The Shintani descents of Suzuki groups and their consequences

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Abstract

The main aim of this paper is to associate to every cuspidal unipotent character of the Suzuki group its root of unity and to show that the Fourier matrix defined by Geck and Malle for the family of the cuspidal unipotent characters of this group satisfies a conjecture of Digne and Michel. To this end we determine the character table of the extension of $\text{Sp}_4$ by the exceptional graph automorphism in characteristic 2 and compute the Shintani descents of Suzuki groups.

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1. Introduction

Let $G$ be a connected reductive group defined over the finite field with $q$ elements and let $F: G \to G$ be a generalized Frobenius map. Let $W$ be the Weyl group of $G$ which respect to an $F$-stable maximal torus of $G$ contained in an $F$-stable Borel subgroup of $G$. We denote by $G^F$ the finite group of fixed points under $F$. For $w \in W$ we have a corresponding generalized Deligne–Lusztig character $R_w$ of $G^F$. We denote by $\mathcal{U}(G^F)$ the set of unipotent characters of $G^F$, that is the irreducible constituent of the $R_w$ for $w \in W$. Lusztig [8] and Digne and Michel [4] attached to every $\chi \in \mathcal{U}(G^F)$ a root of unity $\omega_\chi$. In [7], Lusztig computes such roots of unity corresponding to unipotent characters of finite reductive groups that lie in $R_w$, where $w$ is the Coxeter element. This work is completed by Geck and Malle in [6] for all unipotent characters of the Ree groups and of the twisted groups of type $2D_n$. However for some pairs of complex conjugate characters,
we do not know explicitly the root associated to its unipotent character, because a sign is missing. For example, if $G^F = Sz(2^{2n+1})$ (where $n$ is a non-negative integer) is the Suzuki group with parameter $2^{2n+1}$ then $G^F$ has two cuspidal unipotent complex conjugate characters $\mathcal{W}$ and $\overline{\mathcal{W}}$ of degree $2^n(2^{2n+1} - 1)$; Lusztig’s method only gives that $\omega_{\mathcal{W}} = \sqrt{2} (-1 \pm \sqrt{-1})$. On the other hand, using the almost characters of $G^F$, Lusztig has shown that the unipotent characters of $G^F$ can be distributed in families. In [8] Lusztig associated to most of the families a matrix, the so-called Fourier matrix of the family. However the Suzuki and Ree groups do not have Fourier matrices in Lusztig’s sense! On the other hand, Geck and Malle have axiomatized Fourier matrices and give in [6] candidates for these groups.

The aim of this paper is to compute the roots of unity attached to the cuspidal unipotent characters $\mathcal{W}$ and $\overline{\mathcal{W}}$ of the Suzuki group with parameter $2^{2n+1}$ and to show that the Fourier matrix associated to the family $\{ \mathcal{W}, \overline{\mathcal{W}} \}$ by Geck and Malle in [6] satisfies a conjecture of Digne and Michel. To this end, we compute the Shintani descents of the Suzuki group and use results of Digne and Michel [4].

Let $n$ be a non-negative integer and let $G$ be a simple group of type $B_2$ defined over $\bar{\mathbb{F}}_2$. Let $F$ be the generalized Frobenius map such that $G^F$ is the Suzuki group with parameter $2^{2n+1}$. The finite group $G^{F^2} = B_2(2^{2n+1})$ is the finite “untwisted” group of type $B_2$ with parameter $2^{2n+1}$.

This paper is organized as follows: in Section 2, we recall some definitions and generalities. In Section 3, we explicitly compute the character table of the finite group $B_2(2^{2n+1}) \rtimes \langle \sigma \rangle$, where $\sigma$ is the restriction of $F$ to $B_2(2^{2n+1})$. The main result of this part is:

**Theorem 1.1.** Let $n$ be a non-negative integer. We set $q = 2^{2n+1}$ and let $\sigma$ the exceptional graph automorphism of $B_2(q)$ such that its fixed points subgroup is the Suzuki group with parameter $q$. Then the group $B_2(q) \rtimes \langle \sigma \rangle$ has $(2q + 6)$ irreducible extensions of $(q + 3)$ irreducible $\sigma$-stable characters of $B_2(q)$. The values of these extensions are given in Table 6.

In the last section, we compute the Shintani descents of the Suzuki group with parameter $2^{2n+1}$. We then obtain two consequences on the unipotent characters of the Suzuki group: we first explicitly compute the root of unity associated to their unipotent characters and secondly we show that the Fourier matrices for the families of these groups satisfy a Digne and Michel conjecture.

2. Generalities

2.1. Finite reductive groups

Let $G$ be a connected reductive group defined over the finite field with $q = p^f$ elements. Let $F$ be a generalized Frobenius map over $G$. We recall that the finite subgroup $G^F = \{ x \in G \mid F(x) = x \}$ is a so-called finite reductive group. Let $H$ be an $F$-stable maximal torus of $G$ contained in an $F$-stable Borel subgroup $B$ of $G$. We set $W = N_G(H)/H$ the Weyl group of $G$. The map $F$ induces an automorphism of $W$ (also denoted by $F$ for simplicity). We denote by $\delta$ the order of this automorphism.

We fix $w \in W$ and we define the corresponding Deligne–Lusztig variety by:

$$X_w = \{ xB \mid x^{-1}F(x) \in BwB \}.$$
We recall that for every positive integer \( i \), we can associate a \( \bar{\mathbb{Q}}_\ell \)-space to \( X_w \), the \( i \)th \( \ell \)-adic cohomology space with compact support \( H^i_c(X_w, \bar{\mathbb{Q}}_\ell) \) over the algebraic closure \( \bar{\mathbb{Q}}_\ell \) of the \( \ell \)-adic field (here, \( \ell \) is a prime not dividing \( q \)). The group \( G^F \) acts on \( X_w \). This action induces a linear action on \( H^i_c(X_w, \bar{\mathbb{Q}}_\ell) \). Thus these spaces are \( \bar{\mathbb{Q}}_\ell G^F \)-modules. We define the generalized Deligne–Lusztig character by:

\[
\forall g \in G^F, \quad R_w(g) = \sum_{i \geq 0} (-1)^i \text{Tr}(g, H^i_c(X_w, \bar{\mathbb{Q}}_\ell)).
\]

The set of irreducible characters of \( G^F \) is denoted by \( \text{Irr}(G^F) \) and we denote by \( (\langle, \rangle)_{G^F} \) the usual scalar product on the space \( C(G^F) \) of the \( \bar{\mathbb{Q}}_\ell \)-valued class functions of \( G^F \). We define the set \( \mathcal{U}(G^F) \) of unipotent characters of \( G^F \) by:

\[
\mathcal{U}(G^F) = \{ \chi \in \text{Irr}(G^F) \mid \exists w \in W, \langle R_w, \chi \rangle_{G^F} \neq 0 \}.
\]

### 2.1.1. The root of a unipotent character

The group \( \langle F^\delta \rangle \) acts on \( X_w \). This action induces a linear endomorphism on \( H^1_c(X_w, \bar{\mathbb{Q}}_\ell) \). We also fix an eigenvalue \( \lambda \) of \( F^\delta \) on \( H^1_c(X_w, \bar{\mathbb{Q}}_\ell) \) and we denote by \( F_{\lambda,i} \) its generalized eigenspace. The actions of \( G^F \) and of \( \langle F^\delta \rangle \) on \( H^1_c(X_w, \bar{\mathbb{Q}}_\ell) \) commute, thus \( F_{\lambda,i} \) is a \( \bar{\mathbb{Q}}_\ell G^F \)-module. Moreover, the irreducible constituents which occur in the character associated to this \( \bar{\mathbb{Q}}_\ell G^F \)-module are unipotent characters of \( G^F \). Now let \( \chi \in \mathcal{U}(G^F) \). Then there exists \( w \in W, \lambda \in \bar{\mathbb{Q}}_\ell^\times \) and \( i \in \mathbb{N} \) such that \( \chi \) occurs in the character associated to \( F_{\lambda,i} \). Lusztig has shown that \( \lambda \), up to a power of \( q^{1/2} \), is a root of unity which depends only on \( \chi \) (denoted by \( \omega_\chi \)). Thus there exists \( s \in \mathbb{N} \) such that \( \lambda = \omega_\chi q^{s/2} \) (see [4]).

### 2.1.2. Fourier matrices

We assume that \( \delta \neq 1 \). We recall that \( \langle F \rangle \) acts on \( \text{Irr}(W) \). More precisely if \( \rho \in \text{Irr}(W) \), we define \( \rho^F \) by \( \rho^F(w) = \rho(F(w)) \) for every \( w \in W \). Let \( \rho \in \text{Irr}(W) \) such that \( \rho^F = \rho \), i.e., the inertial group of \( \rho \) in \( W \rtimes \langle F \rangle \) is \( W \rtimes \langle F \rangle \). It follows that \( \rho \) has extensions to \( W \rtimes \langle F \rangle \). Let \( \tilde{\rho} \) be such an extension; we define the almost character associated to \( \tilde{\rho} \) by:

\[
\mathcal{R}_{\tilde{\rho}} = \frac{1}{|W|} \sum_{w \in W} \tilde{\rho}(w, F) R_w,
\]

where the elements of \( W \rtimes \langle F \rangle \) are denoted by \( (w, x) \) for every \( w \in W \) and \( x \in \langle F \rangle \). Let \( \chi, \chi' \in \mathcal{U}(G^F) \). The characters \( \chi \) and \( \chi' \) are in the same family if and only if there exists \( (\chi_i)_{i=1,...,m} \), where \( \chi_i \in \mathcal{U}(G^F) \) such that:

- We have \( \chi_1 = \chi \) and \( \chi_m = \chi' \).
- For every \( 1 \leq i \leq m - 1 \), there exists an \( F \)-stable character \( \rho_i \in \text{Irr}(W) \) such that

\[
\langle \chi_i, \mathcal{R}_{\tilde{\rho}_i} \rangle_{G^F} \neq 0 \quad \text{and} \quad \langle \chi_{i+1}, \mathcal{R}_{\tilde{\rho}_i} \rangle_{G^F} \neq 0.
\]

Let \( \mathcal{F} \) be a family of unipotent characters of \( G^F \) obtained in this way. Except in the cases where \( G^F \) is a Suzuki group or a Ree group of type \( G_2 \) or \( F_4 \), Lusztig has shown that we can associate a matrix \( M_{\mathcal{F}} \) to \( \mathcal{F} \). We refer to [8] for details. In the case where \( G^F \) is a Suzuki group or a
2.1.3. Shintani descents

