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## Field theory at a Lifshitz point

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## ARTICLE INFO

## Article history:

Received 17 October 2009

Received in revised form 6 December 2009

Accepted 10 December 2009

Available online 18 December 2009

Editor: M. Cvetič

## ABSTRACT

We construct the general renormalizable actions for the scalar field and the gauge field at a Lifshitz point characterized by the dynamical critical exponent  $z$ . The Lorentz invariance is broken down in the UV region, but is recovered in the IR limit. Even though the theories are UV complete, the speed of light is related to the momentum by  $z(k/M)^{z-1}$  which can go to infinity in the UV limit for  $z \geq 2$ . Since the Lorentz invariance is broken down, the dispersion relation is modified and the time delays in gamma-ray bursts can be easily explained. In addition, we also discuss the thermal dynamics and the size of causal patch in a FRW universe for the field theory at a Lifshitz point.

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## 1. Introduction

Recently Horava proposed a quantum field theory of gravity with the dynamical critical exponent equal to  $z = 3$  in the UV region [1]. Though the spatial isotropy is still assumed to be kept, the isometry between space and time is got lost. The degree of anisotropy between space and time is measured by the dynamical critical exponent  $z$ ,

$$\vec{x} \rightarrow b\vec{x}, \quad t \rightarrow b^z t. \quad (1)$$

The theory proposed in [1] describes the interaction of non-relativistic gravitons at short distances, and recovers nearly the Einstein's gravity in the IR region with some highly suppressed higher-spatially-derivative modifications. Such a theory is at least power-counting renormalizable in the  $(3+1)$ -dimensional space-time. Some solutions of Horava gravity theory were given in [2–4]. Since Horava gravity has a very nice UV behavior, it has been applied to investigate the physics in the early universe in [5–11]. An interesting result is that the perturbation of the scalar field with  $z = 3$  is scale-invariant in the universe where the scale factor goes like  $a(t) \sim t^p$  with  $p > 1/3$  [8]. It may provide an alternative model to the inflation. But there are still many open questions in this area, for example how to solve the flatness problem without inflation. Other related works are given in [12–20].

In fact, the first field theory model exhibiting the above anisotropic scale invariance (1) has been known for a long time.

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It is the so-called Lifshitz scalar field theory with the critical exponent  $z = 2$  [21],

$$\mathcal{L} = \int d^2x dt \left( (\partial_t \phi)^2 - \lambda (\Delta^2 \phi)^2 \right). \quad (2)$$

It has a line of fixed points parameterized by  $\lambda$ . Such fixed points with anisotropic scale invariance are usually called the Lifshitz points. The Lifshitz scalar field theory and its generalizations have been used to study quantum phase transitions in various strongly correlated electron systems [22]. Moreover, the nontrivial gauge theories with the Lifshitz fixed points in  $2+1$  dimensions has been discussed in [23]. And in [12] a different construction on the non-Abelian gauge theories with  $z = 2$  in arbitrary dimensions was presented.

In this Letter we temporarily forget about the gravity and only focus on the classical field theory at a possible Lifshitz point with arbitrary dynamical critical exponent  $z$  and figure out the most general renormalizable actions for the scalar and the gauge fields. Due to the anisotropic scaling, the power counting of the fields is different from the one in usual field theory. As a result, for a field theory with  $z \geq 2$ , it has the marginal terms with higher spatially derivatives and also has more renormalizable interactions. This leads to the modification of the dispersion relation in the UV limit. And more importantly, due to the breaking of Lorentz invariance, the speed of light at UV may turn to infinity. The fact that the Lorentz invariance just appears as accidental symmetry at IR provide a natural mechanism of Lorentz symmetry breaking. As an application, the issue of time delays in gamma-ray bursts could be addressed in this context.

Our Letter is organized as follows. The general renormalizable actions for the scalar field and the gauge field are proposed in Sec-

tions 2 and 3 respectively. As an application, we provide a possible explanation for the time delays in gamma-ray bursts due to the modification of the dispersion relation in Section 4. The thermal dynamics and the size of causal patch in a FRW universe for the field theory at a Lifshitz point are discussed in Sections 5 and 6 respectively. Some inspired discussions are included in Section 7.

