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Bernoulli polynomials and Pascal matrices in the context of Clifford analysis

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ABSTRACT

This paper describes an approach to generalized Bernoulli polynomials in higher dimensions by using Clifford algebras. Due to the fact that the obtained Bernoulli polynomials are special hypercomplex holomorphic (monogenic) functions in the sense of Clifford Analysis, they have properties very similar to those of the classical polynomials. Hypercomplex Pascal and Bernoulli matrices are defined and studied, thereby generalizing results recently obtained by Zhang and Wang (Z. Zhang, J. Wang, Bernoulli matrix and its algebraic properties, Discrete Appl. Math. 154 (11) (2006) 1622–1632).

1. Introduction

In the last decade, a surprisingly amount of papers appeared proposing new generalizations of the classical Bernoulli polynomials $B_n(x)$ to several real and complex variables or treating other topics related to Bernoulli polynomials (cf. [2,4,6,7,13,17]). In general, the starting point for these works is a modification of the classical exponential generating function given by

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^{t}-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<2\pi. \tag{1}$$

For instance, some of the generalized Bernoulli polynomials mentioned in [4,13] are $B_n^{\alpha}(x)$, $B_{n,\alpha}(x)$, $B_{n;h,w}^{\alpha}(x)$, and $B_n^{[m-1]}(x)$, $(m\geq 1)$, which have been obtained by choosing as exponential generating functions

$$\frac{t^{\alpha}\mathrm{e}^{xt}}{(\mathrm{e}^{t}-1)^{\alpha}},\quad \frac{(\mathrm{i}z)^{\alpha}\,\mathrm{e}^{(x-1/2)z}}{2^{2\alpha}\Gamma(\alpha+1)J_{\alpha}(\mathrm{i}z/2)},\quad \frac{(ht)^{\alpha}\,(1+wt)^{x/w}}{\left[(1+wt)^{h/w}-1\right]^{\alpha}},\quad \frac{t^m\mathrm{e}^{xt}}{\mathrm{e}^{t}-\sum\limits_{h=0}^{m-1}(t^h/h!)},$$

(where J_{α} is the Bessel function of the first kind of order α), respectively.

Particularly, if there generalizations to several variables are considered, the approach mainly takes advantage of the use of some modification of the involved exponential function. As an example we mention the approach [4], motivated by Gould-Hopper polynomials (also known as Hermite–Kampé de Fériet polynomials) $H_n^{(j)}(x,y), j\geq 2$, whose generating

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function is e^{xt+yt^j}. Consequently, bidimensional Bernoulli polynomials $B_n^{(j)}(x,y), j \geq 2$ are therefore obtained by means of the generating function $\frac{te^{xt+yt^j}}{e^t-1}$. Other recently obtained generalizations can be found in [14,15].

In this article we are following similar ideas combined with methods of generalized power series representations used in the theory of hypercomplex holomorphic (monogenic) functions, which are generalized complex holomorphic functions in the context of Clifford Analysis. As far as the authors know, this is done for the first time. The paper is organized in the following way. After some preliminaries on the use of Clifford algebras in higher dimensional Euclidean spaces (Section 2.1), we introduce in Section 2.2 generalized Bernoulli polynomials of n hypercomplex variables. Their main properties, analogous to the classical Bernoulli polynomials ones, are proved. The fact of being polynomials in several variables implies also new properties with relevance in applications, which will be studied in more detail in a forthcoming paper. Finally, in Section 3, we focus on some interesting generalizations of Pascal and Bernoulli matrices referred in [5,17]. For this purpose we restrict our study to the 3-dimensional real Euclidean space, which corresponds to the use of two hypercomplex variables.

2. Hypercomplex generalizations of Bernoulli polynomials

2.1. Preliminaries

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \quad k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. This non-commutative product generates the 2^n-dimensional Clifford algebra $Cl_{0,n}$ over $\mathbb{R}$ and the set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$, $1 \leq h_1 \leq \cdots \leq h_n$, $e_\emptyset = e_0 = 1$, forms a basis of $Cl_{0,n}$. The real vector space $\mathbb{R}^{n+1}$ will be embedded in $Cl_{0,n}$ by identifying the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with

$$z = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n \in \mathcal{A} \equiv span_{\mathbb{R}} \{e_0, \ldots, e_n\} \cong \mathbb{R}^{n+1}.$$

As natural generalization of the complex Cauchy–Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$$

is given by the operator

$$D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \cdots + \frac{\partial}{\partial x_n} e_n,$$

and the equation

$$Df = 0$$

defines hypercomplex holomorphic (or monogenic) functions $f = f(z)$ as Clifford algebra valued functions in the kernel of this generalized Cauchy–Riemann operator (cf. [3]). Since the operator D can be applied both from the left and from the right hand side of f, it is usual to refer to left monogenic function and right monogenic function, respectively. For simplicity, from now on we only deal with left monogenic functions. The case of right monogenic functions can be treated completely analogously.

