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Traces on the skein algebra of the torus

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Abstract

For a surface F, the Kauffman bracket skein module of $F \times [0,1]$, denoted $K(F)$, admits a natural multiplication which makes it an algebra. When specialized at a complex number t, nonzero and not a root of unity, we have $K_t(F)$, a vector space over $\mathbb{C}$. In this paper, we will use the product-to-sum formula of Frohman and Gelca to show that the vector space $K_t(T^2)$ has five distinct traces. One trace, the Yang–Mills measure, is obtained by picking off the coefficient of the empty skein. The other four traces on $K_t(T^2)$ correspond to the four singular points of the moduli space of flat $SU(2)$-connections on the torus.

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Keywords: Skein algebra; Trace; Torus

1. Introduction

Skein modules were introduced independently by Przytycki [8] and Turaev [10] and have been an active topic of research since their introduction. In particular, skein modules underlie quantum invariants [7,6] and are connected to the representation theory of the fundamental group of the manifold [1,2,9].

The skein module is spanned by the equivalence classes of framed links in the 3-manifold. The skein module of the cylinder over a surface has a multiplication that comes from laying one framed link on top of the other. With this multiplication, the skein module of the cylinder over a surface is an algebra.

In this paper, we will consider the skein algebra of the torus specialized at a complex number and describe the five distinct traces on this vector space. One trace, the Yang–Mills measure, has been explored in [3] and [5]. The other four traces correspond to the singular points of the moduli space of flat $SU(2)$-connections on the torus.

2. Preliminaries

Let M be an orientable 3-manifold. A framed link in M is the embedding of disjoint annuli into M. A framed link is depicted by drawing the core of each annulus. One typically uses the blackboard framing to produce the annulus from its core. We will use $M = F \times I$ for a surface F. In these cases, we will use the framing given by the surface to produce the annulus from its core.

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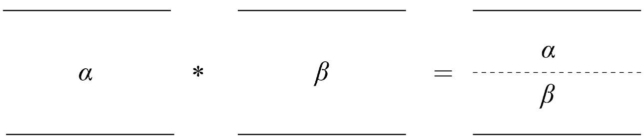

Fig. 1. The product structure on $K(F \times I)$.

Equivalence of framed links in M is up to regular isotopy. That is, using only isotopy and Reidemeister's II and III moves. A Reidemeister I move corresponds to a twist in the annulus and thus such a move does not preserve the equivalence class of a framed link.

Let $\mathcal{L}(M)$ denote the equivalence class of framed links in M, including the empty link, ϕ. Let $R = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials. Consider the free module $R\mathcal{L}(M)$ with basis $\mathcal{L}(M)$. Define $S(M)$ to be the smallest subspace of $R\mathcal{L}(M)$ containing all expressions of the form $\times - t \asymp - t^{-1})($ and $\bigcirc + t^2 + t^{-2}$ where the framed links in each expression are identical outside the region pictured in the diagrams. The *Kauffman bracket skein module* $K(M)$ is the quotient $R\mathcal{L}(M)/S(M)$.

Because $K(M)$ is defined using local relations on framed links, two homeomorphic manifolds have isomorphic skein modules. Thus $K(M)$ is an invariant of the 3-manifold M.

Let F be a compact, orientable surface and let $I = [0, 1]$ be the unit interval. $K(F \times I)$ has an algebra structure that comes from laying one link on top of the other. Given skein elements $\alpha, \beta \in K(F \times I)$, we can represent α, β with the links $L_\alpha, L_\beta \subset F \times I$. Use isotopy to move L_α to $F \times (\frac{1}{2}, 1]$ and L_β to $F \times [0, \frac{1}{2})$. The product $\alpha * \beta$ is the skein element represented by $L_\alpha \cup L_\beta$. A schematic of this product structure is shown in Fig. 1.

To simplify notation, and to emphasize that the algebra structure is determined by F rather than by $F \times I$, we will use $K(F)$ to denote the skein module $K(F \times I)$ with the algebra structure described above. We will refer to $K(F)$ as the *skein algebra* of the surface F.

With the relations used to define $S(M)$, we can represent any skein element as the linear combination of *simple* diagrams, these are diagrams with no crossings and no trivial components. The skein elements induced by such diagrams form a basis for $K(F)$ [8,10].