## 2. The renormalizable scalar field theory at a Lifshitz point in $d + 1$ dimensions

In this section we will construct the most general renormalizable action for the scalar field theory with a dynamical critical exponent  $z$  in  $d + 1$  dimensions. The spacetime is assumed to be  $R \times R^d$  with the coordinates

$$(t, \vec{x}) \equiv (t, x^i), \quad (3)$$

for  $i = 1, 2, \dots, d$ . The spacetime metric takes the form

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \quad (4)$$

where  $g_{ij}$  are the  $d$ -dimensional spatial metric of signature  $(+\dots+)$ ,  $N$  is the lapse function, and  $N^i$  is the shift factor. The field theory is assumed to have a UV fixed point with the scaling properties given in Eq. (1). In the case of general  $z$ , the classical scaling dimensions of the coordinates in the unit of the spatial momenta are

$$[t]_s = -z, \quad [\vec{x}]_s = -1, \quad [\Delta \equiv \partial^i \partial_i]_s = 2, \quad (5)$$

and the classical scaling dimensions of the fields are

$$[g_{ij}]_s = 0, \quad [N^i]_s = z - 1, \quad [N]_s = 0. \quad (6)$$

The prototype of a quantum field theory is the theory of a single Lifshitz scalar  $\phi(t, \vec{x})$  whose dynamics is supposed to be governed by the following action,

$$S = \frac{1}{2} \int dt d^d x N \sqrt{g} \left[ \frac{1}{N^2} (\partial_t \phi - N^i \partial_i \phi)^2 - \sum_{J \geq 2} \mathcal{O}_J \star \phi^J \right], \quad (7)$$

where  $\mathcal{O}$  is an operator which can be expanded by

$$\mathcal{O}_J = \sum_{n=0}^{n_J} (-1)^n \frac{\lambda_{J,n}}{M^{2n + \frac{d-1}{2}J - d - 1}} \Delta^n, \quad (8)$$

here  $\lambda_{J,n}$  are the energy dimensionless coupling constant. The  $\star$  product in Eq. (7) contains all possible independent combinations of  $\Delta$  and  $\phi$  up to a total derivative. For example,

$$\Delta^3 \star \phi^3 = c_1 (\Delta \phi)^3 + c_2 (\Delta^2 \phi)(\Delta \phi)\phi + c_3 (\Delta^3 \phi)\phi^2, \quad (9)$$

where  $c_1, c_2, c_3$  are the dimensionless parameters. For simplicity we can assume  $c_1 = 1$ . Here we mainly work in the Minkowski spacetime and then have  $g_{ij} = \delta_{ij}$ ,  $N^i = 0$  and  $N = 1$ . From the kinetic term in the action (7), the scalar field  $\phi$  has the scaling dimension

$$[\phi]_s = \frac{d-z}{2}. \quad (10)$$

The case of  $z = d$  corresponds to a very special field theory in which the scalar field is dimensionless and the power of  $\phi$  can be arbitrary large. The action for the scalar field with  $z = d = 3$  has been written down in [7]. In general, the scaling dimension of the coupling constant  $\lambda_{J,n}$  in the unit of the spatial momenta is

$$[\lambda_{J,n}]_s = z + d + \frac{z-d}{2} J - 2n. \quad (11)$$

In order that this theory is power-counting renormalizable,  $[\lambda_{J,n}]_s$  is required to be not less than zero, namely

$$n \leq \frac{z+d}{2} + \frac{z-d}{4} J. \quad (12)$$

Therefore

$$n_J = \max \left\{ n \in Z \mid n \leq \frac{z+d}{2} + \frac{z-d}{4} J \right\}. \quad (13)$$

If  $z < d$ ,  $n \geq 0$  implies  $J \leq 2(z+d)/(d-z)$ . If  $z \geq d$ , there is no upper bound on  $J$ .<sup>1</sup> For  $J = 2$ , we have  $n \leq z$ .

In the UV limit, the operator  $\mathcal{O}_J$  is dominated by

$$(-1)^{n_J} \frac{\lambda_{J,n_J}}{M^{2n_J + \frac{d-1}{2}J - d - 1}} \Delta^{n_J}, \quad (14)$$

which takes the form of

$$\lambda_{J,n_J} \frac{k^{2n_J}}{M^{2n_J + \frac{d-1}{2}J - d - 1}} \quad (15)$$

in the momentum space, where  $k = |\vec{k}|$ . Therefore the stability of the field theory in the UV limit requires that  $\lambda_{J,n_J}$  be positive.