We recall that the Lang map associated to a generalized Frobenius map $F$ is the map $L_F : G \rightarrow G$, $x \mapsto x^{-1} F(x)$. Since $G^{F^δ}$ is a finite $F$-stable subgroup and $F$ is an automorphism of the abstract group $G$, it follows that $F$ restricts to an automorphism of $G^{F^δ}$, also denoted by $F$. Since $G$ is connected, the maps $F$ and $F^δ$ have Lang’s property, that is, their associated Lang maps is surjective. Using this fact, we can establish a correspondence $N_{F/F^δ}$ between $G^F$ and $G^{F^δ} \rtimes \langle F \rangle$, the so-called Shintani correspondence [4, §I.7]. More precisely, let $g \in G^F$; by the surjectivity of the Lang map $L_{F^δ}$, there exists $x \in G$ such that $g = L_{F^δ}(x)$. Therefore we have $(LF^{-1}(x), F) \in G^{F^δ} \rtimes \langle F \rangle$. This correspondence induces a bijection between the conjugacy classes of $G^F$ and the conjugacy classes of $G^{F^δ} \rtimes \langle F \rangle$ which consist of elements of the form $(g, F)$, with $g \in G^{F^δ}$. Moreover, we have:

$$\forall g \in G^F, \quad |C_{G^{F^δ} \rtimes \langle F \rangle}(N_{F/F^δ}(g))| = \delta |C_{G^F}(g)|. \quad (1)$$

Using this correspondence, we can associate a class function of $G^F$ to every class function of $G^{F^δ} \rtimes \langle F \rangle$. Indeed let $\psi \in C(G^{F^δ} \rtimes \langle F \rangle)$; we then define the Shintani descent of $\psi$ by $Sh_{F^δ/F^δ} \psi = \psi \circ N_{F/F^δ}$. We refer to [4] for further details.

2.1.4. The link between Shintani descents, Roots and Fourier matrices

The set of the irreducible constituents of $\text{Ind}_{B^{F^δ}}^{G^{F^δ}} 1_{B^{F^δ}}$ is the so-called principal series of $G^{F^δ}$. There exists a 1–1 correspondence between the irreducible characters of $W$ and the characters of the principal series of $G^{F^δ}$ (see [3, Proposition 10.1.2]). Let $\rho \in \text{Irr}(W)$, then we denote by $\chi_\rho$ its corresponding character. Similarly Malle (see [9, 1.5]) has shown that there is a 1–1 correspondence between $\text{Irr}(W \rtimes \langle F \rangle)$ and the irreducible components of $\text{Ind}_{B^{F^δ}}^{G^{F^δ} \rtimes \langle F \rangle} 1_{B^{F^δ}}$. We now assume that $\rho = \rho^F$, therefore $\rho$ has irreducible extensions in $W \rtimes \langle F \rangle$. We fix such an extension $\tilde{\rho}$ and we denote by $\chi_{\tilde{\rho}}$ the corresponding character. Then $\chi_{\tilde{\rho}}$ is an extension of $\chi_\rho$ to $G^{F^δ} \rtimes \langle F \rangle$. We have:

**Theorem 2.1.** (Digne and Michel [4]) Let $\rho \in \text{Irr}(W)$ such that $\rho^F = \rho$. Let $\tilde{\rho} \in \text{Irr}(W \rtimes \langle F \rangle)$ be an extension of $\rho$. Then we have:

$$\text{Sh}_{F^δ/F^δ} \chi_{\tilde{\rho}} = \sum_{V \in \mathcal{U}(G^F)} \langle R_{\tilde{\rho}}, V \rangle_{G^F} \omega_V V.$$

**Remark 2.1.** The theorem is proved in [4] in the case where $F$ is a Frobenius map. But the arguments are the same when $F$ is a generalized Frobenius map.

We now recall some conjectures of Digne and Michel (see [4]):

**Conjecture 2.1.** Let $\chi \in \mathcal{U}(G^{F^δ})$ such that $\chi^F = \chi$. Let $\tilde{\chi}$ be an extension of $\chi$ to $G^{F^δ} \rtimes \langle F \rangle$. Then:
1. The irreducible constituents of $\text{Sh}_{F^\delta/F\tilde{\chi}}$ are unipotent characters of $G^F$ and lie in the same family $\mathcal{F}$.

2. There exists a root of unity $u$ such that

$$\pm u \text{Sh}_{F^\delta/F\tilde{\chi}} = \sum_{V \in \mathcal{F}} a_V \omega_V V.$$

In this case, the coefficients $a_V$ give (up to a sign) a row of the Fourier matrix associated to the family $\mathcal{F}$.

2.2. Suzuki groups

Let $G$ be a simple group of type $B_2$ defined over $\mathbb{F}_2$. The root system of $G$ is $\Phi = \{-a, -b, -a - b, -2a - b, a, b, a + b, 2a + b\}$, where $\Pi = \{a, b\}$ is chosen as a fundamental root system. We denote by $\Phi^+ = \{a, b, a + b, 2a + b\}$ the set of positive roots with respect to $\Pi$. The Weyl group $W$ of $G$ is the dihedral group with 8 elements. We denote by $x_r(t)$ ($r \in \Phi$, $t \in \overline{\mathbb{F}}_2$) the Chevalley generators. It is convenient to identify $G$ with the symplectic group of dimension 4 over the algebraic closure of $\mathbb{F}_2$ defined by:

$$\text{Sp}_4(\overline{\mathbb{F}}_2) = \{ A \in M_4(\overline{\mathbb{F}}_2) \mid ^tAJA = J \}, \quad \text{where } J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We denote by $H = \{h(z_1, z_2) \mid z_1, z_2 \in \overline{\mathbb{F}}_2\}$ the subgroup of diagonal matrices of $G$ and we set $N = N_G(H)$. We then have $N = \langle n_a, n_b, H \rangle$, where the elements $n_a$ and $n_b$ of $N$ have the representing matrices

$$n_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad n_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We recall the Chevalley relations of $G$; for every $u, v, z_1, z_2 \in \overline{\mathbb{F}}_2$, we have:

$$x_a(u)x_b(v) = x_b(v)x_a(u)x_{a+b}(uv)x_{2a+b}(u^2v),$$

$$x_{a+b}(u)x_{2a+b}(v) = x_{2a+b}(v)x_{a+b}(u),$$

$$h(z_1, z_2)x_a(u)h(z_1, z_2)^{-1} = x_a(z_1u^2z_2^{-1}),$$

$$h(z_1, z_2)x_b(u)h(z_1, z_2)^{-1} = x_b(z_2^2u),$$

$$h(z_1, z_2)x_{a+b}(u)h(z_1, z_2)^{-1} = x_{a+b}(z_1uz_2),$$

$$h(z_1, z_2)x_{2a+b}(u)h(z_1, z_2)^{-1} = x_{2a+b}(z_1^2u),$$

$$n_a h(z_1, z_2)n_a^{-1} = h(z_2, z_1),$$

$$n_b h(z_1, z_2)n_b^{-1} = h(z_1, z_2^{-1}).$$

(2)
Let $n$ be a positive integer. We define $F_{2n}$ to be the Frobenius map with parameter $2^n$ of $G$, hence it raises the coefficients of a matrix to their $2^n$th powers. The group $G$ has a graph endomorphism $\alpha$ described in [2, Proposition 12.3.3]. It is given on generators by:

\[
\begin{align*}
\alpha(x_a(t)) &= x_b(t^2), \\
\alpha(x_b(t)) &= x_a(t), \\
\alpha(x_{a+b}(t)) &= x_{2a+b}(t^2), \\
\alpha(x_{2a+b}(t)) &= x_{a+b}(t), \\
\alpha(h(z_1, z_2)) &= h(z_1z_2, z_1z_2^{-1}), \\
\alpha(n_a) &= n_b, \\
\alpha(n_b) &= n_a.
\end{align*}
\]

We set $\theta = 2^n$ and $q = 2\theta^2$. We define the map:

\[ F = F_\theta \circ \alpha. \]

Since $F^2 = F_q$, it follows that $F$ is a generalized Frobenius map of $G$. The automorphism of $W$ induced by $F$ has order 2 (that is $\delta = 2$ with the preceding notations). The finite subgroup $G^F$ is the Suzuki group with parameter $q$. Using [10], this group is the same as the one studied in [11]. Moreover, we have $G^{F^2} = G^{F_q} = \text{Sp}_4(\mathbb{F}_q)$. This is a finite “untwisted” group of type $B_2$ with parameter $q$. We denote by $B$ the Borel subgroup of the upper triangular matrices of $G$ and we set $U$ to be the unipotent radical of $B$. The groups $H$, $U$ and $B$ are $F^2$-stable. We then define $H = H^{F^2}$, $U = U^{F^2}$ and $B = B^{F^2}$. We denote by $\sigma$ the restriction of $F$ to $G^{F^2}$. In the following we set $G^F = \text{Sz}(q)$ and $G^{F^2} = G$, and we remark that

\[ G^\sigma = \text{Sz}(q). \]

Now the aim is to obtain results about the unipotent characters of $\text{Sz}(q)$ by using Shintani descents between $G \rtimes \langle \sigma \rangle$ and $\text{Sz}(q)$. Before we do this we must compute the irreducible characters of $G \rtimes \langle \sigma \rangle$.

### 3. The irreducible characters of $B_2(q) \rtimes \langle \sigma \rangle$

We use the notation of the preceding section. In this section we compute the irreducible characters of the extension $\tilde{G} = G \rtimes \langle \sigma \rangle$. This group is an extension by an automorphism of $G$ of order 2. For generalities on character tables of extensions by an automorphism of order 2 we refer to [1, §1]. We recall some definitions and general properties. The group $G$ is a normal subgroup of $\tilde{G}$. Thus a conjugacy class of $\tilde{G}$ is either contained in $G$ or it has no element in $G$. A class in the first case is called an inner class and it is called an outer class in the second case. A character $\psi$ of $\tilde{G}$ is called an inner character if there exists an element $(g, \sigma)$ such that $\psi(g, \sigma) \neq 0$. We denote by $\varepsilon$ the linear character of $\tilde{G}$ with kernel $G$. Clifford theory shows that the irreducible characters of $\tilde{G}$ can be parameterized by the irreducible characters of $G$ as
follows: let $\chi \in \text{Irr}(G)$, then either $\chi^\sigma \neq \chi$ and $\text{Ind}_G^\tilde{G} \chi \in \text{Irr}(\tilde{G})$, or $\chi^\sigma = \chi$ and $\chi$ has two extensions in $\tilde{G}$ which differ up to multiplication by $\varepsilon$. Since the values of extensions on $(1, \sigma)$ are integers, in the case where this value is non-zero, we denote by $\tilde{\chi}$ the extension of $\chi$ such that $\tilde{\chi}(1, \sigma) > 0$.

