2\left(\overline{\mathcal{T}}(w_2) - \overline{\mathcal{T}}(w_1)\right) + 2\left(\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}\left(w_2''\right)\right) \\
&\quad + 2\left(\overline{\mathcal{H}}(w_2) - \overline{\mathcal{H}}(w_1)\right) + 2\left(\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}\left(w_2''\right)\right) \\
&\quad + \sum_{j=-\infty}^{k} \left(|c_{2,j}| - |c_{1,j}|\right) - \left(|T|+1\right)B - B - 2\left(|T|+2\right) \\
&> -8L + \left(|T|+2\right)B + 8L + 1 + 2\left(|T|+2\right) \\
&\quad - \left(|T|+1\right)B - B - 2\left(|T|+2\right) \\
&= 1
\end{aligned}$$

as desired. □

Claim 4. *We can modify the w_2 produced in Claim* 3 *so that conditions* (1)–(5) *are satisfied.*

Proof. Assume that for some k we have

$$\sum_{j=-\infty}^{k} \left(|c_{1,j}| - |c_{2,j}|\right) > \left(|T|+2\right)B + 8L + 1 + 2\left(|T|+2\right).$$

Pick this k to be the minimal k with this property. Since $|c_{1,j}| - |c_{2,j}| \leqslant (|T|+2)B$, we have

$$\sum_{j=-\infty}^{k} |c_{1,j}| - |c_{2,j}| < \big((|T|+2)B + 8L + 1\big) + 2\big(|T|+2\big) + \big(|T|+2\big)B.$$

We will construct from w_2 a w_2'' satisfying conditions (1)–(4) whose divergence from w_1 is bounded by K. Again, the construction of w_2'' is a two-step process. The idea of the first part of our construction is that since the initial portion of w_2 is so much smaller than the corresponding portion of w_1 we can use only the initial portion of q to get a word w_2' which is definitely smaller than w_1. Set

$$t_2' = t_1 + \big(1 - Tz + z^2\big) \sum_{j=-\infty}^{k} d_j z^j.$$

Expand this as $t_2' = \sum_j c_{2,j}' z^j$ and set $w_2' = \psi^{-1}(t_2', h)$. Observe that

$$c_{2,j}' = \begin{cases} c_{2,j} & \text{if } j \leqslant k, \\ c_{1,j} & \text{if } j > k+2 \end{cases}$$

and

$$\big||c_{2,j}'| - |c_{1,j}|\big| \leqslant \begin{cases} (|T|+1)B & \text{if } j = k+1, \\ B & \text{if } j = k+2. \end{cases}$$

Now, like in Claim 3 we may have inadvertently shortened the tail or head of w_2 so much that w_2' no longer satisfies condition (3). To fix this, create a new Laurent polynomial t_2'' by adding $(E_1 z^{M_1} + E_2 z^{M_2})(1 - Tz + z^2)$ with $E_1, E_2 \in \{-1, 0, 1\}$ and $M_1, M_2 \in \mathbb{Z}$ to t_2' in such a way as to assure that (setting $w_2'' = \psi^{-1}(t_2'', h)$) we have $w_2'' \in L_N$ and

$$\big|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}\big(w_2''\big)\big|, \big|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}\big(w_2''\big)\big| \leqslant L.$$

Observe that the divergence of w_2' and w_2'' is bounded by $2(|T|+2)$. This implies that the divergence of w_1 and w_2'' is bounded by $2(|T|+2)$ plus the divergence of w_1 and w_2'. The above formulas plus the minimality of k imply that the divergence of w_1 and w_2' is bounded by

$$\begin{aligned} &\big((|T|+2)B + 8L + 1\big) + 2\big(|T|+2\big) + \big(|T|+2\big)B + \big(|T|+1\big)B + B \\ &\quad = \big(|T|+2\big)(3B+2) + 8L + 1. \end{aligned}$$

We conclude that the divergence of w_1 and w_2'' is bounded by

$$\big(|T|+2\big)(3B+2) + 8L + 1 + 2\big(|T|+2\big) = \big(|T|+2\big)(3B+4) + 8L + 1 = K$$

as desired. It is enough, therefore, to prove that $\|w_2''| < \|w_1\|$. By the above,

$$\begin{aligned}
\|w_1\| - \|w_2''\| &\geqslant 2\big(\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2')\big) + 2\big(\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2')\big) \\
&\quad + \left(\sum_j \big(|c_{1,j}| - |c'_{2,j}|\big) - 2\big(|T|+2\big)\right) \\
&> -2L - 2L + \big(\big(|T|+2\big)B + 8L + 1 + 2\big(|T|+2\big) - 2\big(|T|+2\big)\big) \\
&\quad - \big(|T|+1\big)B - B \\
&= 4L + 1 > 0
\end{aligned}$$

as desired. □

These claims complete the proof of the first conclusion of the theorem.