Without loss of generality, we assume

$$\lambda_{2,z} = 1. \quad (16)$$

The effective mass term corresponds to  $J = 2$ , namely

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^z (-1)^n \frac{\lambda_{2,n}}{M^{2n-2}} \phi \Delta^n \phi \\ &= \frac{1}{2} \sum_{2 \leq n \leq z} (-1)^n \frac{\lambda_{2,n}}{M^{2n-2}} \phi \Delta^n \phi - \frac{1}{2} \lambda_{2,1} \phi \Delta \phi \\ & \quad + \frac{1}{2} \lambda_{2,0} M^2 \phi^2. \end{aligned} \quad (17)$$

In the IR fixed point, the mass square is nothing but  $m^2 = \lambda_{2,0} M^2$  and the speed of light is given by  $c = \sqrt{\lambda_{2,1}}$ . Here we assume that the Lorentz invariance of the field theory is recovered in the IR limit, which requires  $\lambda_{2,1} = 1$ . Now the dispersion relation for this field theory can be written down by

$$\omega^2 = m^2 + \vec{k}^2 + \sum_{2 \leq n \leq z} \frac{\lambda_{2,n}}{M^{2n-2}} \vec{k}^{2n}. \quad (18)$$

For  $z = 1$ , the last term in the above equation does not exist and the standard dispersion relation is recovered. For  $z \geq 2$  the dispersion relation is changed. The group velocity is given by

$$v_g = \frac{k}{\omega} \left[ 1 + \sum_{2 \leq n \leq z} n \lambda_{2,n} \left( \frac{k}{M} \right)^{2n-2} \right]. \quad (19)$$

In the UV limit ( $k \gg M$ ),

$$v_g \simeq z \left( \frac{k}{M} \right)^{z-1}, \quad (20)$$

which goes to infinity for  $k \rightarrow \infty$  if  $z \geq 2$ . It is not surprised because the special relativity is broken down in the UV limit. In the IR region, the speed of light is modified to be

$$c_g = 1 + \frac{3}{2} \lambda_{2,2} \left( \frac{k}{M} \right)^2 + \mathcal{O}((k/M)^4). \quad (21)$$

<sup>1</sup> A similar result was obtained in [13].

If  $z \geq 3$ ,  $\lambda_{2,2}$  can be positive or negative. As long as  $\omega^2$  is positive, the field theory is always stable.

Of particular interest is the case when  $z = 3$ ,  $d = 3$ . In this case, the scalar field could couple to Horava–Lifshitz gravity and provide an alternative to inflation. Note that in this case, the scalar field is dimensionless and renormalizability gives no constraint on the scalar potential  $V(\phi)$ .

### 3. The renormalizable Yang–Mills theory at a Lifshitz point in $d + 1$ dimensions

In this section we switch to the Yang–Mills theory with an arbitrary dynamical critical exponent  $z$  in  $d + 1$  dimensions. The gauge field is a one-form in  $(d + 1)$ -dimensional spacetime, with the spatial components  $A_i = A_i^a(t, \vec{x})T_a$  and a time component  $A_0 = A_0^a(t, \vec{x})T_a$ . The Lie algebra generators  $T_a$  of the gauge group  $\mathcal{G}$  satisfy

$$[T_a, T_b] = if_{ab}{}^c T_c. \quad (22)$$

The Lie algebra is normalized to be  $\text{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$ . The gauge transformations are

$$\begin{aligned} \delta_\epsilon A_0 &= \partial_t \epsilon - i[A_0, \epsilon], \\ \delta_\epsilon A_i &= (\partial_i \epsilon^a + f_{bc}{}^a A_i^b \epsilon^c) T_a \equiv D_i \epsilon. \end{aligned} \quad (23)$$

The gauge-invariant field strengths are given by

$$\begin{aligned} E_i &= (\partial_t A_i^a - \partial_i A_0^a + f_{bc}{}^a A_i^b A_0^c) T_a \\ &= \partial_t A_i - \partial_i A_0 - i[A_i, A_0], \end{aligned} \quad (24)$$

$$\begin{aligned} F_{ij} &= (\partial_i A_j^a - \partial_j A_i^a + f_{bc}{}^a A_i^b A_j^c) T_a \\ &= \partial_i A_j - \partial_j A_i - i[A_i, A_j]. \end{aligned} \quad (25)$$