3.1. The outer classes of $B_2(q) \rtimes \langle \sigma \rangle$

The Suzuki group $Sz(q)$ has three maximal tori that are cyclic groups: $\langle \pi_0 \rangle$, $\langle \pi_1 \rangle$ and $\langle \pi_2 \rangle$ of order $(q - 1)$, $(q + 2\theta + 1)$ and $(q - 2\theta + 1)$, respectively (see [11]). We denote by $E_0$ (respectively $E_1$ and $E_2$) the set of non-zero classes modulo the equivalence relation $\sim$ on $\mathbb{Z}/(q - 1)\mathbb{Z}$ (respectively $\mathbb{Z}/(q + 2\theta + 1)\mathbb{Z}$ and $\mathbb{Z}/(q - 2\theta + 1)\mathbb{Z}$) defined by $j \sim i \Leftrightarrow j \equiv \pm i \mod (q - 1)$ (respectively $j \equiv \pm i, \pm qi \mod (q + 2\theta + 1)$ and $j \equiv \pm i, \pm qi \mod (q - 2\theta + 1)$). We put:

$$E = \left\{ \pi_0^i, \pi_1^i, \pi_2^k \mid i \in E_0, \ j \in E_1, \ k \in E_2 \right\}.$$ 

The conjugacy classes of $G$ are given in [5]. To simplify notation, we denote $x_r(1)$ by $x_r$, where $r \in \Phi$. We have:

**Theorem 3.1.** Let $n$ a non-negative integer. We put $\theta = 2^n$ and $q = 20^2$. Let $G = B_2(q)$ and $\sigma$ the exceptional automorphism of $G$ that defines $Sz(q)$. Then the group $\tilde{G} = B_2(q) \rtimes \langle \sigma \rangle$ has $(q + 3)$ outer classes. We give in Table 1 a system of representatives of the outer classes of $\tilde{G}$ and their centralizer orders.

**Proof.** The Suzuki group with parameter $q$ has $(q + 3)$ conjugacy classes (see [11]). Using the Shintani correspondence, it follows that $\tilde{G}$ has $(q + 3)$ outer classes. It is easy to show that the elements of $E$ are not conjugate in $G$. We obtain in this way $(q - 1)$ distinct outer classes of $\tilde{G}$ with representative $(x, \sigma)$, $x \in E$. Moreover, we have

$$C_{\tilde{G}}(x, \sigma) = C_{Sz(q)}(x) \times \langle (1, \sigma) \rangle \quad \text{for } x \in E.$$ 

In [11, Proposition 16] it is proven that for every $i \in E_0$, $j \in E_1$ and $k \in E_2$, we have $C_{Sz(q)}(\pi_0^i) = \langle \pi_0 \rangle$, $C_{Sz(q)}(\pi_1^i) = \langle \pi_1 \rangle$ and $C_{Sz(q)}(\pi_2^k) = \langle \pi_2 \rangle$.

Now we prove that $(1, \sigma)$, $(x_a, \sigma)$, $(x_{a+b}, \sigma)$ and $(x_a x_{a+b}, \sigma)$ are not conjugate in $\tilde{G}$. They are of order 2, 8, 4 and 8, respectively. It then suffices to prove that $(x_a, \sigma)$ and $(x_a x_{a+b}, \sigma)$ are

<table>
<thead>
<tr>
<th>Representative</th>
<th>Centralizer order</th>
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<tbody>
<tr>
<td>$(1, \sigma)$</td>
<td>$2q^2(q - 1)(q^2 + 1)$</td>
</tr>
<tr>
<td>$(x_{a+b}, \sigma)$</td>
<td>$2q^2$</td>
</tr>
<tr>
<td>$(x_a, \sigma)$</td>
<td>$4q$</td>
</tr>
<tr>
<td>$(x_a x_{a+b}, \sigma)$</td>
<td>$4q$</td>
</tr>
<tr>
<td>$(\pi_0^i, \sigma), \ i \in E_0$</td>
<td>$\frac{1}{2}(q - 2)$</td>
</tr>
<tr>
<td>$(\pi_1^j, \sigma), \ j \in E_1$</td>
<td>$\frac{1}{4}(q - \theta)$</td>
</tr>
<tr>
<td>$(\pi_2^k, \sigma), \ j \in E_2$</td>
<td>$\frac{1}{4}(q + \theta)$</td>
</tr>
</tbody>
</table>
not conjugate in \( \tilde{G} \). Furthermore, using the Bruhat decomposition of \( G \), we can show that two elements in \( (U, \sigma) \) are conjugate in \( \tilde{G} \) if and only if they are conjugate by an element of \( B \). Suppose there exists \( b = uh \in B \) such that \( (b, 1)(x_a, \sigma) = (x_a x_{a+b}, \sigma)(b, 1) \), that is \( bx_a = x_a x_{a+b} \sigma(b) \). Let \( z_1, z_2 \in \mathbb{F}_q \) such that \( h = h(z_1, z_2) \) and we set \( z_0 = z_1/z_2 \). Then \( uhx_a = ux_a(z_0)h \). Moreover:

\[
x(x_a(z_0)) = x_a(t_a) x_b(t_b) x_{a+b}(t_{2a+b}) x_a(z_0)
\]

\[
= x_a(t_a + z_0) x_b(t_b) x_{a+b}(t_{a+b} + z_0 t_b) x_{2a+b}(t_{2a+b} + z_0^2 t_b),
\]

\[
x_a x_{a+b} \sigma(u) = x_a(1 + t_b^0) x_b(t_a^2) x_{a+b}(1 + t_{2a+b}^0 + t_{2a+b}^2),
\]

\[
\times x_{2a+b}(t_{a+b}^2 + t_{2a+b}^2).
\]

By the uniqueness of the decomposition of the elements of \( B \), we deduce that \( \sigma(h) = h \) and

\[
\begin{align*}
t_a + z_0 &= t_b^0 + 1, \\
t_a^2 &= t_b, \\
z_0 t_b + t_{a+b} &= t_{a+b}^2 t_b^0 + t_{2a+b}^2 + 1, \\
z_0^2 t_b + t_{2a+b} &= t_{a+b}^2 t_b^2 + t_{a+b}^2.
\end{align*}
\]

Then we obtain \( z_0 = 1 \) and \( z_1 = z_2 \). Furthermore, since \( \sigma(h) = h \) we obtain \( z_1 = z_2 = 1 \). It follows that \( h = 1 \). Now we deduce from these relations that \( t_b^0 + t_b = 1 \) and that \( t_b^2 + t_b + 1 = 0 \). Therefore \( t_b^0 t_b = t_b^2 \) and \( t_b^0 \) is a root of \( X^2 + X \), that is \( t_b^0 \in \{0, 1\} \). We then obtain a contradiction in the relation \( t_b^0 t_b = 1 \). Therefore \( (x_a, \sigma) \) and \( (x_a x_{a+b}, \sigma) \) are not conjugate in \( \tilde{G} \).

We now compute the centralizer in \( \tilde{G} \) of these elements. First we remark that \( C_{\tilde{G}}(\sigma) = Sz(q) \times \langle \sigma \rangle \). Now let \( x \in \{x_a, x_{a+b}, x_{2a+b}\} \). We therefore have \( |C_{\tilde{G}}(x, \sigma)| = 2 |\{g \in G \mid g x \sigma(g^{-1}) = x\}| \). Using the Bruhat decomposition and a similar calculation as above, we prove that \( \{g \in G \mid g x \sigma(g^{-1}) = x\} \subset U \). Now the required result follows by using the Chevalley relations (2). \( \square \)

3.2. The character table of \( B_2(q) \times \langle \sigma \rangle \)

The aim of this section is to prove the following theorem:

**Theorem 3.2.** We let \( \tilde{G} = B_2(q) \times \langle \sigma \rangle \) as before. Then the character table of \( \tilde{G} \) is as given in Table 6.

Using the notation of [5] we set \( \chi_{\pi_0}(i) = \chi_1(i, (2 \theta - 1)i), i \in E_0, \chi_{\pi_1}(j) = \chi_5((q - 2 \theta + 1)j), j \in E_1 \) and \( \chi_{\pi_2}(k) = \chi_5((q + 2 \theta + 1)k), k \in E_2 \). The \( (q + 3) \) \( \sigma \)-stable irreducible characters of \( B_2(q) \) are \( 1_G, \theta_1, \theta_4, \theta_5, \chi_{\pi_0}(i), i \in E_0, \chi_{\pi_1}(j), j \in E_1 \), and \( \chi_{\pi_2}(k), k \in E_2 \). The values of these characters are given in Table 7.

3.2.1. Irreducible characters obtained by induction from \( B \times \langle \sigma \rangle \)

The group \( B \) is \( \sigma \)-stable, thus \( \tilde{B} = B \rtimes \langle \sigma \rangle \subset \tilde{G} \). We now induce some characters of \( \tilde{B} \) to \( \tilde{G} \) which permit to obtain the outer values of \( \chi_{\pi_0}(i) (i \in E_0) \) and of \( \theta_4 \). Let \( \gamma_0 \) the primitive \( (q - 1) \)th
root of unity given in Table 7. We define the primitive \((q - 1)\)th root of unity \(\varepsilon_0 = \gamma_0^{(4-4\theta)}\) and we set \(e_0^i(\pi l_0^i) = e_0^i + e_0^{-i}\).

