Spectral rational variation in two places
for adjacency matrix is impossible

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Abstract

Let \( G = (V, E) \) be a simple graph and \( \{\lambda_1(G), \ldots, \lambda_n(G)\} \) be its adjacency spectrum. It is easy to see that if an edge is added between two isolated vertices, then one zero eigenvalue increases by 1, and another zero eigenvalue decreases by 1. Let \( G^+ \) be a connected graph obtained from \( G \) by adding an edge \( e \notin E(G) \). In this paper, it will be proved that the spectrum of \( G^+ \) is different from that of \( G \) only in two places with one eigenvalue increases by \( m \) and another eigenvalue decreases by \( m \), where \( m > 0 \) is a rational number, if and only if \( G \) is an empty graph with order 2. It will also be proved that one cannot construct a new adjacency integral connected graph with order \( n \geq 3 \) from a known one by adding an edge.

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1. Introduction

Throughout the paper, all considered graphs are undirected graphs without loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Its adjacency matrix $A(G)$ is defined as an $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if the vertices $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ if they are non-adjacent. Thus $A(G)$ is a symmetric matrix with zero diagonal.

The Laplacian (or admittance) matrix of $G$ (with respect to an ordering of $V$) is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix whose $i$th diagonal entry is the degree $d_i(G)$ of the vertex $v_i$.

Let $G^+$ denote the graph obtained from $G$ by adding an edge $e \notin E(G)$. With a suitable ordering, we have

$$L(G^+) = L(G) + H,$$

where $H = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \oplus 0_{n-2}$. Since $H$ is a positive semidefinite symmetric matrix with trace equals to 2, we know that by adding an edge, none of the eigenvalues of the corresponding Laplacian can decrease, and that the sum of those eigenvalues will move up by 2. So [9] raised the problem of determining the circumstances under which the addition of an edge to a graph will cause the Laplacian eigenvalues to change only by integer quantities. Evidently there are just two possible scenarios where that can happen: (1) one eigenvalue of $L(G)$ will increase by 2 (and $n-1$ eigenvalues remain unchanged); (2) two eigenvalues of $L(G)$ will increase by 1 (and $n-2$ eigenvalues remain unchanged). Fan [4] and Kirkland [7] referred to the two scenarios by saying that Laplacian spectral integral variation occurs in one place by adding an edge, and Laplacian spectral integral variation occurs in two places by adding an edge, respectively.

So [9] characterized the case that Laplacian spectral integral variation occurs in one place; Kirkland [7] characterized all the graphs for which Laplacian spectral integral variation occurring in two places. Hence the problem raised by So in [9] was settled completely.

In this article, we shall concentrate on the adjacency matrix of a graph $G$. The terminology in [7] will be extended here for the adjacency matrix of $G$. Since the adjacency matrix $A(G)$ is a real symmetric matrix, all of its eigenvalues are real, and they are called the eigenvalues of $G$. The spectrum of $G$ is defined by the multi-set $\text{Spec}(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. When an edge is added, the resulted adjacency matrix $A(G^+)$ can be written as

$$A(G^+) = A(G) + K$$

with a suitable ordering, where $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-2}$. Observe that the spectrum of $K$ is $\{1, 0, 0, \ldots, 0, -1\}$. If the eigenvalues of $A(G^+)$ are the sums of the eigenvalues
of $A(G)$ and $K$ in a suitable order, then we denote this situation by $\text{Spec}(A(G^+)) = \text{Spec}(A(G)) + \text{Spec}(K)$.

**Lemma 1.1.** Let $G$ be a graph and $G^+$ a graph obtained from $G$ by adding a new edge. Suppose that $\text{Spec}(A(G^+)) = \text{Spec}(A(G)) + \text{Spec}(K)$. If the two changed eigenvalues of $G$ are $\lambda_i(G)$ and $\lambda_j(G)$, then $\lambda_i(G) = \lambda_j(G)$.