Since the symmetry between space and time is broken down for  $z \neq 1$ , we will write the action in terms of the electric field strength  $E_i$  and the magnetic field strength  $F_{ij}$ . The engineering dimensions of the gauge field components at the Lifshitz point are

$$[A_0]_s = z, \quad [A_i]_s = 1, \quad (26)$$

and then the engineering dimensions of the field strengths become

$$[E_i]_s = z + 1, \quad [F_{ij}]_s = 2. \quad (27)$$

Similar to [12], we choose a natural gauge-fixing condition,

$$A_0 = 0, \quad \text{and} \quad \partial_i A_i = 0. \quad (28)$$

In order to keep the unitarity, the Lagrangian should contain a kinetic term which is only quadratic in the first time derivatives of the gauge field. Here the only choice is  $\text{Tr}(E_i E_i)$ . The action in terms of the gauge field strength  $E_i$  and  $F_{ij}$  could be of the form,

$$S = \frac{1}{2} \int dt d^d x \left[ \frac{1}{g_E^2} \text{Tr}(E_i E_i) - \sum_{J \geq 2} \mathcal{O}_J \star F^J \right], \quad (29)$$

where

$$\mathcal{O}_J = \frac{1}{g_E^J} \sum_{n=0}^{n_J} (-1)^n \frac{\lambda_{J,n}}{M^{2n + \frac{d+1}{2}J - d - 1}} D^{2n}. \quad (30)$$

Here  $F$  and  $D$  are the abbreviated denotation for the field strength  $F_{ij}$  and the covariant derivative  $D_k$  respectively, and  $\lambda_{J,n}$  are the coupling with zero energy dimension. Similarly  $D^{2n} \star F^J$  also contains all possible independent combinations of  $D_k$  and  $F_{ij}$ . Now the scaling dimensions of  $g_E$  and  $\lambda_{J,n}$  are respectively given by

$$\begin{aligned} [g_E]_s &= \frac{z-d}{2} + 1, \\ [\lambda_{J,n}]_s &= z + d + \frac{z-d-2}{2} J - 2n. \end{aligned} \quad (31)$$

The renormalizable condition for  $E_i$  is  $[g_E]_s \geq 0$ , namely

$$z \geq d - 2. \quad (32)$$

For  $z = 1$ , the gauge theory is renormalizable only when  $d \leq 3$ . Since there is no symmetry relating the kinetic term and the potential terms, we still need to find out the renormalizable conditions for the potential terms. A simple way to work them out is to rescale the gauge field  $A_i^a$  to the canonical one  $\tilde{A}_i^a$  which is related to  $A_i^a$  by

$$\tilde{A}_i^a = A_i^a / g_E, \quad (33)$$

and then the gauge field strengths become

$$\tilde{E}_i = \partial_t \tilde{A}_i = E_i / g_E, \quad (34)$$

$$\tilde{F}_{ij} = \partial_i \tilde{A}_j - \partial_j \tilde{A}_i - ig_E [\tilde{A}_i, \tilde{A}_j] = F_{ij} / g_E. \quad (35)$$

The action for the canonical gauge field is

$$\begin{aligned} S &= \frac{1}{2} \int dt d^d x \left[ \text{Tr}(\tilde{E}_i \tilde{E}_i) \right. \\ &\quad \left. - \sum_{J \geq 2} \sum_{n=0}^{n_J} (-1)^n \frac{\lambda_{J,n}}{M^{2n + \frac{d+1}{2}J - d - 1}} \tilde{D}^{2n} \star \tilde{F}^J \right]. \end{aligned} \quad (36)$$

The renormalizable conditions for the potential terms are  $[\lambda_{J,n}]_s \geq 0$  which implies

$$n_J = \max \left\{ n \in \mathbb{Z} \mid n \leq \frac{z+d}{2} + \frac{z-d-2}{4} J \right\}. \quad (37)$$

For  $J = 2$ ,  $n \leq z - 1$ . In order to recover the  $z = 1$  gauge theory in the IR limit, we set  $\lambda_{2,0} = 1$ . On the other hand, the UV stability requires that  $\lambda_{J,n_J}$  should be positive and  $\lambda_{2,z-1}$  can be set to be 1 for simplicity.