Since $Dz = 1 - n$ it is evident that the function $f(z) = z \in \mathcal{A}$ is only monogenic if $n = 1$, i.e., in the case of $\mathcal{A} = \mathbb{C}$. This implies significant differences between the cases $n = 1$ and $n > 1$. Moreover, powers of z, i.e., $f(z) = z^k$, $k = 2, \ldots$, are not monogenic which means that they cannot be considered appropriate as hypercomplex generalizations of the complex power z^k, $z \in \mathbb{C}$. These facts are the reason for generalized power series of a special structure, which we are going to use in the following subsection.

To overcome the mentioned situation in [8] has been considered another hypercomplex structure for $\mathbb{R}^{n+1}$ based on an isomorphism between $\mathbb{R}^{n+1}$ and

$$\mathcal{H}^n = \left\{\vec{z} : \vec{z} = (z_1, \ldots, z_n), z_k = x_k - x_0 e_k, x_0, x_k \in \mathbb{R}, k = 1, \ldots, n\right\}.$$

Whereas the components of the vector $\vec{z}$, i.e. the hypercomplex variables z_k themselves are monogenic, their ordinary products $z_i z_k$, $i \neq k$, are not monogenic. But a n-ary operation, namely their permutational (symmetric) product resolves the problem (cf. [8]).

Definition 2.1. Let $V_{+,\cdot}$ be a commutative or non-commutative ring, $a_k \in V$ $(k = 1, \ldots, n)$, then the symmetric "$\times$"-product is defined by

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1,\ldots,i_n)} a_{i_1} a_{i_2} \cdots a_{i_n} \tag{2}$$

where the sum runs over **all** permutations of all $(i_1, \ldots, i_n)$.

Additionally, the following convention has been introduced in [8].

Convention

If the factor a_j occurs σ_j-times in (2), we briefly write

$$\underbrace{a_1 \times \cdots \times a_1}_{\sigma_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\sigma_n} = {a_1}^{\sigma_1} \times \cdots \times {a_n}^{\sigma_n} = \vec{a}^{\sigma} \tag{3}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ and set parentheses if the powers are understood in the ordinary way.

Formula (3) simply allows to work with a polynomial formula exactly in the same way as in the case of several commutative variables. It holds (see [9,10])

$$(z_1 + \cdots + z_n)^k = \sum_{|\sigma|=k} \binom{k}{\sigma} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} = \sum_{|\sigma|=k} \binom{k}{\sigma} \vec{z}^{\sigma}, \quad k \in \mathbb{N} \tag{4}$$

with polynomial coefficients defined as usual by $\binom{k}{\sigma} = \frac{k!}{\sigma!}$ where $\sigma! = \sigma_1! \cdots \sigma_n!$.

Moreover, all functions of the form $f(z) = \vec{z}^{\sigma}$, are left and right monogenic and $Cl_{0,n}$- linear independent. Therefore, they can be used as basis for generalized power series. Following [9,10] it has been shown, that the generalized power series of the form

$$P(\vec{z}) = \sum_{k=0}^{\infty} \left(\sum_{|\sigma|=k} \vec{z}^{\sigma} c_{\sigma} \right), c_{\sigma} \in Cl_{0,n}$$

generates in the neighborhood of the origin a monogenic function $f(\vec{z})$ and coincides in the interior of its domain of convergence with the Taylor series of $f(\vec{z})$, i.e., in a neighborhood of the origin we have

$$f(\vec{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{|\sigma|=k} \vec{z}^{\sigma} \binom{k}{\sigma} \frac{\partial^{|\sigma|} f(\vec{0})}{\partial \vec{x}^{\sigma}} \right),$$

where $\vec{x} = (x_1, \ldots, x_n)$.

In [9] has been shown that the partial derivatives of $\vec{z}^{\sigma}$ with respect to x_k are obtained as

$$\frac{\partial \vec{z}^{\sigma}}{\partial x_k} = \sigma_k \vec{z}^{\sigma - \tau_k} \tag{5}$$

where τ_k is the multiindex with 1 at place k and zero otherwise.