**Proof.** Let $\text{Tr}(A)$ denote the trace of the matrix $A$. It is easy to see that the $i$th diagonal entry of $A^2(G)$ is equal to the degree $d_i(G)$ of the vertex $v_i$ of $G$. Then,

$$
\sum_{k=1}^{n} \lambda_k^2(G) = \text{Tr}(A^2(G)) = \sum_{k=1}^{n} d_k(G) = 2|E(G)|, \tag{1}
$$

$$
\sum_{k=1}^{n} \lambda_k^2(G^+) = \sum_{k \neq i, j} \lambda_k^2(G) + (\lambda_i(G) + 1)^2 + (\lambda_j(G) - 1)^2
\begin{aligned}
&= \sum_{k=1}^{n} \lambda_k^2(G) + 2(\lambda_i(G) - \lambda_j(G)) + 2 \\
&= \text{Tr}(A^2(G^+)) = \sum_{k=1}^{n} d_k(G^+) = 2|E(G^+)| = 2(|E(G)| + 1). \tag{2}
\end{aligned}
$$

Combining (1) and (2), we know that $\lambda_i(G) = \lambda_j(G)$. \hfill $\Box$

Similar to articles [7,9], we will study the circumstances under which the spectrum of $G^+$ is from that of $G$ to be

$$
\{\lambda_1(G), \lambda_2(G), \ldots, \lambda_i(G)+1, \ldots, \lambda_j(G)-1, \ldots, \lambda_n(G)\},
$$

where $i < j$.

In Theorem 2.6, it will be proved that if $G^+$ is a connected graph with order greater than 2, then $\text{Spec}(A(G^+)) \neq \text{Spec}(A(G)) + \text{Spec}(K)$.

Note that $K$ is a symmetric matrix with trace equals to 0. Someone may want to know what is the circumstances under which the spectrum of $G^+$ is different from that of $G$ only in two places with one eigenvalue increases by $m$ and another eigenvalue decreases by $m$, where $m > 0$ is a real number. If so, when $m$ is a rational number, it will be proved in Theorem 2.7 that the two changed eigenvalues must be integers and that $m$ must be the integer 1. Recall Theorem 2.6, then we know that the adjacency spectral rational variation will not occur in two places by adding a new edge if $G^+$ is a connected graph with order $n \geq 3$. In Section 2, start with Lemma 2.4, we will also get a result (Corollary 2.5) that one cannot construct a new adjacency integral connected graph with order $n \geq 3$ from a known one by adding an edge.
2. Main results

From now on, if an $n \times n$ matrix $X$ has real eigenvalues then its decreasingly ordered eigenvalues are denoted by $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$. The following inequalities are well known.

**Lemma 2.1** (Weyl [10]). Let $A$ and $B$ be $n \times n$ symmetric matrices. Then for integers $r, s$ and $i$ such that $1 \leq r \leq i \leq s \leq n$. Then

$$\lambda_s(A) + \lambda_{i-s+n}(B) \leq \lambda_i(A + B) \leq \lambda_r(A) + \lambda_{i-r+1}(B).$$

Moreover, $\lambda_s(A) + \lambda_{i-s+n}(B) = \lambda_i(A + B)$ if and only if there exists a unit real vector $x \neq 0$ such that $Ax = \lambda_s(A)x$, $Bx = \lambda_{i-s+n}(B)x$ and $(A + B)x = \lambda_i(A + B)x$; and $\lambda_s(A + B) = \lambda_r(A) + \lambda_{i-r+1}(B)$ if and only if there exists a unit real vector $x \neq 0$ such that $Ax = \lambda_r(A)x$, $Bx = \lambda_{i-r+1}(B)x$, and $(A + B)x = \lambda_i(A + B)x$.

**Lemma 2.2.** Let $G$ be a graph with order $n$ and $G^+$ a graph obtained from $G$ by adding a new edge between two non-adjacent vertices $v_k$ and $v_l$. For $1 \leq i \leq n$,

(i) $\lambda_i(G) - 1 \leq \lambda_i(G^+) \leq \lambda_i(G) + 1$,

(ii) If $\lambda_i(G) \pm 1 = \lambda_i(G^+)$, then $\lambda_i(G) = 0$.

**Proof.** (i): By Lemma 2.1, it follows from $A(G^+) = A(G) + K$ that

$$\lambda_i(A(G)) + \lambda_n(K) \leq \lambda_i(A(G^+)) \leq \lambda_i(A(G)) + \lambda_1(K).$$

Note that $\lambda_1(K) = 1$ and $\lambda_n(K) = -1$. Thus (i) follows.