Now we can easily write down the dispersion relation for a free gauge field theory as follows

$$\omega^2 = k^2 \left[ 1 + \sum_{1 \leq n \leq z-1} \lambda_{2,n} \left( \frac{k}{M} \right)^{2n} \right], \quad (38)$$

where  $k = |\vec{k}|$ . The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{k}{\omega} \left[ 1 + \sum_{1 \leq n \leq z-1} (n+1) \lambda_{2,n} \left( \frac{k}{M} \right)^{2n} \right]. \quad (39)$$

In the UV limit ( $k \gg M$ ), we have

$$v_g \simeq z \left( \frac{k}{M} \right)^{z-1}. \quad (40)$$

The speed of light goes to infinity for  $k \rightarrow \infty$  if  $z \geq 2$ . In the IR regime,

$$v_g \simeq 1 + \frac{3}{2} \lambda_{2,1} \left( \frac{k}{M} \right)^2. \quad (41)$$

Here a negative  $\lambda_{2,1}$  is allowed as long as  $\omega^2$  is positive definitely for  $z \geq 3$ . In the next section the above modified speed of light can be used to explain the time delays in gamma-ray bursts.

#### 4. An explanation for the time delays in gamma-ray bursts

Recently the Fermi LAT and Fermi GBM Collaborations reported that the photon with energy  $E_h = 13.22^{+1.70}_{-1.54}$  GeV arrived at the Earth is 16.54 s later than the low-energy photon from GRB 08916C with measured redshift of  $z' = 4.35 \pm 0.15$  [25].<sup>2</sup> If the high-energy photon was emitted at the same time as the low-energy photon, this delay may encode the information of Lorentz symmetry violation [26–28]. In [26–28], the dispersion relation of the photon is proposed to be modified by the effect of quantum gravity. Some other possible explanations were suggested in [29, 30]. In Section 4 we saw that the dispersion relation and the speed of light of the photon field at a Lifshitz point was modified. This fact suggests a natural way to explain the time delays in gamma-ray bursts.

Here we would like to give a general discussion about the time delays in the gamma-ray bursts. Assume that the velocity of the photon with physical momentum  $k$  is given by

$$c_g(k) = 1 + \lambda \left( \frac{k}{M} \right)^\alpha. \quad (42)$$

This deformed velocity of the photon implies that the simultaneously emitted photons from the source of the gamma-ray bursts reach the Earth at different times. In the FRW universe, the momentum of the photon is redshifted by the expansion of the universe. The scale factor is related to the redshift factor  $z'$  by  $a = (1 + z')^{-1}$  and the speed of light at the time of  $z'$  becomes

$$c_g(k, z') = 1 + \lambda \left( \frac{k/a}{M} \right)^\alpha = 1 + \lambda (1 + z')^\alpha \left( \frac{k}{M} \right)^\alpha. \quad (43)$$

The comoving distance between the source of the gamma-ray burst and the Earth is  $x_c$  which is given by

$$x_c = \int_{t_\gamma}^{t_k} c_g(k) \frac{dt}{a}, \quad (44)$$

where  $t_\gamma$  is the time when the photon was emitted. If the high-energy and low-energy photons were emitted at the same time  $t_\gamma$ , the time delay can be easily obtained,

$$\delta t \equiv t_{k_h} - t_{k_l} \simeq -\lambda \frac{\delta k^\alpha}{M^\alpha} \int_0^{z_\gamma} \frac{(1 + z')^\alpha}{H(z')} dz', \quad (45)$$

where

$$\delta k^\alpha \equiv k_h^\alpha - k_l^\alpha. \quad (46)$$

For  $\Lambda$ CDM model, we have

$$H(z') = H_0 \sqrt{\Omega_m^0 (1 + z')^3 + \Omega_\Lambda^0}, \quad (47)$$

where  $H_0$  is the present Hubble parameter, and then

$$\delta t \simeq -\lambda H_0^{-1} \frac{\delta E^\alpha}{M^\alpha} \int_0^{z_\gamma} \frac{(1 + z')^\alpha}{\sqrt{\Omega_m^0 (1 + z')^3 + \Omega_\Lambda^0}} dz', \quad (48)$$