**Proposition 3.1.** The values of \(\tilde{\chi}_{\pi_0}(i)\) and \(\tilde{\theta}_4\) in Table 6 are correct.

**Proof.** We have \(U < \tilde{B}\) and we denote by \(\pi_U : \tilde{B} \to \tilde{B}/U\) the canonical map. Let \(\phi \in \text{Irr}(\tilde{B}/U)\); then \(\phi \circ \pi_U\) is an irreducible character of \(\tilde{B}\). We have \(\tilde{B}/U \simeq H \rtimes \langle \sigma \rangle\). We now construct outer characters of \(H \rtimes \langle \sigma \rangle\). Since \(H \simeq \mathbb{F}_q^\times \times \mathbb{F}_q^\times\), it follows that the irreducible characters of \(H\) are:

\[
\phi_k,i(\gamma^i, \gamma^j) = \gamma_0^{ik + jl} \quad \text{for} \ 1 \leq k, l \leq q - 1.
\]

The \(\sigma\)-stable irreducible characters of \(H\) are \(\phi_{i, (2q-1)i} (i \in \{1, \ldots, q-1\})\). Using [1, Lemma 3.5], we construct \((q - 2)\) linear characters of \(H \rtimes \langle \sigma \rangle\) defined by \(\phi_i(h, x) = \phi_{i, (2q-1)i}(h)\), where \(x \in \{1, \sigma\}\). We write \(\phi_{i, \tilde{B}} = \phi_i \circ \pi_U\); we thus obtain \((q - 2)\) linear characters of \(\tilde{B}\). Moreover, \(\tilde{B}\) has \((q + 2)\) outer classes, which are \((\pi_0^i, \sigma)\) \((1 \leq l \leq q - 2)\) and \((1, \sigma), (x_a, \sigma), (x_{a+b}, \sigma)\) and \((x_a x_{a+b}, \sigma)\). It is then easy to compute the values of \(\phi_{i, \tilde{B}}\). Moreover, using the Mackey formula, it follows that \(\text{Ind}_{\tilde{B}}^G(\phi_{i, \tilde{B}})\) is an irreducible extension of \(\chi_{\pi_0}(i)\). To obtain the outer values of these characters, we will induce \(\phi_{i, \tilde{B}}\) from \(\tilde{B}\) to \(\tilde{G}\). To this end, we give the corresponding induction formula. Except for \((1, \sigma)\), the centralizers of the outer elements of \(\tilde{B}\) are the same as their centralizers in \(\tilde{G}\). We have \(\text{Cl}_{\tilde{B}}(1, \sigma) = B^\sigma \times \langle \sigma \rangle\) of order \(2q^2(q-1)\) and

\[
\text{Cl}_{\tilde{G}}(\pi_0, \sigma) \cap \tilde{B} = \text{Cl}_{\tilde{G}}(\pi_0, \sigma) \cup \text{Cl}_{\tilde{B}}(\pi_0^{-1}, \sigma).
\]

This permits to obtain the induction formula from \(\tilde{B}\) to \(\tilde{G}\) given in the following table:

<table>
<thead>
<tr>
<th>((1, \sigma))</th>
<th>((x_a, \sigma))</th>
<th>((x_{a+b}, \sigma))</th>
<th>((x_a x_{a+b}, \sigma))</th>
<th>((\pi_0, \sigma))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Ind}_{\tilde{B}}^G\phi)</td>
<td>((q^2 + 1)\phi(1, \sigma))</td>
<td>(\phi(x_a, \sigma))</td>
<td>(\phi(x_{a+b}, \sigma))</td>
<td>(\phi(x_a x_{a+b}, \sigma))</td>
</tr>
</tbody>
</table>

Using this formula, we compute the values of \(\tilde{\chi}_{\pi_0}(i)\). We have

\[
\text{Ind}_{\tilde{B}}^G 1_{\tilde{B}} = 1_{\tilde{G}} + \theta_4 + 2\theta_1 + \theta_2 + \theta_3.
\]

We have \(\text{Res}_{\tilde{G}}^G(\text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}) = \text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}\) and write \(\xi\) for the constituent of \(\text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}\) which is an extension of \(\theta_1\). Since \(\langle \text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}, \theta_1 \rangle_{\tilde{G}} = 2\), it follows that \(\xi\) occurs in \(\text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}\) with multiplicity 1 or 2. If this multiplicity is 2, then

\[
\langle \text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}, \text{Ind}_{\tilde{B}}^G 1_{\tilde{B}} \rangle_{\tilde{G}} \geq 6.
\]

But a direct calculation gives \(\langle \text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}, \text{Ind}_{\tilde{B}}^G 1_{\tilde{B}} \rangle_{\tilde{G}} = 5\). Thus it follows that \(\xi + \varepsilon \xi\) is a constituent of \(\text{Ind}_{\tilde{B}}^G 1_{\tilde{B}}\). Decomposing the character \(\text{Ind}_{\tilde{B}}^G (1_{\tilde{B}}) - \xi - \varepsilon \xi\), we find an irreducible extension of \(\theta_4\). We use the preceding induction formula to compute its values. \(\Box\)
3.2.2. Irreducible characters obtained by induction from \( S\tilde{z}(q) \times \langle \sigma \rangle \)

Let \( \rho_0 = x_a x_{a+b} \in S\tilde{z}(q) \) and \( \sigma_0 = x_{a+b} x_{2a+b} \in S\tilde{z}(q) \). Let \( \tau_0 \) be the complex primitive root of order \( q^2 + 1 \) which appears in the character table of \( G \). We recall that \( \{ 1, \sigma_0, \rho_0, \rho_0^{-1}, \pi_i^j, i \in E_0, \pi_i^j; j \in E_1, \pi_i^k; k \in E_2 \} \) is a system of representatives of the classes of \( S\tilde{z}(q) \) (see [11]).

We put \( \varepsilon_1 = \tau_0(q^{-2a+1})^2 \) and \( \varepsilon_2 = \tau_0(q^{2a+1})^2 \). The character table of \( S\tilde{z}(q) \) is computed in [11] and reprinted in the appendix for the convenience of the reader. Since \( S\tilde{z}(q) \) is the subgroup of fixed points under \( \sigma \), it follows that \( \tilde{S}\tilde{z}(q) = S\tilde{z}(q) \times \langle \sigma \rangle \subseteq \tilde{G} \). Thus the classes and the character table of \( \tilde{S}\tilde{z}(q) \) are directly obtained using the classes and the character table of \( S\tilde{z}(q) \). We give the induction formula from \( S\tilde{z}(q) \) to \( \tilde{G} \) in Table 2. Let \( \tilde{\phi} \in \text{Irr}(\tilde{S}\tilde{z}(q)) \); to simplify we denote by the same symbol its induced character of \( \tilde{G} \). The outer values of the induced characters from \( S\tilde{z}(q) \) to \( \tilde{G} \) are given in Table 3.

Let \( \psi \) be a generalized character of \( \tilde{G} \). We recall that we can associate to \( \psi \) its \( \sigma \)-reduction \( \rho(\psi) \), which is a character of \( \tilde{G} \) such that \( \rho(\psi)(g, \sigma) = \psi(g, \sigma) \) (for every \( g \in G \)) and the irreducible constituents of \( \rho(\psi) \) are outer characters of \( \tilde{G} \). We refer to [1, §3.2.1] for details. We now define:

\[
X_0 = \rho\left( \tilde{I}_{S\tilde{z}(q)} - 1_{\tilde{G}} - \tilde{\theta}_4 - \sum_{i \in E_0} \tilde{x}_\pi_0(i) \right),
\]

\[
W_0 = \rho\left( \varepsilon \tilde{W} - \theta(\tilde{I}_{S\tilde{z}(q)} - 1_{\tilde{G}}) \right).
\]

**Proposition 3.2.** For every \( k \in E_1 \) (respectively \( k \in E_2 \)), there exists an extension \( \psi_k \) (respectively \( \psi'_k \)) of \( \chi_{\pi_1}(k) \) (respectively \( \chi_{\pi_2}(k) \)) such that:

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Induction formula from ( S\tilde{z}(q) ) to ( \tilde{G} )</td>
</tr>
<tr>
<td>( \text{Ind}_{S\tilde{z}(q)}^G(\phi) )</td>
</tr>
<tr>
<td>( A_1 )</td>
</tr>
<tr>
<td>( q^2(q+1)(q^2-1)\phi(1) )</td>
</tr>
<tr>
<td>( (\pi_0, \sigma) )</td>
</tr>
<tr>
<td>( \phi(\pi_0, \sigma) )</td>
</tr>
<tr>
<td>( A_{32} )</td>
</tr>
<tr>
<td>( q^2\phi(\sigma_0) )</td>
</tr>
<tr>
<td>( (1, \sigma) )</td>
</tr>
<tr>
<td>( \phi(1, \sigma) + (q-1)(q^2+1)\phi(\sigma_0, \sigma) )</td>
</tr>
<tr>
<td>( A_\sigma )</td>
</tr>
<tr>
<td>( q(\phi(\rho_0) + \phi(\rho_0^{-1})) )</td>
</tr>
<tr>
<td>( (x_{a+b}, \sigma) )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>( \pi_0 )</td>
</tr>
<tr>
<td>( (q-1)\phi(\pi_0) )</td>
</tr>
<tr>
<td>( (x_{a+b}, \sigma) )</td>
</tr>
<tr>
<td>( \frac{q}{2}(\phi(\rho_0, \sigma) + \phi(\rho_0^{-1}, \sigma)) )</td>
</tr>
<tr>
<td>( \pi_1 )</td>
</tr>
<tr>
<td>( (q-r+1)\phi(\pi_1) )</td>
</tr>
<tr>
<td>( (x_{a+b}, \sigma) )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>( \pi_2 )</td>
</tr>
<tr>
<td>( (q+r+1)\phi(\pi_2) )</td>
</tr>
<tr>
<td>( (\pi_2, \sigma) )</td>
</tr>
<tr>
<td>( \phi(\pi_2, \sigma) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer values of the induced characters of ( S\tilde{z}(q) )</td>
</tr>
<tr>
<td>( (1, \sigma) )</td>
</tr>
<tr>
<td>( (x_{a+b}, \sigma) )</td>
</tr>
<tr>
<td>( (\pi_0, \sigma) )</td>
</tr>
<tr>
<td>( (\pi_1, \sigma) )</td>
</tr>
<tr>
<td>( (\pi_2, \sigma) )</td>
</tr>
<tr>
<td>( \tilde{I}_{S\tilde{z}(q)} )</td>
</tr>
<tr>
<td>( q(q^2-q+1) )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>( \tilde{S}_I )</td>
</tr>
<tr>
<td>( q^2 )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
</tr>
<tr>
<td>( \tilde{X}_i )</td>
</tr>
<tr>
<td>( q(q^2+1) )</td>
</tr>
<tr>
<td>q</td>
</tr>
<tr>
<td>( \varepsilon_0^i(\pi_0) )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>( \tilde{Y}_j )</td>
</tr>
<tr>
<td>( (q-1)(q+q^2(\theta-1)) )</td>
</tr>
<tr>
<td>q</td>
</tr>
<tr>
<td>( -\varepsilon_0^i(\pi_1) )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>( \tilde{Z}_k )</td>
</tr>
<tr>
<td>( (q-1)(q-q^2(\theta+1)) )</td>
</tr>
<tr>
<td>-q</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>( -\varepsilon_0^i(\pi_2) )</td>
</tr>
<tr>
<td>( \tilde{V}_\psi )</td>
</tr>
<tr>
<td>( -\frac{1}{2}q^2\theta(q-1) )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>-1</td>
</tr>
</tbody>
</table>
$$X_0 = \sum_{k \in E_1} \psi_k + \sum_{k \in E_2} \psi_k',$$

$$W_0 = \sum_{k \in E_1} \psi_k - \sum_{k \in E_2} \psi_k'.$$

We give the values of $X_0$ and $W_0$ in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$X_0$</th>
<th>$W_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, \sigma)$</td>
<td>$q(q-1)^2/2$</td>
<td>$-\theta(q-1)$</td>
</tr>
<tr>
<td>$(x_a, \sigma)$</td>
<td>$q/2$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$(x_{a+b}, \sigma)$</td>
<td>$-q/2$</td>
<td>$-\theta(q-1)$</td>
</tr>
<tr>
<td>$(\pi_0, \sigma)$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(\pi_1, \sigma)$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$(\pi_2, \sigma)$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