(ii): If $\lambda_i(G^+) = \lambda_i(G) + 1$ then $\lambda_i(A(G^+)) = \lambda_i(A(G)) + \lambda_1(K)$ and it follows from Lemma 2.1 that there is a unit real vector $x = (x_1, x_2, \ldots, x_n)^T \neq 0$ such that

$$\begin{align*}
A(G^+)x &= \lambda_i(G^+)x, \quad (3) \\
A(G)x &= \lambda_i(G)x, \quad (4) \\
Kx &= x. \quad (5)
\end{align*}$$

It is easy to see that the equality (5) implies that the elements $x_t$ of $x$ are equal to 0, for $t \neq k, l$. Then from (4), we have $\lambda_i(G) = xA(G)x^T = 0$. By the same argument, we can know that if $\lambda_i(G^+) = \lambda_i(G) - 1$, then $\lambda_i(G) = 0$. Thus (ii) follows.

**Lemma 2.3.** Let $A$ and $B$ be $n \times n$ non-negative matrices. If $B - A$ is a non-negative matrix, then

(i) $\lambda_1(A) \leq \lambda_1(B)$;
(ii) if $B$ is irreducible and $\lambda_1(A) = \lambda_1(B)$, then $A = B$. 
Proof. (i): It is easy to see that (i) follows from Theorem 8.1.18 of [6].

(ii): Suppose to the contrary that \( A \neq B \). Let \( C = B - A \), then \( C \neq 0 \) and \( C \) is a non-negative matrix. For any real number \( \varepsilon \), define \( f(\varepsilon) = A + \varepsilon C \), then \( B = f(1) \) and \( A = f(0) \). Since \( B \) is irreducible, we know that if \( \varepsilon > 0 \), then \( f(\varepsilon) \) is irreducible and non-negative. By Perron–Frobenius theory, the spectral radius of a non-negative irreducible matrix strictly increases with each element. Since \( C \neq 0 \), then \( \lambda_1(B) = \lambda_1(f(1)) > \lambda_1(f\left(\frac{1}{2}\right)) > \cdots > \lambda_1(f\left(\frac{1}{k}\right)) > \cdots \), with \( \lim_{k \to \infty} \lambda_1(f\left(\frac{1}{k}\right)) \). It is well known that the eigenvalues of a matrix are continuous functions of the elements, so \( \lim_{k \to \infty} \lambda_1(f\left(\frac{1}{k}\right)) = \lambda_1(f(0)) = \lambda_1(A) \). So \( \lambda_1(B) > \lambda_1(A) \). It is a contradiction to (ii). Thus \( A = B \). □

Lemma 2.4. Let \( G \) be a graph with order \( n \geq 3 \) and \( G^+ \) a connected graph obtained from \( G \) by adding a new edge between two non-adjacent vertices \( v_k \) and \( v_l \). Then

(i) \( \lambda_1(G^+) > \lambda_1(G) \);
(ii) \( \lambda_2(G^+) \neq \lambda_1(G) \);
(iii) \( \lambda_i(G^+) = \lambda_i(G) + 1 \), for \( i = 1, 2, \ldots, n \).

Proof. (i): Since \( G^+ \) is connected, \( A(G^+) \) is non-negative irreducible. \( A(G^+) - A(G) = K \neq 0 \). Hence we know from Lemma 2.3 that \( \lambda_1(A(G^+)) > \lambda_1(A(G)) \). Hence (i) follows.

(ii): Suppose to the contrary that \( \lambda_2(G^+) = \lambda_1(G) \).
Since \( n \geq 3 \), \( \lambda_2(K) = 0 \). Then \( \lambda_2(G^+) = \lambda_1(G) + \lambda_2(K) \). By Lemma 2.1, there exists a unit real vector \( x \neq 0 \) such that

\[
\begin{align*}
A(G^+)x &= \lambda_2(G^+)x, \\
A(G)x &= \lambda_1(G)x, \\
Kx &= \lambda_2(K)x.
\end{align*}
\]

Since \( G^+ \) is connected, \( A(G^+) \) is a non-negative irreducible matrix. Apply Perron–Frobenius theory [6], we know that \( \lambda_2(G^+) \) is not the spectral radius of \( A(G^+) \), the vector \( x \) must contain both positive and negative elements. Then it follows from (7) that \( G \) is not connected with two connected components \( G_1 \) and \( G_2 \). Thus \( A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix} \). Write \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) with \( x_1 \neq 0 \), to confirm to the partition of \( A(G) \).