here  $E \simeq k$  is the photon energy measured on the Earth and  $\delta E^\alpha \simeq E_h^\alpha$ . In order to explain the time delays,  $\lambda$  should be negative. Here  $H_0^{-1}$  is roughly the same as the age of the universe, but  $\delta t$  is only about 16.54 s, and hence  $M$  should be much larger

than  $E_h$  if  $|\lambda|$  is not so small. The WMAP 5 yr data [31] indicates that  $H_0 = 70.5 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ,  $\Omega_\Lambda^0 = 0.726$  and  $\Omega_m^0 = 0.274$ . Taking Eq. (41) into account, we have  $\lambda = \frac{3}{2} \lambda_{2,1}$  and  $\alpha = 2$ . For  $E_h = 13.22 \text{ GeV}$  and  $\delta t = 16.54 \text{ s}$ , we get

$$M \simeq 60 |\lambda_{2,1}|^{\frac{1}{2}} \frac{1}{\sqrt{H_0 \delta t}} \text{ GeV} \simeq 9.8 \times 10^9 |\lambda_{2,1}|^{\frac{1}{2}} \text{ GeV}. \quad (49)$$

Usually it is expected that  $|\lambda_{2,1}| \sim \mathcal{O}(1)$  and then a conservative estimation of  $M$  is roughly not lower than  $10^{10} \text{ GeV}$ . It would be better to take this result as the constraint on the scale of Lorentz symmetry breaking in the Lifshitz gauge field theory, taking into account of the fact that there exist possible astrophysical sources accounting for the time delays of the gamma-ray bursts.

Before closing this section, we need to stress that a negative  $\lambda$  in Eq. (42) implies that the theory becomes unstable and ill-defined in the UV region. However, for a field theory at Lifshitz point with  $z \geq 3$  it is a UV well-defined field theory which can easily explain the time delays in gamma-ray bursts.

#### 5. The thermal dynamics of the field theory at the Lifshitz point

It would be interesting to study the thermal dynamics of the above field theories at Lifshitz point. From the discussions in Sections 2 and 3, the dispersion relations for both the scalar field and the gauge field are given by

$$\omega^2 = m^2 + k^2 + \dots + \frac{k^{2z}}{M^{2z-2}}. \quad (50)$$

The energy density at finite temperature  $T$  is

$$\rho \sim \int_0^\infty \omega e^{-\omega/T} k^{d-1} dk. \quad (51)$$

In the high temperature limit ( $T \gg M$ ), the dispersion relation can be simplified to be  $w \simeq k^z/M^{z-1}$ , and hence

$$\rho \sim M^{(z-1)d/z} T^{1+d/z}. \quad (52)$$

Similarly, the entropy density is found to be

$$s \sim M^{(z-1)d/z} T^{d/z}. \quad (53)$$

The above scaling behaviors imply that the field theory seems living in a  $(d_s = 1 + d/z)$ -dimensional spacetime. For  $z = d$ ,  $\rho \sim M^{d-1} T^2$  and  $s \sim M^{d-1} T$ . We see that the thermal behaviors of these field theories are quite different from the ones of a relativistic field theory.

We are also interested in the equation of state of matter at the Lifshitz point with  $z$  in a FRW universe. Considering  $[E]_s = z$  and the spatial volume has dimension  $[V]_s = -d$ , we have

$$[\rho]_s = \left[ \frac{E}{V} \right]_s = z + d. \quad (54)$$

The metric of a FRW universe is

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2. \quad (55)$$

Taking Eq. (54) into account, we have

$$\rho \propto a^{-(z+d)}. \quad (56)$$

In the FRW universe, the energy density of matter with the equation of state  $w$  goes like  $\rho \sim a^{-d(1+w)}$ . Therefore the equation of state of matter at the Lifshitz point with  $z$  is

$$w = \frac{z}{d}. \quad (57)$$

<sup>2</sup> In this Letter, we use  $z'$  to denote the redshift.

Obviously, when  $z = 1$ , this is exactly the equation of state of relativistic matter in a FRW universe. Since the energy density  $\rho \propto T^{1+d/z}$ ,  $T \propto a^{-z}$ . The temperature of the radiation with  $z > 1$  decreases faster than that of the relativistic matter in an expanding universe. On the other hand, the entropy density  $s \propto T^{d/z}$  and then  $s \propto a^{-d}$ . This is reasonable because the entropy density is inversely proportional to the physical volume  $a^d$ .