It is well known, that for complex holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ the complex derivative $f' = \frac{\mathrm{d}f}{\mathrm{d}z}$ exists and coincides with the complex partial derivative

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - \mathrm{i} \frac{\partial f}{\partial y} \right).$$

The analogous situation is true in the hypercomplex case (cf. [10]). A real differentiable function $f(\vec{z})$ is left (right) hypercomplex derivable in $\Omega \subset \mathcal{H}^n$ if and only if f is left (right) monogenic in $\Omega \subset \mathcal{H}^n$. In the case of its existence, the hypercomplex derivative is given by

$$\frac{1}{2}\overline{D}f \quad \text{resp. } \frac{1}{2}f\overline{D}$$

with the conjugated generalized Cauchy–Riemann operator

$$\overline{D} = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} e_1 - \cdots - \frac{\partial}{\partial x_n} e_n.$$

Furthermore, like in the complex case, where the complex derivative satisfies

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial f}{\partial x}$$

the left (right) hypercomplex derivative of f at $\vec{z}$ is exactly

$$\frac{1}{2}\overline{D}f = \frac{1}{2}f\overline{D} = \frac{\partial f}{\partial x_0}. \tag{6}$$

2.2. Hypercomplex Bernoulli polynomials

In the previous subsection we mentioned already that the ordinary product of two monogenic variables $z_k = x_k - x_o e_k$, $k = 1, 2, \ldots n$ is not again a monogenic function. The same is true, in general, for the ordinary product of two arbitrary monogenic functions. This situation complicates to start with a direct generalization of the generating function (1) and we preferred to use the equivalent form

$$e^{tx} = \sum_{j=0}^{\infty} \frac{1}{j!}(xt)^j = \sum_{j=0}^{\infty} \left(\sum_{r+n=j} \frac{1}{(r+1)!n!} B_n(x) \right) t^j. \tag{7}$$

Thus, following this point of view, let $\vec{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $\vec{z} = (z_1, \ldots, z_n) \in \mathcal{H}^n$, and define a hypercomplex exponential function by an everywhere convergent series of the form

$$\mathbf{Exp}(\vec{t}, \vec{z}) := \exp(t_1 z_1 + \cdots + t_n z_n) = \sum_{k=0}^{\infty} \frac{1}{k!}(t_1 z_1 + \cdots + t_n z_n)^k.$$

Immediately we arrive to

Definition 2.2. The hypercomplex Bernoulli polynomials $B_{j_1,\ldots,j_n}(z_1, \ldots, z_n)$, $j_k \in \mathbb{N}_0, k = 1, \ldots, n$ are defined as the coefficients of a multiple power series ordered with respect to the degree of homogeneity by the following relation:

$$\mathbf{Exp}(\vec{t}, \vec{z}) = \left(\sum_{r=0}^{\infty} \frac{1}{(r+1)!}(t_1 + \cdots + t_n)^r \right) \left(\sum_{|j|=0}^{\infty} \frac{1}{j!} B_{j_1,\ldots,j_n}(z_1, \ldots, z_n) t_1^{j_1} \ldots t_n^{j_n} \right). \tag{8}$$

Due to the one-dimensional case, the second series on the right hand side of (8) is convergent in the n-dimensional parallelepiped

$$M = \{\vec{t} : |t_k| < 2\pi, k = 1, \ldots, n\}.$$

Applying (4), the formula (8) is equivalent to

$$\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} t_1^{\sigma_1} \ldots t_n^{\sigma_n} = \left(\sum_{|s|=0}^{\infty} \frac{1}{(|s|+1)s!} t_1^{s_1} \ldots t_n^{s_n} \right) \left(\sum_{|j|=0}^{\infty} \frac{1}{j!} B_{j_1,\ldots,j_n}(z_1, \ldots, z_n) t_1^{j_1} \ldots t_n^{j_n} \right)$$

or

$$\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} t_1^{\sigma_1} \ldots t_n^{\sigma_n} = \sum_{|\sigma|=0}^{\infty} \left[\sum_{s+j=\sigma} \frac{B_{j_1,\ldots,j_n}(z_1, \ldots, z_n)}{(|s|+1)s!j!} \right] t_1^{\sigma_1} \ldots t_n^{\sigma_n}$$

which generalize (7) to the hypercomplex case. Comparing both sides gives the relationship of hypercomplex Bernoulli polynomials to the generalized powers

$$\sum_{s+j=\sigma} \frac{1}{(|s|+1)s!j!} B_{j_1,\ldots,j_n}(z_1, \ldots, z_n) = \frac{1}{\sigma!} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n}, \tag{9}$$

for $\sigma_k = 0, 1, \ldots, k = 1, \ldots, n$.