**Proof.** We write $\tilde{K} = Sz(q)$. By computing the scalar products of $\tilde{K}$ with the irreducible characters obtained in Proposition 3.1, we deduce that $\tilde{\chi}_{\pi_0}(i)$ is a character. Furthermore, we have

$$\langle Ind^G_K(1_K), \theta \rangle_G = \langle Ind^G_K(1_K), \theta \rangle_G = 0,$$

$$\langle Ind^G_K(1_K), \chi_{\pi_1}(k) \rangle_G = \langle Ind^G_K(1_K), \chi_{\pi_2}(k) \rangle_G = 1.$$  

This proves that $X_0$ is exactly the summand of one extension of $\chi_{\pi_1}(k)$ (denoted by $\psi_k$) and one extension of $\chi_{\pi_2}(k)$ (denoted by $\psi'_k$). Now we compute the scalar products of $\tilde{\chi}$ with the known irreducible characters of $\tilde{G}$. Thus $\tilde{\chi} = \theta(\tilde{\chi}_{\pi_0}(i))$ is a character of $\tilde{G}$ and we denote by $\Theta$ its $\sigma$-reduction. We compute that $\langle Res^G_{\tilde{G}}(\tilde{\chi}), \theta \rangle_G = \langle Res^G_{\tilde{G}}(\tilde{\chi}), \theta \rangle_G = 0$, $\langle Res^G_{\tilde{G}}(\tilde{\chi}), \chi_{\pi_1}(k) \rangle_G = \theta - 1$ and $\langle Res^G_{\tilde{G}}(\tilde{\chi}), \chi_{\pi_2}(k) \rangle_G = \theta + 1$. Since $X_0$ and $\tilde{\chi}$ have no common constituents (because $\langle X_0, \tilde{\chi} \rangle_G = 0$) we deduce that $\Theta = (\theta - 1)\sum_{k \in E_1} \psi_k \epsilon + (\theta + 1)\sum_{k \in E_2} \psi'_k \epsilon$. We remark that $W_0 = \Theta \epsilon - \theta X_0$ and deduce that

$$W_0 = \sum_{k \in E_2} \psi'_k - \sum_{k \in E_1} \psi_k.$$  

For every $j \in E_1$ and $k \in E_2$, we put $\varphi_j = \rho(\tilde{Y}_j - \tilde{Y}_1)$ and $\vartheta_k = \rho(\tilde{Z}_k - \tilde{Z}_1)$.

**Proposition 3.3.** Using the preceding notation we have:

$$\varepsilon \tilde{\chi}_{\pi_1}(1) = \frac{4}{q + 2\theta} \left( \sum_{j \in E_1} \varphi_j + \frac{1}{2}(X_0 - W_0) \right),$$

$$\varepsilon \tilde{\chi}_{\pi_2}(1) = \frac{4}{q - 2\theta} \left( \sum_{k \in E_2} \vartheta_k + \frac{1}{2}(X_0 + W_0) \right).$$

Consequently we deduce that the values of $\tilde{\chi}_{\pi_1}(j)$ and $\tilde{\chi}_{\pi_2}(k)$ in Table 6 are correct.
Proof. Let $j \in E_1$ such that $j \neq 1$; we have $\langle \text{Res}_{\hat{G}}(\varphi_j), \chi_{\pi_1}(j) \rangle_G = -1$ and $\langle \text{Res}_{\hat{G}}(\varphi_j), \chi_{\pi_1}(1) \rangle_G = 1$. Moreover, for every other $\sigma$-stable character $\chi$ of $G$ we have $\langle \text{Res}_{\hat{G}}(\varphi_j), \chi \rangle_G = 0$. This yields $\text{Res}_{\hat{G}}(\varphi_j) = \chi_{\pi_1}(1) - \chi_{\pi_1}(j)$. We similarly prove that $\text{Res}_{\hat{G}}(\varphi_k) = \chi_{\pi_1}(1) - \chi_{\pi_1}(k)$. We denote by $\theta_j$ (respectively $\theta_{1,j}$) the extension of $\chi_{\pi_1}(j)$ (respectively $\chi_{\pi_1}(1)$), which is an irreducible component of $\varphi_j$. In particular we have $\varphi_j = \theta_{1,j} - \theta_j$. First we prove that $\theta_{1,j}$ is independent of $j$. Indeed, we immediately compute that $\langle \varphi_j - \varphi_2, \varphi_j - \varphi_2 \rangle_{\hat{G}} = 2$ (where $j \geq 3$). Furthermore, we have $\varphi_j - \varphi_2 = \theta_2 - \theta_j + \theta_{1,j} - \theta_{1,2}$. Moreover, since $\text{Res}_{\hat{G}}(\varphi_j - \varphi_2) = \chi_{\pi_1}(2) - \chi_{\pi_1}(j)$, the characters $\theta_2$ and $\theta_j$ are constituents of $\varphi_j - \varphi_2$. Since $\varphi_j - \varphi_2$ has two constituents, it follows that $\theta_{1,j} - \theta_{1,2} = 0$, i.e., $\theta_{1,j} = \theta_{1,2}$ for $j \geq 2$.

We denote by $\theta_1$ this common constituent. We compute that $\langle \varphi_i, X_0 \rangle_{\hat{G}} = 0$. Also, if $\theta_1 = \psi_1$ then for every $j \geq 2$, we have $\theta_j = \psi_j$. Indeed, if this is not the case we have $\langle \varphi_i, X_0 \rangle_{\hat{G}} = 1 \neq 0$. With a similar argument we prove that if $\theta_1 = \epsilon \psi_1$ then for every $j \geq 2$, we obtain $\theta_j = \epsilon \psi_j$. In summary we have:

$$
\begin{cases}
\theta_j = \psi_j & \forall j \in E_1, \\
\text{or} \quad \theta_j = \psi_j \epsilon & \forall j \in E_1.
\end{cases}
$$

Moreover,

$$
\frac{1}{4}(q + 2\theta)\theta_1 = \frac{1}{4}(q + 2\theta)\theta_1 - \sum_{j \in E_1} \theta_j + \sum_{j \neq 1} \theta_j = \sum_{j \neq 1} \varphi_j + \sum_{j \in E_1} \theta_j.
$$

Thus

$$
\theta_1 = \frac{4}{q + 2\theta} \left( \sum_{j \neq 1} \varphi_j + \sum_{j \in E_1} \theta_j \right).
$$

We immediately deduce from Proposition 3.2 that,

$$
\sum_{j \in E_1} \psi_j = \frac{1}{2} (X_0 - W_0).
$$

Hence we have,

$$
\sum_{j \in E_1} \theta_j = \frac{1}{2} (X_0 - W_0) \quad \text{or} \quad \sum_{j \in E_1} \theta_j = \frac{1}{2} (X_0 - W_0) \epsilon.
$$

We set

$$
f_1 = \frac{4}{q + 2\theta} \left( \sum_{j \neq 1} \varphi_j + \frac{1}{2} (X_0 - W_0) \right), \quad f_2 = \frac{4}{q + 2\theta} \left( \sum_{j \neq 1} \varphi_j + \frac{1}{2} (X_0 - W_0) \epsilon \right).
$$
Either \( f_1 \) or \( f_2 \) is an irreducible character of \( \tilde{G} \). But \( f_1(\pi_1, \sigma) \) is not an algebraic integer. Thus \( f_1 \) is not a character. It follows that \( f_2 = \varepsilon' \psi_1 \). We similarly prove that

\[
\varepsilon' \psi_1' = \frac{4}{q - 2\theta} \left( \sum_{k \neq 1} \theta_k + \frac{1}{2}(X_0 + W_0)\varepsilon \right).
\]

Since we have \( \psi_j = \varepsilon \phi_j \), we immediately deduce the values of \( \psi_j \) using the relation \( \psi_j = \psi_1 - \varepsilon \phi_j \). Similarly we compute the values of \( \psi_k' \) using the relation \( \psi_k' = \psi_1' - \varepsilon \theta_k \).

The induced characters of \( \tilde{S_2(q)} \) are not sufficient to obtain the outer values of the extensions of \( \tilde{\theta}_1 \) and \( \tilde{\theta}_5 \). However decomposing \( \tilde{Y}_i \) we obtain:

**Lemma 3.1.** The outer values of \( \tilde{\theta}_1 + \tilde{\theta}_5 \) are:

<table>
<thead>
<tr>
<th>( (1, \sigma) )</th>
<th>( (\sigma, \sigma) )</th>
<th>( (\sigma_a + b, \sigma) )</th>
<th>( (\sigma_a, \sigma_a + b, \sigma) )</th>
<th>( (\pi_0, \sigma) )</th>
<th>( (\pi_1, \sigma) )</th>
<th>( (\pi_2, \sigma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\theta}_1 + \tilde{\theta}_5 )</td>
<td>( 2\theta(q-1) )</td>
<td>( 0 )</td>
<td>( -2\theta )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>

### 3.2.3. The outer values of \( \tilde{\theta}_1 \) and \( \tilde{\theta}_5 \)

Let \( U_0 = \langle x_a \rangle \langle x_b \rangle X_{a+b}X_{2a+b} \subseteq U \). Since \( U_0 \) is \( \sigma \)-stable, we have \( \tilde{U}_0 = U_0 \times \langle \sigma \rangle \subseteq \tilde{B} \). We recall that the irreducible characters of \( B \) are given on p. 87 of [5]. We use the same notation. Let \( \theta_5(1) \) be the irreducible character of \( B \) of degree \( \frac{1}{2}q(q - 1)^2 \). Using the character tables of \( G \) and \( B \) we deduce that \( \text{Res}^B_G \theta_5 = \tilde{\theta}_3(1) \). Thus \( \text{Res}^G_B \tilde{\theta}_5 = \tilde{\theta}_3(1) \). We now construct \( \tilde{\theta}_3(1) \) using \( \tilde{U}_0 \).