Then from \( A(G)x = \lambda_1(G)x \), we have

\[
\begin{align*}
A(G_1)x_1 &= \lambda_1(G)x_1, \\
A(G_2)x_2 &= \lambda_1(G)x_2.
\end{align*}
\]

Without loss of generality, we may assume that \( \lambda_1(G) = \lambda_1(G_1) \). This implies that all the elements of \( x_1 \) are of the same sign (\( \neq 0 \)) and hence that there is at least one
element of $x_2$ is of the opposite sign of $x_1$. So $x_2$ is a non-zero vector and the equality (10) implies that $\lambda_1(G)$ must be one of the eigenvalues of $A(G_2)$. Thus $\lambda_1(G) = \lambda_1(G_1) = \lambda_1(G_2)$. Since $A(G_2)$ is a non-negative irreducible matrix, apply Perron–Frobenius theory again, we know that there is no element of $x_2$ equal to 0 from (10). Thus there is no element of $x$ is equal to 0, and hence that $\lambda_2(K) = x^T K x = 0$. It is impossible once again. Hence (ii) follows.

(iii): If $\lambda_i(G +) = \lambda_i(G) + 1$, for some $1 \leq i \leq n$, then $\lambda_i(A(G +)) = \lambda_i(A(G)) + \lambda_1(K)$ and it follows from Lemma 2.1 that there is a unit real vector $x = (x_1, \ldots, x_n)^T \neq 0$ such that the equalities (3)–(5) hold. If $\lambda_1(G +) = \lambda_1(G) + 1$, then the equality (3) implies that all the elements of $x$ are of the same sign ($\neq 0$). But now $\lambda_1(G +) = \lambda_1(G) + \lambda_1(K) = \lambda_1(G) + x^T K x = \lambda_1(G) + 2x_k x_l \leq \lambda_1(G) + x_k^2 + x_l^2 < \lambda_1(G) + 1$. Impossible. If $\lambda_i(G +) = \lambda_i(G) + 1$, for some $i > 1$, then the equality (3) implies that $x$ is an eigenvector corresponding to $\lambda_i(G +)$ of $A(G +)$ and it follows from equality (5) that $x_k = x_l = 0$, for $i \neq k, l$. Since $x \neq 0$, $x_k = x_l \neq 0$. Therefore, $x$ is not orthogonal to any eigenvector corresponding to $\lambda_1(G +)$ of $A(G +)$. Impossible again. Hence (iii) follows. □

A graph $G$ is said to be adjacency integral if Spec($G$) consists of entirely of integers. In 1970s, Harary and Schwenk [5] posed the problem of determining the graphs $G$ whose $A(G)$ has integral spectrum. Up to now there are many results on adjacency integral graphs in the literature, see for instance [1,2,3,8,11]. If an edge is added between two isolated vertices, then adjacency spectral integral variation occurs in two places, with one zero eigenvalue increases by 1, and another zero eigenvalue decreases by 1. Since the adjacency spectrum of a disconnected graph is a union of those of connected components, it is enough to handle the graph $G$ with $G^+$ connected. The following Corollary 2.5 will tell us that one cannot construct a new adjacency integral connected graph with order $n \geq 3$ from a known one by adding a new edge.

**Corollary 2.5.** Let $G$ be a adjacency integral graph and $G^+$ a connected graph obtained from $G$ by adding a new edge. Then $G^+$ is adjacency integral if and only if $G$ is an empty graph with order 2.

**Proof.** Since all the eigenvalues of $G^+$ and $G$ are integers, by Lemma 2.2 (i), and Lemma 2.4 (i), we know that if the order of $G$ is greater than 2, then $\lambda_1(G +) = \lambda_1(G) + 1$. It is a contradiction to Lemma 2.4 (iii). Hence the order of $G$ is 2. Conversely, if the order of $G$ is 2, it is easy to see that this corollary follows. □

**Theorem 2.6.** Let $G$ be a graph with order $n \geq 3$ and $G^+$ a connected graph obtained from $G$ by adding a new edge. Then Spec($A(G^+)$) $\neq$ Spec($A(G)$) + Spec($K$).