Obviously, the set of the hypercomplex Bernoulli polynomials contains n copies of the classical Bernoulli polynomials that are obtained when all the indices j_k, $k = 1, \ldots, n$ in (9) are equal to zero or only one of them is different from zero.

For example, some hypercomplex Bernoulli polynomials given by (9), with $r, s \in \mathbb{N}$ and $r, s \leq n$, are:

$$B_{0,\ldots,0}(z_1, \ldots, z_n) = 1$$

$$B_{0,\ldots,\underbrace{1}_{r},\ldots,0}(z_1, \ldots, z_n) = z_r - \frac{1}{2}$$

$$B_{0,\ldots,\underbrace{1}_{r},\ldots,\underbrace{1}_{s},\ldots,0}(z_1, \ldots, z_n) = z_r \times z_s - \frac{1}{2}(z_r + z_s) + \frac{1}{6}$$

$$B_{0,\ldots,\underbrace{2}_{r},\ldots,0}(z_1, \ldots, z_n) = z_r^2 - z_r + \frac{1}{6}$$

$$B_{0,\ldots,\underbrace{2}_{r},\ldots,\underbrace{1}_{s},\ldots,0}(z_1, \ldots, z_n) = z_r^2 \times z_s - \frac{1}{2} z_r^2 - z_r \times z_s + \frac{1}{3} z_r + \frac{1}{6} z_s$$

$$B_{0,\dots,\underbrace{3}_{r},\dots,0}(z_1,\dots,z_n) = z_r^3 - \frac{3}{2}z_r^2 + \frac{1}{2}z_r$$

$$B_{0,\dots,\underbrace{2}_{r},\dots,\underbrace{2}_{s},\dots,0}(z_1,\dots,z_n) = z_r^2 \times z_s^2 - z_r^2 \times z_s - z_r \times z_s^2 + \frac{2}{3}z_r \times z_s + \frac{1}{6}\left(z_r^2 + z_s^2\right) - \frac{1}{30}$$

$$\vdots$$

The values of $B_n(x)$ for $x = 0$ easily can be recognized as the ordinary *Bernoulli numbers* and simply designated by B_n. Similarly, using the notation

$$B_{\sigma_1,\dots,\sigma_n} := B_{\sigma_1,\dots,\sigma_n}(0,\dots,0),$$

for the values of the generalized Bernoulli polynomials in the origin, we can see that all $B_{\sigma_1,\dots,\sigma_n}$ with the same norm of the multiindex $|\sigma| = k$ coincide and are equal to an ordinary Bernoulli number, for example:

$$\begin{aligned}
B_{\sigma_1,\dots,\sigma_n} &= 1, && |\sigma| = 0\\
B_{\sigma_1,\dots,\sigma_n} &= -\frac{1}{2}, && |\sigma| = 1\\
B_{\sigma_1,\dots,\sigma_n} &= \frac{1}{6}, && |\sigma| = 2\\
B_{\sigma_1,\dots,\sigma_n} &= 0, && |\sigma| = 2k+1, k = 1,\dots\\
B_{\sigma_1,\dots,\sigma_n} &= -\frac{1}{30}, && |\sigma| = 4\\
&\vdots
\end{aligned}$$

In general, this means

$$B_{\sigma_1,\dots,\sigma_n}(0,\dots,0)\,|_{\sigma_1+\dots+\sigma_n=k} = B_k,\ k = 0, 1, \dots. \tag{10}$$

Before concluding this section, we emphasize some of the most interesting relations and properties of hypercomplex Bernoulli polynomials and Bernoulli numbers. Some of them are analogous to the ones in the classical case (cf. [1,12]).

Proposition 2.1.

$$B_{\sigma_1,\dots,\sigma_n}(1,\dots,1) = (-1)^{|\sigma|}B_{\sigma_1,\dots,\sigma_n}. \tag{11}$$

Proof. Making use of the definition of hypercomplex Bernoulli polynomials,

$$F\left(\vec{t},\vec{z}\right) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(z_1,\dots,z_n)\, t_1^{\sigma_1}\dots t_n^{\sigma_n},$$

where

$$F\left(\vec{t},\vec{z}\right) = \frac{(t_1+\dots+t_n)\exp(t_1z_1,\dots,t_nz_n)}{\exp(t_1,\dots,t_n)-1},$$

and taking $(z_1,\dots,z_n) = (0,\dots,0)$ and $(z_1,\dots,z_n) = (1,\dots,1)$ we get,

$$F\left(\vec{t},\vec{0}\right) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n} t_1^{\sigma_1}\dots t_n^{\sigma_n}$$

and

$$F\left(\vec{t},\vec{1}\right) = \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(1,\dots,1)\, t_1^{\sigma_1}\dots t_n^{\sigma_n},$$

respectively.