In this paper we will make a number of simplifying assumptions. Namely, F is the standard 2-torus T^2 and t is a complex number that is nonzero and not a root of unity. Thus the polynomials in $R = \mathbb{Z}[t, t^{-1}]$ are evaluated at the complex number t and the skein algebra $K(F)$ is *specialized* at t to form $K_t(F)$, a vector space over $\mathbb{C}$. Throughout this paper, we will divide by expressions of the form $(t^n - t^{-n})$. Choosing t to be a complex number that is not a root of unity allows this type of division without requiring the use of rational functions.

Let $C_t(F) = [K_t(F), K_t(F)]$ be the vector space over $\mathbb{C}$ with basis consisting of the commutators on $K_t(F)$. A *trace* on the algebra $K_t(F)$ is a linear functional $\varphi : K_t(F) \to \mathbb{C}$ satisfying $\varphi(\alpha * \beta) = \varphi(\beta * \alpha)$ for all $\alpha, \beta \in K_t(F)$. Since a trace is linear, this condition can also be written $\varphi(\alpha * \beta - \beta * \alpha) = 0$. Then a trace φ on $K_t(F)$ has $C_t(F) \subset \ker(\varphi)$. So φ descends to be a linear functional on the quotient $K_t(F)/C_t(F)$.

3. Examples of skein algebras

Before we explore the traces on $K_t(T^2)$ in detail, let's look a few examples of skein algebras.

Let F be the 2-dimensional disk D^2. Since every diagram in D^2 that has no crossings is trivial, the only simple diagram in D^2 is the empty skein ϕ. Thus the skein algebra $K(D^2)$ is one-dimensional with basis $\{\phi\}$.

Let F be the annulus $A = S^1 \times [0, 1]$. The simple diagrams of A consist of the empty skein along with any number of parallel copies of the core of the annulus. Denote n parallel copies of the core of the annulus by z^n with $z^0 = \phi$. Then a basis for $K(A)$ is $\{z^0, z^1, z^2, \ldots\}$ and hence $K(A)$ is isomorphic to the algebra of polynomials in z with coefficients from $R = \mathbb{Z}[t, t^{-1}]$.

Now let F be the torus T^2. The collection of simple diagrams has a more intricate structure. We can have any (p, q)-curve with p and q relatively prime and also any number of parallel copies of a given (p, q)-curve. Thus a basis for $K(T^2)$ is $\{\phi, (p, q)^n \mid \gcd(p, q) = 1, n \in \mathbb{N}\}$, and we use the convention that $(p, q)^0 = \phi$.

Properties of the algebra $K(T^2)$ are explored by Frohman and Gelca in [4]. In particular, they give a basis for $K(T^2)$ that behaves nicely under multiplication. Let $(p,q)_T$ be the (p,q)-curve when $\gcd(p,q)=1$ and let

$$(p,q)_T = T_{\gcd(p,q)}\left(\left(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)}\right)\right)$$

when $\gcd(p,q) \neq 1$. Here $T_n(x)$ is the nth Chebyshev polynomial defined recursively by $T_0(x)=2$, $T_1(x)=x$, and $T_{n+1}(x) = T_n(x)T_1(x) - T_{n-1}(x)$. Using this basis for $K(T^2)$, we have the following *product-to-sum* formula.

Theorem 1 *(Frohman–Gelca).*

$$(p,q)_T * (r,s)_T = t^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p+r,q+s)_T + t^{-\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p-r,q-s)_T,$$

where $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|$ *is the determinant.*

4. Traces on $K_t(T^2)$

Let t be a fixed complex number, nonzero and not a root of unity. Let $A = K_t(T^2)$ and let $\delta : A \otimes A \to A$ be defined by $\delta(a \otimes b) = ab - ba$. The image of δ in A is the subalgebra of A generated by these commutators. Denote this subalgebra by $C(A)$. To understand the nature of the traces on A, we will focus our attention on the commutator quotient $A/C(A)$ because the space of traces is dual to this quotient.

As a vector space $A/C(A)$ is spanned by the cosets of all $(p,q)_T \in A$. To narrow this spanning set to a basis, we use the product-to-sum formula.

As cosets, $(x,y)_T + C(A) = (z,w)_T + C(A)$ if and only if $(x,y)_T - (z,w)_T \in C(A)$. $(x,y)_T - (z,w)_T \in C(A)$ if and only if $(x,y)_T - (z,w)_T$ is equal to some linear combination of commutators. The simplest case would be if

$$(x,y)_T - (z,w)_T = \lambda\big((p,q)_T * (r,s)_T - (r,s)_T * (p,q)_T\big), \quad \lambda \in \mathbb{C}.$$

We can use the product-to-sum formula to find the integers p, q, r, s when we are given x, y, z, w.