We set \( \lambda : \mathbb{F}_q \to \{-1, 1\} \), such that \( \lambda(x) = 1 \) if \( x^2 + x \) has a root in \( \mathbb{F}_q \) and \( \lambda(x) = -1 \) otherwise. Let \( k, l \in \{0, 1\} \) and define the linear character \( \lambda(k, l) \) of \( U_0 \) on the generators by \( \lambda(k, l)(x_a) = (-1)^k, \lambda(k, l)(x_b) = (-1)^l \) and \( \lambda(k, l)(x_{a+b}(u)x_{2a+b}(v)) = \lambda(u+v) \).

**Lemma 3.2.** The characters \( \lambda(0, 0) \) and \( \lambda(1, 1) \) are \( \sigma \)-stable.

**Proof.** Since \( \lambda(v) = 1 \) if and only if \( \lambda(v^\theta) = 1 \) and since \( \lambda(v) = -1 \) if and only if \( \lambda(v^\theta) = -1 \), it follows that

\[
\lambda(k, l)^\sigma(x_{a+b}(u)x_{2a+b}(v)) = \lambda(k, l)(x_{a+b}(u)x_{2a+b}(v)).
\]

We remark that if \( (k, l) \in \{(0, 0), (1, 1)\} \), then \( \lambda(k, l) \) has the same values on \( x_a \) and \( x_b \). Thus \( \lambda(0, 0) \) and \( \lambda(1, 1) \) are \( \sigma \)-stable.

Using [1, Lemma 3.5] the characters \( \lambda(0, 0) \) and \( \lambda(1, 1) \) can be extend to \( \tilde{U}_0 \) and we thus obtain \( \tilde{\lambda}(0, 0) \) and \( \tilde{\lambda}(1, 1) \).

**Lemma 3.3.** The conjugacy classes of \( U_0 \) (respectively of \( \tilde{U}_0 \)) are given in Table 4 (respectively in Table 5).

**Lemma 3.4.** There exists an extension \( \xi \) of \( \theta_5(1) \) such that

\[
\text{Ind}_{\tilde{U}_0}^B \tilde{\lambda}(1, 1) = \frac{1}{4}(q + 2\theta)\xi + \frac{1}{4}(q - 2\theta)\xi \varepsilon.
\]
Table 4
Conjugacy classes of $U_0$

| Representatives          | Number | $|C_{U_0}(x)|$ |
|--------------------------|--------|--------------|
| 1                        | 1      | $4q^2$       |
| $x_a$                    | 1      | $2q^2$       |
| $x_b$                    | 1      | $2q^2$       |
| $x_{a+b}(u)$             | $q-1$ | $4q^2$       |
| $x_{2a+b}(v)$            | $q-1$ | $4q^2$       |
| $x_ax_b$                 | 1      | $2q^2$       |
| $x_{a+b}(u)x_{2a+b}(v)$  | $(q-1)^2$ | $4q^2$     |
| $u \neq 0$ $x_{a+b}(u)$ | $q-1$ | $2q^2$       |
| $u \neq 0$ $x_{2a+b}(v)$| $q-1$ | $2q^2$       |
| $v \neq 0, 1$ $x_{a+b}(u)$ | $q-2$ | $2q^2$       |
| $v \neq 0, 1$ $x_{a+b}(v)$ | $q-2$ | $2q^2$       |
| $u \neq 0$ $x_{a+b}(u)$  | $q-1$ | $2q^2$       |
| $v \neq 0, 1$ $x_{a+b}(v)$ | $q-2$ | $2q^2$       |

$u \neq 0$ $x_{a+b}(u)x_{2a+b}(v)$

Table 5
Outer classes of $\tilde{U}_0$

| Representatives          | Number | $|C_{\tilde{U}_0}(x)|$ |
|--------------------------|--------|------------------------|
| $(1, \sigma)$            | 1      | $4q$                   |
| $(x_a x_{a+b}(u), \sigma)$ | $q$   | $4q$                   |
| $u \neq 0$ $(x_{a+b}(u), \sigma)$ | $q-1$ | $4q$                   |

Proof. We set $\chi = \text{Ind}_{\tilde{U}_0}^\tilde{B} \tilde{\lambda}(1, 1)$. Using the character table of $B$ (see [5, p. 87]) we prove that $\langle \text{Res}_B^\tilde{B} \chi, \theta_3(1) \rangle_B = \frac{1}{2} q$. Thus there exists an extension $\xi$ of $\theta_3(1)$, integers $n$ and $n_\varepsilon$ where $n \geq n_\varepsilon$, such that $n + n_\varepsilon = \frac{1}{2} q$ and $\chi = n \xi + n_\varepsilon \xi \varepsilon$. We have $\chi \varepsilon = n_\varepsilon \xi + n \xi \varepsilon$. We deduce that $\chi - \chi \varepsilon = (n - n_\varepsilon)(\xi - \xi \varepsilon)$. It follows that $\langle \chi - \chi \varepsilon, \chi - \chi \varepsilon \rangle_B = 2(n - n_\varepsilon)^2$. Furthermore, since $\langle \chi - \chi \varepsilon, \chi - \chi \varepsilon \rangle_B = q$ we get $2(n - n_\varepsilon)^2 = 2\theta^2$ and hence $(n - n_\varepsilon)^2 = \theta^2$. This yields $n - n_\varepsilon = \theta$. Solving the system

$$
\begin{cases}
    n + n_\varepsilon = \frac{1}{2} q, \\
    n - n_\varepsilon = \theta,
\end{cases}
$$

we find $n = \frac{1}{4}(q + 2\theta)$ and $n_\varepsilon = \frac{1}{4}(q - 2\theta)$. □

Lemma 3.5. The outer values of $\tilde{\theta}_3(1)$ given in Table 6 are correct.

Proof. To compute the fusion of the classes of $\tilde{U}_0$ in $\tilde{B}$, we use that $\text{Ker} \lambda = \{t + t^\theta \mid t \in \mathbb{F}_q\}$. Thus the values of $\text{Ind}_{\tilde{U}_0}^\tilde{B} \tilde{\lambda}(1, 1)$ are obtained using Lemma 3.3 and the relation $\sum_{t \in \mathbb{F}_q} \lambda(t) = 0$. 

Table 6
Values of the outer characters $B_2(q) \rtimes \langle \sigma \rangle$ on outer classes

| $|C_{\tilde{G}}|$ | $(1, \sigma)$ | $(x_{a+b}, \sigma)$ | $(x_a, \sigma)$ | $(x_a x_{a+b}, \sigma)$ | $(\pi_0, \sigma)$ | $(\pi_1, \sigma)$ | $(\pi_2, \sigma)$ |
|----------------|----------------|-----------------|----------------|----------------|----------------|----------------|----------------|
|                | $2q^2(q-1)(q^2+1)$ | $2q^2$ | $4q$ | $4q$ | $2(q-1)$ | $2(q+2\theta+1)$ | $2(q-2\theta+1)$ |
| 1              | 1              | 1              | 1              | 1              | 1              | 1              | 1              |
| $\tilde{\theta}_4$ | 1              | $q^2$          | 0              | 0              | 0              | 1              | −1             | −1             |
| $\tilde{\theta}_1$ | 1              | $\theta(q-1)$  | $-\theta$      | $\theta$       | $-\theta$      | 0              | 1              | −1             |
| $\tilde{\delta}_5$ | 1              | $\theta(q-1)$  | $-\theta$      | $-\theta$      | $\theta$       | 0              | 1              | −1             |
| $\tilde{\chi}_{\pi_0}(i)$ | $i \in E_0$ | $q^2+1$        | 1              | 1              | 1              | $\varepsilon_0^j(\pi_0)$ | 0              | 0              |
| $\tilde{\chi}_{\pi_1}(j)$ | $j \in E_1$ | $(q-1)(q-2\theta+1)$ | $2\theta-1$ | $-1$          | $-1$          | 0              | $-\varepsilon_1^j(\pi_1)$ | 0              |
| $\tilde{\chi}_{\pi_2}(k)$ | $k \in E_2$ | $(q-1)(q+2\theta+1)$ | $-2\theta-1$ | $-1$          | $-1$          | 0              | 0              | $-\varepsilon_2^k(\pi_2)$ |

Using Lemma 3.4, we then compute the values of \( \xi \). For example, to compute \( \xi(1, \sigma) \) we set \( \xi(1, \sigma) = \alpha \). Then we have:

\[
\frac{1}{4}(q + 2\theta)\alpha - \frac{1}{4}(q - 2\theta)\alpha = \frac{1}{2}q(q - 1),
\]

and we deduce \( \alpha = \theta(q - 1) \). \( \square \)

We now set \( \phi = \text{Res}^\tilde{G}_B(\tilde{\theta}_1 + \tilde{\theta}_5) \). We have \( \langle \phi, \tilde{\theta}_3(1) \rangle_B = 1 \) and \( \langle \phi, \varepsilon\tilde{\theta}_3(1) \rangle_B = 0 \). Moreover, \( \text{Res}^\tilde{G}_B \tilde{\theta}_5 \) is an extension of \( \tilde{\theta}_3(1) \). Thus \( \text{Res}^\tilde{G}_B \tilde{\theta}_5 = \tilde{\theta}_3(1) \) and \( \text{Res}^\tilde{G}_B \tilde{\theta}_1 = \phi - \tilde{\theta}_3(1) \). We then obtain the values of \( \tilde{\theta}_5 \) and \( \tilde{\theta}_1 \) on \( (\pi_0, \sigma), (1, \sigma), (x_a, \sigma), (x_a+b, \sigma) \) and \( (x_a x_a+b, \sigma) \). This leaves us with the values of \( \tilde{\theta}_1 \) and \( \tilde{\theta}_5 \) on \( (\pi_1, \sigma) \) and \( (\pi_2, \sigma) \) which still need to be computed.

**Lemma 3.6.** We have \( \tilde{\theta}_1(\pi_1, \sigma) = \tilde{\theta}_5(\pi_1, \sigma) = 1 \) and \( \tilde{\theta}_1(\pi_2, \sigma) = \tilde{\theta}_5(\pi_2, \sigma) = -1 \).