Moreover $F\left(\vec{t},\vec{1}\right) = F\left(-\vec{t},\vec{0}\right)$, that is,

$$\begin{aligned}
\sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(1,\dots,1)\, t_1^{\sigma_1}\dots t_n^{\sigma_n} &= \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(-t_1)^{\sigma_1}\dots(-t_n)^{\sigma_n}\\
&= \sum_{|\sigma|=0}^{\infty} \frac{1}{\sigma!} B_{\sigma_1,\dots,\sigma_n}(-1)^{|\sigma|}\, t_1^{\sigma_1}\dots t_n^{\sigma_n}.
\end{aligned}$$

Hence,

$$B_{\sigma_1,\ldots,\sigma_n}(1,\ldots,1) = (-1)^{|\sigma|} B_{\sigma_1,\ldots,\sigma_n}. \quad \square$$

The equality (11) generalizes the property $B_n(1) = (-1)^n B_n$, $n \in \mathbb{N}_0$, already known in the classical case.

Proposition 2.2.

$$B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{j_1=0}^{\sigma_1} \cdots \sum_{j_n=0}^{\sigma_n} \binom{\sigma_1}{j_1} \cdots \binom{\sigma_n}{j_n} B_{j_1,\ldots,j_n} z_1^{\sigma_1 - j_1} \times \cdots \times z_n^{\sigma_n - j_n}. \tag{12}$$

Proof. Using the definitions of hypercomplex Bernoulli polynomials and the Bernoulli numbers, we can write

$$\sum_{|\sigma|=0}^{\infty} \left(\sum_{j+k=\sigma} B_{j_1,\ldots,j_n} \frac{z_1^{k_1} \times \cdots \times z_n^{k_n}}{j_1! \ldots j_n! k_1! \ldots k_n!} \right) t_1^{\sigma_1} \ldots t_n^{\sigma_n} = \sum_{|\sigma|=0}^{\infty} \frac{B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)}{\sigma_1! \ldots \sigma_n!} t_1^{\sigma_1} \ldots t_n^{\sigma_n}$$

which yields

$$B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{j_1=0}^{\sigma_1} \cdots \sum_{j_n=0}^{\sigma_n} B_{j_1,\ldots,j_n} \frac{\sigma_1! \ldots \sigma_n! z_1^{\sigma_1 - j_1} \times \cdots \times z_n^{\sigma_n - j_n}}{j_1! \ldots j_n! (\sigma_1 - j_1)! \ldots (\sigma_n - j_n)!}$$

that is,

$$B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{j_1=0}^{\sigma_1} \cdots \sum_{j_n=0}^{\sigma_n} \binom{\sigma_1}{j_1} \cdots \binom{\sigma_n}{j_n} B_{j_1,\ldots,j_n} z_1^{\sigma_1 - j_1} \times \cdots \times z_n^{\sigma_n - j_n}. \quad \square$$

With (12) we found a generalization for another property of the classical Bernoulli polynomials $B_n(x)$:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \; n \in \mathbb{N}_0.$$

Moreover, due to formula (10) the relation (12) is nothing else than the explicit expression of the generalized Bernoulli polynomials with ordinary Bernoulli numbers as coefficients.

Proposition 2.2 still allows to introduce a new type of Bernoulli numbers, where one of the arguments is equal to one and the others are equal to zero, which is a situation different from that one in Proposition 2.1, which describes the symmetry relation between $B_{\sigma_1,\ldots,\sigma_n}(1,\ldots,1)$ and $B_{\sigma_1,\ldots,\sigma_n}$.

Proposition 2.3. *Let us call k-* Bernoulli numbers, $B^k_{\sigma_1,\ldots,\sigma_n}$*, those that are obtained by calculating the hypercomplex Bernoulli polynomials in* $(0,\ldots,\underbrace{1}_{k},\ldots,0)$, $k = 1,\ldots,n$*, i.e.,*

$$B^k_{\sigma_1,\ldots,\sigma_n} = B_{\sigma_1,\ldots,\sigma_n}(0,\ldots,\underbrace{1}_{k},\ldots,0).$$

Then these k- Bernoulli numbers *can be represented as a linear combination of the ordinary Bernoulli numbers,*

$$B^k_{\sigma_1,\ldots,\sigma_n} = \sum_{j_k=0}^{\sigma_k} \binom{\sigma_k}{j_k} B_{\sigma_1,\ldots,j_k,\ldots,\sigma_n}.$$

Proof. The proof follows immediately from (12) by taking $z_k = 1$ e $z_i = 0$, $i = 1,\ldots,n$, $i \neq k$. $\square$