Lemma 1. *Pick a complex number t that is nonzero and not a root of unity and let $A = K_t(T^2)$ with basis $\{(p,q)_T\}$ as described above. Then $(x,y)_T + C(A) = (z,w)_T + C(A)$ if $x+z$ is even, $y+w$ is even, and $\left|\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right| \neq 0$.*

Proof. Suppose we have integers x, y, z, w such that $x+z$ is even, $y+w$ is even and $\left|\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right| \neq 0$. Let $p = \frac{x+z}{2}$, $q = \frac{y+w}{2}$, $r = \frac{x-z}{2}$, and $s = \frac{y-w}{2}$. Since $\left|\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right| \neq 0$, elementary matrix operations lead to $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| \neq 0$. Let $\alpha = \left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| \neq 0$. Then

$$(p,q)_T * (r,s)_T = t^{\alpha}(x,y)_T + t^{-\alpha}(z,w)_T,$$
$$(r,s)_T * (p,q)_T = t^{-\alpha}(x,y)_T + t^{\alpha}(-z,-w)_T.$$

Since orientation does not matter in A, $(-z,-w)_T = (z,w)_T$ and therefore

$$(p,q)_T * (r,s)_T - (r,s)_T * (p,q)_T = (t^{\alpha} - t^{-\alpha})\big((x,y)_T - (z,w)_T\big).$$

Since $\alpha \neq 0$ and t is not a root of unity, we can divide by $(t^{\alpha} - t^{-\alpha})$ to get

$$(x,y)_T - (z,w)_T = \frac{1}{t^{\alpha} - t^{-\alpha}}\big((p,q)_T * (r,s)_T - (r,s)_T * (p,q)_T\big) \in C(A).$$

Thus $(x,y)_T + C(A) = (z,w)_T + C(A)$ as cosets in $A/C(A)$. □

In other words, if x and z have the same parity, y and w have the same parity, and (x,y) and (z,w) are linearly independent, then $(x,y)_T$ is equivalent to $(z,w)_T$ in $A/C(A)$. This fact allows us to reduce our basis for $A/C(A)$ somewhat, and it suggests that the parity of (p,q) will determine the class of $(p,q)_T$ in $A/C(A)$.

Theorem 2. *Pick a fixed complex number t, nonzero and not a root of unity. Let $A = K_t(T^2)$ and let $C(A)$ be the subalgebra of A generated by commutators. Then $A/C(A)$ is a five-dimensional vector space over $\mathbb{C}$.*

Proof. Motivated by Lemma 1, we define a map

$$\varphi : A \to \mathbb{C}\{\phi, ee, eo, oe, oo\}$$

by

$$(p,q)_T \mapsto \begin{cases} \phi & \text{if } p=0,\ q=0, \\ ee & \text{if } p \text{ even},\ q \text{ even}, \\ eo & \text{if } p \text{ even},\ q \text{ odd}, \\ oe & \text{if } p \text{ odd},\ q \text{ even}, \\ oo & \text{if } p \text{ odd},\ q \text{ odd}, \end{cases}$$

then extend linearly.

We need to show that $\ker(\varphi) = C(A)$. Recall that $C(A)$ is the vector space generated by $(p,q)_T * (r,s)_T - (r,s)_T * (p,q)_T$ for $(p,q)_T, (r,s)_T \in A$.

Choose $(p,q)_T, (r,s)_T \in A$. To show that $C(A) \subset \ker(\varphi)$, it suffices to show that $c = (p,q)_T * (r,s)_T - (r,s)_T * (p,q)_T \in \ker(\varphi)$.

$$\begin{aligned} c &= (p,q)_T * (r,s)_T - (r,s)_T * (p,q)_T \\ &= \left(t^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|} - t^{-\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}\right)\big((p+r, q+s)_T - (p-r, q-s)_T\big). \end{aligned}$$

If $(p+r, q+s)_T = (0,0)_T$, then $(p,q) = -(r,s)$. Hence $(p,q)_T = (r,s)_T$. Thus $c = 0 \in \ker(\varphi)$. If $(p-r, q-s)_T = (0,0)_T$, then $(p,q) = (r,s)$. Hence $(p,q)_T = (r,s)_T$. Thus $c = 0 \in \ker(\varphi)$.