We now prove the second conclusion. We recall that the second conclusion of the theorem is that if w_1 and w_2 are two minimal size representatives of the same element modulo $P_{5|T|}$, then $|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2)| \leqslant L$ and $|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2)| \leqslant L$. Assume that w_1 and w_2 are equal modulo $P_{5|T|}$ and satisfy either $|\overline{\mathcal{H}}(w_1) - \overline{\mathcal{H}}(w_2)| > L$ or $|\overline{\mathcal{T}}(w_1) - \overline{\mathcal{T}}(w_2)| > L$. Without loss of generality assume that either $\overline{\mathcal{H}}(w_2) - \overline{\mathcal{H}}(w_1) > L$ or $\overline{\mathcal{T}}(w_2) - \overline{\mathcal{T}}(w_1) > L$. The proof of Claim 1 tells us then that there exists some w_2' satisfying $\overline{w}_2' = \overline{w}_1$ and $\|w_2'\| < \|w_2\|$. In particular, w_2 was not of minimal size, as desired. □

6. Some questions

As we remarked in the introduction, using the methods of this paper to actually compute growth series would be a long and unpleasant task. However, in an unpublished paper Parry has calculated some growth series for torus bundle groups [14]. We reproduce his formulas here. He considers a torus bundle whose monodromy has trace $2T$. Letting $\langle a, b, t\rangle$ be the natural generators, he calculates the growth series of the finite-index subgroup generated by $\langle a, tat^{-1}, t\rangle$ with respect to that generating set (observe that this is the same subgroup we considered, but with one additional generator. It is not too hard to adapt our proof to this new generating set). He proves that the growth function is $N(z)/D(z)$, where $N(z)$ and $D(z)$ are the following:

$$\begin{aligned}
N(z) &= (1-z)^2(1+z)\big(1 + 3z + 4z^2 + 4z^3 + 3z^4 + z^5 \\
&\quad - z^T - 3z^{T+1} - 14z^{T+2} - 16z^{T+3} - 11z^{T+4} - 5z^{T+5} + 2z^{T+6} \\
&\quad + 2z^{2T+1} - 13z^{2T+2} + 35z^{2T+3} + 40z^{2T+4} + 6z^{2T+5} \\
&\quad - 23z^{2T+6} - 7z^{2T+7} + 4z^{2T+8} + 4z^{2T+9} \\
&\quad - 5z^{3T+2} + 31z^{3T+3} - 40z^{3T+4} - 44z^{3T+5} \\
&\quad + 33z^{3T+6} + 25z^{3T+7} - 12z^{3T+8} - 4z^{3T+9}\big), \\
D(z) &= \big(1 - 2z - z^2 - z^T + 4z^{T+1} - z^{T+2}\big)\big(1 - z - z^2 - z^3 \\
&\quad - z^{T+1} + 3z^{T+2} + z^{T+3} - z^{T+4}\big)^2.
\end{aligned}$$

Since Parry only dealt with the even trace case, we pose the following combinatorial challenge.

Question 1. Explicitly compute the growth series of our finite index subgroups for torus bundles with odd trace monodromy.

The 3-dimensional Sol groups are the fundamental groups of 2-dimensional torus bundles over a circle whose monodromy has no eigenvalues on the unit circle. By considering n-dimensional torus bundles over a circle with the same restriction on the monodromy, we get the $(n+1)$-dimensional Sol groups. It seems difficult to generalize our methods to these groups. This suggests the following question.

Question 2. Do the higher-dimensional Sol groups have rational growth functions?

The fact that we were only able to find a finite index subgroup with rational growth suggests the following question.

Question 3. Does there exist any group G which has irrational growth with respect to all sets of generators but which contains a finite index subgroup G' which has rational growth with respect to some set of generators?

The following more general question also seems interesting.

Question 4. Consider the property of having rational growth with respect to some set of generators. How does this property behave under commensuration? under quasi-isometry?

Remark. Observe that $\widetilde{\mathrm{SL}_2}$ and $\mathbb{H}^2 \times \mathbb{R}$ are quasi-isometric, and hence the fundamental groups of manifolds modeled on these geometries are quasi-isometric. An easy consequence of Cannon's work (see [5]) is that the fundamental groups of manifolds modeled on $\mathbb{H}^2 \times \mathbb{R}$ are rational with respect to any generating set. The work of Shapiro suggests that this is probably false for manifolds modeled on $\widetilde{\mathrm{SL}_2}$ (see [16]), so the property of being rational with respect to *all* generating sets is likely not well-behaved under quasi-isometry. The question of whether the fundamental group of any $\widetilde{\mathrm{SL}_2}$-manifold has rational growth with respect to some generating set is still open.

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