Example.

$$B^1_{2,1} \equiv B_{2,1}(1,0) = \mathbf{1}B_{0,1} + \mathbf{2}B_{1,1} + \mathbf{1}B_{2,1}$$
$$B^1_{3,2} \equiv B_{3,2}(1,0) = \mathbf{1}B_{0,2} + \mathbf{3}B_{1,2} + \mathbf{3}B_{2,2} + \mathbf{1}B_{3,2}$$
$$B^1_{4,3} \equiv B_{4,3}(1,0) = \mathbf{1}B_{0,3} + \mathbf{4}B_{1,3} + \mathbf{6}B_{2,3} + \mathbf{4}B_{3,3} + \mathbf{1}B_{4,3}$$
$$\vdots$$
$$B^2_{1,1} \equiv B_{1,1}(0,1) = \mathbf{1}B_{1,0} + \mathbf{1}B_{1,1}$$
$$B^2_{1,2} \equiv B_{1,2}(0,1) = \mathbf{1}B_{1,0} + \mathbf{2}B_{1,1} + \mathbf{1}B_{1,2}$$
$$B^2_{2,3} \equiv B_{2,3}(0,1) = \mathbf{1}B_{2,0} + \mathbf{3}B_{2,1} + \mathbf{3}B_{2,2} + \mathbf{1}B_{2,3}$$
$$\ddots$$

Proposition 2.4.

$$\frac{\partial}{\partial x_k} B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \begin{cases} \sigma_k B_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n) & ,\sigma_k \neq 0 \\ 0 & ,\sigma_k = 0, \end{cases} \quad k = 1,\ldots,n.$$

Proof. The proof follows directly by partial differentiation with respect to x_k of both sides of (12) together with (5). □

This proposition generalizes for the hypercomplex case the relations $B_n'(x) = nB_{n-1}(x)$, $n \in \mathbb{N}$, used for the differentiation of classical Bernoulli polynomials.

Proposition 2.5.

$$\frac{1}{2}\overline{D}B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = -\sum_{k=1}^{n} \sigma_k B_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n)e_k,$$

where $\frac{1}{2}\overline{D}B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)$ *is the hypercomplex derivative of* $B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)$.

Proof. Considering that the hypercomplex Bernoulli polynomials are monogenic, i.e.,

$$DB_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = 0,$$

we can write

$$\frac{\partial}{\partial x_0} B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k} B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)e_k,$$

that is

$$\frac{1}{2}\overline{D}B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = -\sum_{k=1}^{n} \sigma_k B_{\sigma_1,\ldots,\sigma_k-1,\ldots,\sigma_n}(z_1,\ldots,z_n)e_k. \quad \square$$

Proposition 2.6. *Let*

$$\Delta B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = B_{\sigma_1,\ldots,\sigma_n}(z_1+1,\ldots,z_n+1) - B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)$$

be the (total) difference operator. Then

$$\Delta B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n) = \sum_{k=1}^{n} \sigma_k z_1^{\sigma_1} \times \cdots \times z_k^{\sigma_k-1} \times \cdots \times z_n^{\sigma_n}, \quad \sigma_k \geq 1, k = 1,\ldots,n.$$

Proof. By the definition of hypercomplex Bernoulli polynomials, we have

$$\begin{aligned}
&\sum_{|\sigma|=0}^{\infty} \frac{B_{\sigma_1,\ldots,\sigma_n}(z_1+1,\ldots,z_n+1) - B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)}{\sigma_1!\ldots\sigma_n!} t_1^{\sigma_1}\ldots t_n^{\sigma_n} \\
&\quad = \frac{(t_1+\cdots+t_n)\left[\exp\left(t_1(z_1+1),\ldots,t_n(z_n+1)\right) - \exp\left(t_1z_1,\ldots,t_nz_n\right)\right]}{\exp(t_1,\ldots,t_n) - 1} \\
&\quad = (t_1+\cdots+t_n)\exp(t_1z_1,\ldots,t_nz_n) \\
&\quad = \sum_{k=1}^{n}\left(\sum_{|\sigma|=0}^{\infty} \frac{z_1^{\sigma_1}\times\cdots\times z_n^{\sigma_n}}{\sigma_1!\ldots\sigma_n!} t_1^{\sigma_1}\ldots t_k^{\sigma_k+1}\cdots t_n^{\sigma_n}\right) \\
&\quad = \sum_{k=1}^{n}\left(\sum_{|\sigma|=1,\sigma_k=1}^{\infty} \frac{z_1^{\sigma_1}\times\cdots\times z_k^{\sigma_k-1}\times\cdots\times z_n^{\sigma_n}}{\sigma_1!\cdots(\sigma_k-1)!\cdots\sigma_n!} t_1^{\sigma_1}\cdots t_k^{\sigma_k}\cdots t_n^{\sigma_n}\right).
\end{aligned}$$