If neither $(p+r, q+s)_T$ nor $(p-r, q-s)_T$ is $(0,0)_T$, then since $p+r$ and $p-r$ have the same parity and $q+s$ and $q-s$ have the same parity, we have $\varphi((p+r, q+s)_T) = \varphi((p-r, q-s)_T)$ which implies that $c \in \ker(\varphi)$. Hence $C(A) \subset \ker(\varphi)$.

Now we show that $\ker(\varphi) \subset C(A)$. Take $k \in \ker(\varphi)$. So $k \in A$ and $\varphi(k) = 0$. Then

$$\begin{aligned} k &= \sum_{\text{finite}} \lambda_{(p,q)} (p,q)_T \\ &= \lambda_{(0,0)}(0,0)_T + \sum_{ee} \lambda_{(p,q)}(p,q)_T + \cdots + \sum_{oo} \lambda_{(p,q)}(p,q)_T. \end{aligned} \tag{1}$$

Here we are breaking the sum into five parts, according to the parity of (p,q). In each of these five parts, the coefficients must sum to zero since $k \in \ker(\varphi)$. As a model for the other cases, we work the case where

$$k = \sum_{ee} \lambda_{(p,q)}(p,q)_T$$

and so

$$\sum_{ee} \lambda_{(p,q)} = 0.$$

Now,

$$\sum_{ee} \lambda_{(p,q)}(p,q)_T$$

is a finite sum and $(p,q)_T$ are all of even–even parity, and $(p,q) \neq (0,0)$. Choose integers r and s such that $(r,s)_T$ is of even–even parity and (r,s) is linearly independent to each of the (p,q) in the sum for k. That is, s/r is a rational slope that is different from the finite number of rational slopes q/p. Now, using Lemma 1, each $(p,q)_T = (r,s)_T$ in the quotient $A/C(A)$. Hence

$$\begin{aligned} k &= \sum_{ee} \lambda_{(p,q)}(p,q)_T = \left(\sum_{ee} \lambda_{(p,q)}(r,s)_T\right) + \text{commutators} \\ &= \left(\sum_{ee} \lambda_{(p,q)}\right)(r,s)_T + \text{commutators} = \text{commutators}, \quad \text{since } \sum_{ee} \lambda_{(p,q)} = 0. \end{aligned}$$

Thus $k \in C(A)$.

We could repeat this process for each of the eo, oe, and oo sums given in Eq. (1). Thus for a general $k \in \ker(\varphi)$, we have $k \in C(A)$. Hence $\ker(\varphi) \subset C(A)$. Now $\ker(\varphi) = C(A)$ and we have

$$A/C(A) \cong \mathbb{C}\{\phi, ee, eo, oe, oo\}.$$

Thus $A/C(A)$ is a five dimensional vector space over $\mathbb{C}$. □

Recall that a trace is a linear functional defined on A that is zero on $C(A)$. The space of traces is dual to the quotient $A/C(A)$. Thus Theorem 2 implies that there are five traces on $A = K_t(T^2)$. There is a trace for each $\mathbb{Z}_2$ homology class of T^2, with one more trace for the empty skein. Each trace picks off the coefficients of the basis elements in its corresponding class. We could denote these traces by φ_ϕ, φ_{ee}, φ_{eo}, φ_{oe}, φ_{oo}. The trace φ_ϕ is what Bullock, Frohman, and Kania-Bartoszyńska call the *Yang–Mills measure* in [3] and has been further explored by Frohman and Kania-Bartoszyńska in [5].

5. Further investigation

The moduli space of flat $SU(2)$-connections on the torus is a sphere with four singular points, sometimes called the "pillowcase". The four additional traces on $K_t(T^2)$ correspond to the four singular points on the pillowcase. Though our calculations have been for the torus, it is likely that a similar result will hold for any closed surface F. Namely, that in addition to the Yang–Mills measure there is a trace on $K_t(F)$ corresponding to each singular point on the moduli space of flat $SU(2)$-connections on F.

In [5], Frohman and Kania-Bartoszyńska give a state-sum formula for computing the Yang–Mills measure on $K_t(F)$. It is natural to ask if the traces on $K_t(F)$ corresponding to singular points can be computed similarly.

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