**Proof.** Suppose to the contrary that Spec($A(G^+)$) = Spec($A(G)$) + Spec($K$). There are two cases need to deal with.
Case 1: \( \lambda_1(G) \in \text{Spec}(G^+) \). Then by Lemma 2.4 (i), \( \lambda_1(G) \neq \lambda_1(G^+) \), so \( \lambda_1(G) = \lambda_2(G^+) \). It is a contradiction to Lemma 2.4 (ii).

Case 2: \( \lambda_1(G) \notin \text{Spec}(G^+) \). Then \( \lambda_1(G) \) will be the eigenvalue being changed. By Lemma 1.1, another changed eigenvalue of \( G \) is equal to \( \lambda_1(G) \). Thus \( \lambda_1(G^+) = \lambda_1(G) + 1 \), it is a contradiction to Lemma 2.4 (iii).

Therefore the hypothesis is not true. This theorem follows. \( \square \)

Now let us consider the case that the adjacency spectral rational variation occurs just in two places when an edge is added.

**Theorem 2.7.** Let \( G \) be a graph with order \( n \geq 3 \) and \( G^+ \) a connected graph obtained from \( G \) by adding a new edge. Then the adjacency spectral rational variation will not occur in two places.

**Proof.** Suppose to the contrary that there is an adjacency spectral rational variation occurring in two places, with one eigenvalue \( \lambda_i(G) \) increases by \( m \), and another eigenvalue \( \lambda_j(G) \) decreases by \( m \), \( m(>0) \in \mathbb{Q} \), where \( \mathbb{Q} \) denotes the rational number field.

For the adjacency matrix \( A(G) \), it is easy to see that the \( ii \)-entry \( a_{ii}^{(3)} \) of \( A^3(G) \) is the number of 3-cycles \( v_i v_j v_k v_i \) in \( G \). Each triangle in \( G \) determines six such 3-cycles, because there are three choices of initial vertex and two possible orientations. It follows that the number of triangles in \( G \) is \( \frac{1}{6} \text{Tr} A^3(G) \). Let \( \Delta T \) denote the additional number of triangles of \( G^+ \) from \( G \). Then

\[
\begin{align*}
\sum_{k=1}^{n} \lambda_k^3(G^+) - \sum_{k=1}^{n} \lambda_k^3(G) &= \text{Tr}(A^2(G^+)) - \text{Tr}(A^2(G)) = 2, \\
\sum_{k=1}^{n} \lambda_k^3(G^+) - \sum_{k=1}^{n} \lambda_k^3(G) &= \text{Tr}(A^3(G^+)) - \text{Tr}(A^3(G)) = 6\Delta T.
\end{align*}
\]

\[
\begin{align*}
(\lambda_i(G) - \lambda_j(G))m + m^2 &= 1, \quad (11) \\
(\lambda_i^2(G) - \lambda_j^2(G))m + (\lambda_i(G) + \lambda_j(G))m^2 &= 2\Delta T. \quad (12)
\end{align*}
\]

Solve them out,

\[
\begin{align*}
\lambda_i(G) &= \Delta T + \frac{1}{2} \left( \frac{1}{m} - m \right) \in \mathbb{Q}, \\
\lambda_j(G) &= \Delta T - \frac{1}{2} \left( \frac{1}{m} - m \right) \in \mathbb{Q}.
\end{align*}
\]

Since the characteristic polynomial of the adjacency matrix of a graph is monic and has integral coefficients, we know that every rational eigenvalue of a graph is integral. Thus, the eigenvalues \( \lambda_i(G) \) and \( \lambda_j(G) \) are integral, and \( m \) must be the integer 1. Recall Theorem 2.6, this theorem follows. \( \square \)
**Remark.** Professor Kirkland gives a short, more direct proof of Theorem 2.7 as follows. It can be viewed that $\lambda_1(G^+) = \lambda_1(G) + m$. By Lemma 2.2, $m \leq \lambda_1(K) = 1$. Recall that the difference of two algebraic integers is also an algebraic integer. Since $\lambda_1(G)$ and $\lambda_1(G) + m$ are algebraic integers, we deduce that $m$ is an algebraic integer. Now that $m$ is a rational number, it must be an integer. Thus $m = 1$. So the conclusion follows from Theorem 2.6.

It is still an open problem to characterize the case that adjacency spectral irrational variation occurs just in two places.

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