Therefore, comparing the coefficients of $t_1^{\sigma_1}\cdots t_n^{\sigma_n}$, we obtain

$$\begin{aligned}
&\sum_{\sigma_1=1,\ldots,\sigma_k=1,\ldots,\sigma_n=1}^{\infty} \frac{B_{\sigma_1,\ldots,\sigma_n}(z_1+1,\ldots,z_n+1) - B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)}{\sigma_1!\ldots\sigma_n!} t_1^{\sigma_1}\ldots t_n^{\sigma_n} \\
&\quad = \sum_{\sigma_1=1,\ldots,\sigma_k=1,\ldots,\sigma_n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{z_1^{\sigma_1}\times\cdots\times z_k^{\sigma_k-1}\times\cdots\times z_n^{\sigma_n}}{\sigma_1!\cdots(\sigma_k-1)!\cdots\sigma_n!}\right) t_1^{\sigma_1}\cdots t_n^{\sigma_n}
\end{aligned}$$

and,

$$B_{\sigma_1,\ldots,\sigma_n}(z_1+1,\ldots,z_n+1)-B_{\sigma_1,\ldots,\sigma_n}(z_1,\ldots,z_n)=\sum_{k=1}^{n}\sigma_k z_1^{\sigma_1}\times\cdots\times z_k^{\sigma_k-1}\times\cdots\times z_n^{\sigma_n}.\quad\square$$

The last result generalizes the well-known property of the classical Bernoulli polynomials,

$$B_n(x+1)-B_n(x)=nx^{n-1},\ n\geq 1.$$

3. Hypercomplex Pascal and Bernoulli matrices

Apart from the fact that a Pascal matrix is one of the oldest in the history of Mathematics, the study of its properties is still recent [7]. However, it has been used in a wide scale in different areas of pure and applied mathematics, for instance, in relation to the resolution of differential and difference equations, in the study of special polynomials such as the Bernstein's or the Bernoulli's, in probability problems, in combinatorics, etc.

In various books and papers are defined the Pascal matrix P (see,for instance [2,5,7,17]), the symmetric Pascal matrix PP^{T} [2], where P^{T} means the transpose of P and the inverse of Pascal matrix P^{-1} [5]. Also, they already appeared some generalizations of the Pascal matrix like that found in [17], which Zhang and Wang called generalized Pascal matrix $P[x]$.

Among the many relations concerning the Pascal matrix we will give emphasis to the one found in [17] and that links this matrix with the Bernoulli polynomials. In that paper the authors defined the polynomial Bernoulli matrix $\mathscr{B}(x)=\left[B_{ij}(x)\right]$ by

$$B_{ij}(x)=\begin{cases}\binom{i}{j}B_{i-j}(x), & i\geq j\\ 0, & \text{otherwise}, \quad i,j=0,\ldots,n\end{cases}$$

and called $\mathscr{B}(0)=\mathscr{B}$ the Bernoulli matrix. Afterwards, they established the following relation between $\mathscr{B}(x)$ and $P[x]$:

$$\mathscr{B}(x)=P[x]\mathscr{B}.$$

The goal of this section is to obtain some generalizations, for the hypercomplex case, of the results above referred. In order to achieve this, we start by defining the block Pascal matrix, the hypercomplex Pascal matrix and we allude to some of its properties. Then, we define the hypercomplex polynomial Bernoulli matrix, the hypercomplex Bernoulli matrix and finally we establish a relation between these matrices and the hypercomplex Pascal one.

Definition 3.1. The *block Pascal matrix* is the $(n+1)\times(n+1)$-block matrix, $\mathcal{P}=[\mathcal{P}_{ij}^{sr}]$, where

$$\mathcal{P}_{ij}^{sr}=\begin{cases}\binom{i}{j}\binom{s}{r}, & i\geq j\wedge s\geq r\\ 0, & \text{otherwise}, \quad i,j,s,r=0,\ldots,n.\end{cases}$$

Each block of this matrix is also a $(n+1)\times(n+1)$ matrix.