**Proof.** Setting \( \alpha = \tilde{\theta}_1(\pi_1, \sigma) \) and \( \beta = \tilde{\theta}_5(\pi_1, \sigma) \) we have \( \alpha + \beta = 2 \). The orthogonality relations of the rows give \( |\alpha|^2 + |\beta|^2 = 2 \). Since \( (\pi_1, \sigma) \) and \( (\pi_1, \sigma)^{-1} \) are conjugate, it follows that \( \alpha \) and \( \beta \) are real numbers. Substituting \( \beta \) by \( 2 - \alpha \) we see that \( \alpha \) is a root of \( X^2 - 2X + 1 \) and therefore \( \alpha = 1 \). It follows that \( \beta = 1 \). Similarly we have \( \tilde{\theta}_1(\pi_2, \sigma) = \tilde{\theta}_5(\pi_2, \sigma) = -1 \). \( \square \)

In particular through Lemmas 3.5 and 3.6 and Propositions 3.1 and 3.3 we obtain the character table of \( \tilde{G} \). Thus Theorem 1.1 is proved.

4. Shintani descents

In this section we use the results and notation of Sections 2.1 and 2.2. Just recall that \( G \) is a simple group of type \( B_2 \) and that \( F \) is the generalized Frobenius map such that \( G^F = \Sz(q) \) (where \( q = 2^{2n+1} \)). Recall furthermore that \( \delta = 2 \). We will first give Shintani descents between \( \tilde{G} \) and \( \Sz(q) \) and then obtain some results on the unipotent characters of \( \Sz(q) \).

4.1. Shintani correspondence

**Proposition 4.1.** We have:

- If \( n \) is odd, then \( N_{F/F^2} \Cl(x_a x_b x_{a+b}) = \Cl(x_a, \sigma) \).
- If \( n \) is even, then \( N_{F/F^2} \Cl(x_a x_b x_{a+b}) = \Cl(x_a x_{a+b}, \sigma) \).

**Proof.** We search for a \( g \in G \) such that \( x_a = g^{-1}F(g) \). To this end suppose there exist \( u, v, w, t \in \tilde{F}_2 \) such that \( g = x_a(u) x_b(v) x_{a+b}(w) x_{2a+b}(t) \). Then using the Chevalley relations of \( G \) we immediately deduce:

\[
F(g) = x_a(u) x_b(v) x_{a+b}(w) x_{2a+b}(t),
\]

\[
x_a = x_a(u+1) x_b(v) x_{a+b}(w+v) x_{2a+b}(t+v).
\]
By uniqueness of this decomposition we obtain:

\[
\begin{align*}
    u + 1 &= v^{2n}, \\
    v &= u^{2n+1}, \\
    t^{2n} + v^{2n+1} &= u^{2n+1} + w + v = 0, \\
    w^{2n+1} + t + u^2v &= 0,
\end{align*}
\]

that is:

\[
\begin{align*}
    u + 1 &= v^{2n}, \\
    v &= u^{2n+1}, \\
    t^{2n} + w + uv &= 0, \\
    w^{2n+1} + t + u^2v &= 0.
\end{align*}
\]

Suppose this system has a solution \((u, v, w, t)\). Then we have:

\[
\begin{align*}
    u^q + u + 1 &= 0, \\
    v^q + v + 1 &= 0, \\
    w^q + w + v &= 0, \\
    t^q + t + v &= 0.
\end{align*}
\]

Using these relations we find \(g F^2(g^{-1}) = x_a x_b x_{a+b}(u + v + 1)x_{2a+b}(u^2 + v + 1)\). We now prove that this system has a solution. We write \(h \in \mathbb{F}_2\) for a root of \(X^2 + X + 1\) and \(k \in \mathbb{F}_2\) for a root of \(X^2 + X + h^2\) and we prove formula for \(h^{2m}\): if \(m\) is odd then \(h^{2m} = h + 1\) else \(h^{2m} = h\). We therefore prove that

\[
k^{2m} = \begin{cases} 
    k & \text{if } m \equiv 0 \mod 4, \\
    k + h^2 & \text{if } m \equiv 1 \mod 4, \\
    k + 1 & \text{if } m \equiv 2 \mod 4, \\
    k + h & \text{if } m \equiv 3 \mod 4.
\end{cases}
\]

We now study each of these cases:

- Suppose \(n \equiv 1 \mod 4\). In this case \(n\) is odd. We set \(v = u = h\) and \(t = w = k\). We get \(t^{2n} + t + u v = 0\) and \(t^{2n+1} + t + u^2 v = 0\). The element \(g = x_a(h)x_b(h)x_{a+b}(k)x_{2a+b}(k)\) is a solution and we have \(g F^2(g^{-1}) = x_a x_b x_{a+b}\).

- Suppose \(n \equiv 3 \mod 4\). In this case \(n\) is odd. We set \(v = u = h\) and \(t = w = h + k\). We obtain \(t^{2n} + w + u v = 0\) and \(w^{2n+1} + t + u^2 v = 0\). The element \(g = x_a(h)x_b(h)x_{a+b}(h + k)x_{2a+b}(h + k)\) is a solution and we have \(g F^2(g^{-1}) = x_a x_b x_{a+b}\).

- Suppose \(n \equiv 0 \mod 4\). In this case \(n\) is even. We set \(u = h\), \(v = u + 1\), \(t = k\) and \(w = k + 1\). The element \(g = x_a(h)x_b(h+1)x_{a+b}(k+1)x_{2a+b}(k)\) is a solution and we have \(g F^2(g^{-1}) = x_a x_b x_{a+b}\).

- Suppose \(n \equiv 2 \mod 4\). In this case \(n\) is even. We put \(u = h\), \(v = u + 1\), \(t = k\) and \(w = k\). Then the element \(g = x_a(h)x_b(h+1)x_{a+b}(k)x_{2a+b}(k)\) is a solution and we have \(g F^2(g^{-1}) = x_a x_b x_{a+b}\).

Using the definition of the Shintani correspondence (see Section 2.1.3), the claim is proved. \(\Box\)
4.2. Shintani descents of unipotent characters

We fix $i$ a primitive fourth root of unity. We recall that $\rho_0 = x_a x_{a+b}$ and that $St$ is the Steinberg character of $Sz(q)$. We denote by $\mathcal{W}$ the cuspidal unipotent character of $Sz(q)$ of degree $\theta(q-1)$ such that $\mathcal{W}(\rho_0) = \theta i$. We write $1_{\tilde{G}}, \tilde{\theta}_1, \tilde{\theta}_4$ and $\tilde{\theta}_5$ for the extensions of the unipotent characters of $G$ as above.

**Theorem 4.1.** We have $\text{Sh}_{F_2/F_1} 1_{\tilde{G}} = 1_{Sz(q)}$ and $\text{Sh}_{F_2/F_1} \tilde{\theta}_4 = St$. Setting $\bar{\zeta}_0 = \sqrt{2}(-1 - i)$ we have:

- If $n$ is even, then:
  $$\text{Sh}_{F_2/F_1} \tilde{\theta}_5 = -\frac{\bar{\zeta}_0}{\sqrt{2}} \mathcal{W} - \frac{\bar{\zeta}_0}{\sqrt{2}} \overline{\mathcal{W}},$$
  $$\text{Sh}_{F_2/F_1} \tilde{\theta}_1 = -\frac{\bar{\zeta}_0}{\sqrt{2}} \mathcal{W} - \frac{\bar{\zeta}_0}{\sqrt{2}} \overline{\mathcal{W}}.$$  

- If $n$ is odd, then:
  $$\text{Sh}_{F_2/F_1} \tilde{\theta}_1 = -\frac{\bar{\zeta}_0}{\sqrt{2}} \mathcal{W} - \frac{\bar{\zeta}_0}{\sqrt{2}} \overline{\mathcal{W}},$$
  $$\text{Sh}_{F_2/F_1} \tilde{\theta}_5 = -\frac{\bar{\zeta}_0}{\sqrt{2}} \mathcal{W} - \frac{\bar{\zeta}_0}{\sqrt{2}} \overline{\mathcal{W}}.$$  

**Proof.** We say that a class of $Sz(q)$ is of type $\pi_i$ ($i \in \{0, 1, 2\}$) if there exists some $k \in E_i$ such that $\pi_k^i$ is a representative of the class. Similarly we say that a class of $\tilde{G}$ is of type $\pi_i$ ($i \in \{0, 1, 2\}$) if there exists some $k \in E_i$ such that $(\pi_k^i, \sigma)$ is a representative of the class. Using relation (1) in Section 2.1.3 and Theorem 3.1, we see that the classes of $Sz(q)$ of type $\pi_0, \pi_1$ and $\pi_2$ are sent by $N_{F/F_2}$ to classes of $\tilde{G}$ of type $(\pi_0, \sigma), (\pi_1, \sigma)$ and $(\pi_2, \sigma)$, respectively. Since $1_{\tilde{G}}, \tilde{\theta}_1, \tilde{\theta}_4$ and $\tilde{\theta}_5$ are constant on these classes, we do not explicitly need to know the correspondence of these classes to compute the Shintani descents of these characters. Comparing $|C_{Sz(q)}(1)|$ and $|C_{\tilde{G}}(1, \sigma)|$ (respectively $|C_{Sz(q)}(x_a x_{a+b} x_{2a+b})|$ and $|C_{\tilde{G}}(x_{a+b}, \sigma)|$) we deduce from relation (1) in Section 2.1.3 that

$$N_{F/F_2} \text{Cl}(1) = \text{Cl}(1, \sigma) \quad \text{and} \quad N_{F/F_2} \text{Cl}(x_a x_{a+b} x_{2a+b}) = \text{Cl}(x_{a+b}, \sigma).$$

- Suppose that $n$ is odd. Then we deduce from Proposition 4.1 that
  $$N_{F/F_2} \text{Cl}(\rho_0) = \text{Cl}(x_a, \sigma) \quad \text{and} \quad N_{F/F_2} \text{Cl}(\rho_0^{-1}) = \text{Cl}(x_a x_{a+b}, \sigma).$$