## 6. The size of causal patch for the field theory at the Lifshitz point in the FRW universe

In this section we will figure out a new length scale  $L_H$  which characterizes the proper size of a causal patch in space for the perturbation mode with physical momentum  $p$ . Consider two particles separated by a distance  $L_c$  in the comoving coordinates at the time  $t$  in a flat FRW universe. The proper distance between them is nothing but

$$L = a(t)L_c. \quad (58)$$

If the spatial comoving coordinates of these two particles remain unchanged, the relative speed between them due to the expansion of the universe is

$$\frac{dL}{dt} = \dot{a}L_c = HL. \quad (59)$$

On the other hand, the propagation velocity of the message between these two particles through the field with dynamical critical exponent  $z$  is  $c_g \sim p^{z-1}/M^{z-1}$ . Therefore the size of the causal patch  $L_H$  satisfies

$$HL_H \sim p^{z-1}/M^{z-1}. \quad (60)$$

At the time when the perturbation mode stretches outside its causal patch, we have  $p \sim 1/L_H$  and then we obtain

$$L_H \sim (M^{z-1}H)^{-1/z}. \quad (61)$$

For  $z = 1$ ,  $L_H$  is nothing but the Hubble length. For  $z = d = 3$ , it is the same as the one found in [8].

For a  $(d+1)$ -dimensional FRW universe dominated by the matter with the equation of state  $w$ , the scale factor grows up as

$$a(t) \sim t^{\frac{2}{d(1+w)}}. \quad (62)$$

If  $w < -1 + 2/d$ , the universe is in an inflationary phase. The Hubble parameter decreases as  $1/t$  if  $w > -1$ . In order that a perturbation mode is generated within the causal patch and stretches outside the horizon in the future, we should have  $a(t) > L_H(t)$  for a sufficient large  $t$ , which implies

$$w < w_c = -1 + \frac{2z}{d}. \quad (63)$$

For  $z = 1$ , a causally generated quantum perturbation can stretch outside its causal patch and be frozen to be a classical perturbation only in an inflationary universe. But for  $z > 1$ , it can happen even in a non-inflationary universe. Since the scalar field with the dynamical critical exponent  $z$  has dimension  $\frac{d-z}{2}$ , the perturbation of such a scalar mode with  $z = d$  is expected to be scale-invariant even in a non-inflationary universe. That is why ones claim that the inflation is not necessarily required when the field theory at a Lifshitz point with  $z = 3$  is called for in our  $(3+1)$ -dimensional universe. However, even though the horizon problem in hot big bang model might be released due to the super-luminosity in the UV region, the flatness problem can be solved only in an inflationary universe. It is premature to claim that the Lifshitz field/gravity provides an alternative model to inflation.

## 7. Discussions

In this Letter we constructed the most general power-counting renormalizable actions for the scalar field and the gauge field without considering the detailed balance condition. These field theories at long distance reduce to the field theories with the Lorentz invariance intact, but the symmetry between space and time is broken down at short distance for  $z \geq 2$ . Since only the kinetic terms which is quadratic in the first time derivatives are included, the field theories are still unitary. In this Letter we assumed that the space is isotropic. One can generalize them to the cases with anisotropic space. Here we only proposed that the spatial derivative operators like  $\Delta^n$  appearing in the action, where  $n$  is an integer. Maybe some terms with fractional power of the differential operator  $\Delta$  could be included as well [6]. But the physical meaning of these terms is not well-understood.

In the original Lifshitz scalar field theory and its generalization to non-Abelian gauge field and gravity [14,12,1], one may impose the detailed balance condition to fix the potential. In these cases, the ground state wavefunction of the theory reproduce the partition function of a relativistic theory in lower dimension. This fact may suggest that the theories with the detailed balance condition have quantum critical points. In this Letter, for generality, we did not impose any kind of the detailed balance condition. As a result, even if we only consider the interaction terms with the marginal dimension, the theory is just classically scale-invariant and may not be scale-invariant quantum-mechanically. Obviously, a careful investigation of RG flow and quantum criticality would be a very interesting issue.

In [24], the gravity duals of the anisotropic scale-invariant field theory have been constructed. It would be interesting to investigate the gravity duals of the theories presented in this Letter.

## Acknowledgements

The work was partially supported by NSFC Grant Nos. 10535060, 10775002 and NKBRPC (No. 2006CB805905).

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