Using the values of outer characters obtained in Theorem 1.1 we have:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$\rho_0$</th>
<th>$\sigma_0$</th>
<th>$\rho_0^{-1}$</th>
<th>$\pi_0^i$</th>
<th>$\pi_1^j$</th>
<th>$\pi_2^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Sh}<em>{F_2/F_1} 1</em>{\tilde{G}}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\text{Sh}_{F_2/F_1} \tilde{\theta}_4$</td>
<td>$q^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\text{Sh}_{F_2/F_1} \tilde{\theta}_1$</td>
<td>$\theta(q-1)$</td>
<td>$\theta$</td>
<td>$-\theta$</td>
<td>$-\theta$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\text{Sh}_{F_2/F_1} \tilde{\theta}_5$</td>
<td>$\theta(q-1)$</td>
<td>$-\theta$</td>
<td>$-\theta$</td>
<td>$\theta$</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
This yields $\text{Sh}_{F^2/F} 1_G = 1$ and $\text{Sh}_{F^2/F} \tilde{\theta}_4 = \text{St}$. Moreover, using the character table of $\text{Sz}(q)$ we deduce that

$$\text{Sh}_{F^2/F} \tilde{\theta}_1 = -\xi_0 \sqrt{2}/2 \mathcal{W} - \bar{\xi}_0 \sqrt{2}/2 \bar{\mathcal{W}},$$

$$\text{Sh}_{F^2/F} \tilde{\theta}_5 = -\bar{\xi}_0 \sqrt{2}/2 \mathcal{W} - \xi_0 \sqrt{2}/2 \bar{\mathcal{W}}.$$

• If $n$ is even, we proceed similarly using the identities $N_{F/F^2} \text{Cl}(\rho_0) = \text{Cl}(x_a x_{a+b}, \sigma)$ and $N_{F/F^2} \text{Cl}(\rho_0^{-1}) = \text{Cl}(x_a, \sigma)$.  \hfill \Box

4.3. Roots associated to the unipotent characters of $\text{Sz}(q)$

We denote by $\omega_\mathcal{W}$ and $\omega_{\bar{\mathcal{W}}}$ the roots of unity associated to $\mathcal{W}$ and $\bar{\mathcal{W}}$ as in Section 2.1.1. In [7, §7.4] G. Lusztig shows that $\{\omega_\mathcal{W}, \omega_{\bar{\mathcal{W}}}\} = \{\xi_0, \bar{\xi}_0\}$. We now make this result more precise.

**Theorem 4.2.** The roots associated to $\mathcal{W}$ and $\bar{\mathcal{W}}$ are:

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{W}$</th>
<th>$\bar{\mathcal{W}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ odd</td>
<td>$\xi_0$</td>
<td>$\bar{\xi}_0$</td>
</tr>
<tr>
<td>$n$ even</td>
<td>$\bar{\xi}_0$</td>
<td>$\xi_0$</td>
</tr>
</tbody>
</table>

**Proof.** The $F$-classes of the Weyl group $W$ of $G$ have the representatives 1, $w_a$ and $w_a w_b w_a$. We denote by $\rho_1$, $\rho_2$ and $\rho_3$ the $F$-stable character of $W$. Let $\tilde{\rho}_1$, $\tilde{\rho}_2$ and $\tilde{\rho}_3$ be their extensions to $W \rtimes \langle F \rangle$ such that:

<table>
<thead>
<tr>
<th></th>
<th>$(1, F)$</th>
<th>$(w_a, F)$</th>
<th>$(w_a w_b w_a, F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\rho}_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{\rho}_2$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\tilde{\rho}_3$</td>
<td>0</td>
<td>$-\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
</tr>
</tbody>
</table>

The corresponding almost characters of $\text{Sz}(q)$ are explicitly computed in [6, §2]; we recall that

$$\mathcal{R}_{\tilde{\rho}_1} = 1_{\text{Sz}(q)},$$

$$\mathcal{R}_{\tilde{\rho}_2} = \text{St},$$

$$\mathcal{R}_{\tilde{\rho}_3} = \frac{\sqrt{2}}{2} (\mathcal{W} + \bar{\mathcal{W}}).$$

Therefore we have:

$$\langle \mathcal{R}_{\tilde{\rho}_3}, \mathcal{W} \rangle_{\text{Sz}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \langle \mathcal{R}_{\tilde{\rho}_3}, \bar{\mathcal{W}} \rangle_{\text{Sz}} = \frac{\sqrt{2}}{2}. $$
Moreover, \( \theta_1 \) is in the principal series of \( G^{F^2} \). Using Theorem 2.1 we deduce that:

\[
\sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle_{Sz} = \pm \omega \mathcal{W}.
\]

The sign is due to the fact that \( \text{Sh}_{F^2/F} \tilde{\theta}_1 = \pm \text{Sh}_{F^2/F} \chi_{\tilde{\rho}_3} \). Since \( \omega \mathcal{W} \) is either \( \zeta_0 \) or \( \bar{\zeta}_0 \) and using that \( \bar{\zeta}_0 \neq -\zeta_0 \), we can obtain the root. Indeed

- Either \( \sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle \) is \( \zeta_0 \) or \( \bar{\zeta}_0 \) and in this case \( \omega \mathcal{W} = \sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle \).
- Or \( \sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle \) is neither \( \zeta_0 \) nor \( \bar{\zeta}_0 \) and then \( \omega \mathcal{W} = -\sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle \).

Using 4.1 the claim is proved. \( \square \)

### 4.4. Fourier matrices

The unipotent characters of \( Sz(q) \) are distributed in three families \( \mathcal{F}_1 = \{1\} \), \( \mathcal{F}_2 = \{\text{St}\} \) and \( \mathcal{F}_3 = \{\mathcal{W}, \bar{\mathcal{W}}\} \). The Fourier matrices \( M_i \) \((i = 1, 2, 3)\) associated to the \( \mathcal{F}_i \) are defined by Geck and Malle [6, §5] by

\[
M_1 = M_2 = [1] \quad \text{and} \quad M_3 = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}.
\]

**Proposition 4.2.** The Fourier matrices \( M_i \) \((i = 1, 2, 3)\) associated to the corresponding family \( \mathcal{F}_i \) of unipotent characters of \( Sz(q) \) verify the Conjecture 2.1.

**Proof.** Theorem 4.1 shows that the irreducible components of \( \text{Sh}_{F^2/F} \tilde{\theta}_1 \) and of \( \text{Sh}_{F^2/F} \tilde{\theta}_5 \) are the elements of the family \( \mathcal{F}_3 \). On the other hand, we deduce from Theorem 4.1 that:

If \( n \) even: \( i \text{Sh}_{F^2/F} \tilde{\theta}_5 = -\bar{\zeta}_0 \sqrt{2}/2 \mathcal{W} + \zeta_0 \sqrt{2}/2 \bar{\mathcal{W}} \).

If \( n \) is odd: \( i \text{Sh}_{F^2/F} \tilde{\theta}_5 = \zeta_0 \sqrt{2}/2 \mathcal{W} - \bar{\zeta}_0 \sqrt{2}/2 \bar{\mathcal{W}} \). \( \square \)

### Appendix A

Let \( n \) be a non-negative integer and write \( q = 2^{2n+1} \). The conjugacy classes and the character table of \( B_2(q) \) are given in [5, pp. 92 and 93]. Since \( q \) is even, the two classes of \( B_2(q) \) whose centralizers are of order \( 2q^2 \) are \( \text{Cl}(x_a x_b) \) and \( \text{Cl}(x_a x_b x_{a+b}) \).

We set \( \alpha_i = \gamma_0^i + \gamma_0^{-i} \) and \( \beta_i = v_0^k + v_0^{-k} \). In Table 7, we recall the irreducible characters that we need in this work and correct some errors of [5].

We recall the character table of \( Sz(q) \) which is computed in [11] in Table 8. Here we set \( \theta = 2^n \).
Table 7
Character table of $B_2(q)$

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_{31}$</th>
<th>$A_{32}$</th>
<th>$A_{41}$</th>
<th>$A_{42}$</th>
<th>$B_1(i, j)$</th>
<th>$B_2(i)$</th>
<th>$B_3(i, j)$</th>
<th>$B_4(i, j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>$q(q+1)^2/2$</td>
<td>$q(q+1)/2$</td>
<td>$q(q+1)/2$</td>
<td>$q/2$</td>
<td>$-q/2$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>$q^4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>$q(q-1)^2/2$</td>
<td>$-q(q-1)/2$</td>
<td>$-q(q-1)/2$</td>
<td>$q/2$</td>
<td>$-q/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>$\chi_1(k, l)$</td>
<td>$(q+1)^2(q^2+1)$</td>
<td>$(q+1)^2$</td>
<td>$(q+1)^2$</td>
<td>$2q+1$</td>
<td>1</td>
<td>1</td>
<td>$\alpha_{ik}\alpha_{jl}+\alpha_{il}\alpha_{jk}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4(k, l)$</td>
<td>$(q-1)^2(q^2+1)$</td>
<td>$(q-1)^2$</td>
<td>$(q-1)^2$</td>
<td>$-(2q-1)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\beta_{ik}\beta_{jl}+\beta_{il}\beta_{jk}$</td>
</tr>
<tr>
<td>$\chi_5(k)$</td>
<td>$(q^2-1)^2$</td>
<td>$-(q^2-1)$</td>
<td>$-(q^2-1)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$B_5(i)$</th>
<th>$C_1(i)$</th>
<th>$C_2(i)$</th>
<th>$C_3(i)$</th>
<th>$C_4(i)$</th>
<th>$D_1(i)$</th>
<th>$D_2(i)$</th>
<th>$D_3(i)$</th>
<th>$D_4(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>-1</td>
<td>$q+1$</td>
<td>$q+1$</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>1</td>
<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\theta_5$</td>
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<td>0</td>
<td>$q-1$</td>
<td>$q-1$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_1(k, l)$</td>
<td>0</td>
<td>$(q+1)(\alpha_{ik}+\alpha_{il})$</td>
<td>$(q+1)\alpha_{ik}\alpha_{il}$</td>
<td>0</td>
<td>0</td>
<td>$\alpha_{ik}+\alpha_{il}$</td>
<td>$\alpha_{ik}\alpha_{il}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4(k, l)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-(q-1)(\beta_{ik}+\beta_{il})$</td>
<td>$-(q-1)\beta_{ik}\beta_{il}$</td>
<td>0</td>
<td>0</td>
<td>$\beta_{ik}+\beta_{il}$</td>
<td>$\beta_{ik}\beta_{il}$</td>
</tr>
<tr>
<td>$\chi_5(k)$</td>
<td>$\tau_{ik}+\tau_{-ik}+\tau_{ik}q+\tau_{-ik}q$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
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### Acknowledgment

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### References

[1] O. Brunat, On the extension of $G_2(3^{2n+1})$ by the exceptional graph automorphism, submitted for publication.