Now, we can define the *symmetric block Pascal matrix* $\mathcal{P}\mathcal{P}^{\mathrm{T}}$ through

$$(\mathcal{P}\mathcal{P}^{\mathrm{T}})_{ij}^{sr}=\binom{i+j}{j}\binom{s+r}{r},\ i,j,s,r=0,\ldots,n.$$

Following the idea suggested by [5] about the inverse of the matrix P, we obtain for $\mathcal{P}^{-1}$ the generalization:

Theorem 3.1. *Let the $(n+1)\times(n+1)$-block matrix, $\mathcal{Q}=[\mathcal{Q}_{ij}^{sr}]$, such that*

$$\mathcal{Q}_{ij}^{sr}=\begin{cases}\binom{i}{j}\binom{s}{r}(-1)^{i-j}(-1)^{s-r}, & i\geq j\wedge s\geq r\\ 0, & \textit{otherwise}, i,j,s,r=0,\ldots,n.\end{cases}$$

Then $\mathcal{Q}=\mathcal{P}^{-1}$.

Proof. It is easy to verify that

$$(\mathcal{P}\mathcal{Q})_{ij}^{sr}=\begin{cases}0, & i<j\vee s<r\\ 1, & i=j\wedge s=r.\end{cases}$$

Now, we must show that $(\mathcal{P}\mathcal{Q})_{ij}^{sr} = 0$ for the cases $i > j \wedge s > r$, $i > j \wedge s = r$ and $i = j \wedge s > r$. Supposing that $i > j \wedge s > r$. Then, we can write $i = j + l$ and $s = r + m$ with $l, m > 0$. Hence,

$$\begin{aligned}
(\mathcal{P}\mathcal{Q})_{ij}^{sr} &= \sum_{k_1=0}^{l}\sum_{k_2=0}^{m} \mathcal{P}_{j+l,j+k_1}^{r+m,r+k_2}\mathcal{Q}_{j+k_1,j}^{r+k_2,r} \\
&= \sum_{k_1=0}^{l}\sum_{k_2=0}^{m} \binom{j+l}{j+k_1}\binom{r+m}{r+k_2}\binom{j+k_1}{j}\binom{r+k_2}{r}(-1)^{k_1}(-1)^{k_2} \\
&= \frac{(j+l)!}{j!l!}\frac{(r+m)!}{m!r!}\sum_{k_1=0}^{l}\sum_{k_2=0}^{m}\frac{l!}{(l-k_1)!k_1!}\frac{m!}{(m-k_2)!k_2!}(-1)^{k_1}(-1)^{k_2} \\
&= \binom{j+l}{j}\binom{r+m}{r}\left(\sum_{k_1=0}^{l}\binom{l}{k_1}(-1)^{k_1}\right)\left(\sum_{k_2=0}^{m}\binom{m}{k_2}(-1)^{k_2}\right) \\
&= \binom{i}{j}\binom{s}{r}(1-1)^{l}(1-1)^{m} \\
&= 0.
\end{aligned}$$

The proof is similar for the cases $i > j \wedge s = r$ and $i = j \wedge s > r$. $\square$

Definition 3.2. The *hypercomplex Pascal matrix* is the $(n+1)\times(n+1)$-block matrix, $\mathcal{P}(z_1, z_2) = [\mathcal{P}_{ij}^{sr}(z_1, z_2)]$, such that

$$\mathcal{P}_{ij}^{sr}(z_1, z_2) = \begin{cases} \binom{i}{j}\binom{s}{r} z_1^{i-j}\times z_2^{s-r}, & i \geq j \wedge s \geq r \\ 0, & \text{otherwise}, \qquad i, j, s, r = 0, \dots, n. \end{cases}$$

This matrix extends $P[x]$ to the hypercomplex case.

As it happens to the matrix $P[x]$ (see [5]), also in this case $\mathcal{P}(0, 0) = I$, $\mathcal{P}(1, 1) = \mathcal{P}$ and $\mathcal{P}(-1, -1) = \mathcal{P}^{-1}$.

Definition 3.3. The *hypercomplex polynomial Bernoulli matrix* is the $(n+1)\times(n+1)$-block matrix, $\mathcal{B}(z_1, z_2) = [\mathcal{B}_{ij}^{sr}(z_1, z_2)]$, such that

$$\mathcal{B}_{ij}^{sr}(z_1, z_2) = \begin{cases} \binom{i}{j}\binom{s}{r} B_{i-j,s-r}(z_1, z_2), & i \geq j \wedge s \geq r \\ 0, & \text{otherwise}, \qquad i, j, s, r = 0, \dots, n \end{cases}$$

where $B_{i-j,s-r}(z_1, z_2)$ are hypercomplex Bernoulli polynomials.

The matrix $\mathcal{B} = \mathcal{B}(0, 0)$ will be called *hypercomplex Bernoulli matrix*.

In a similar way as presented in [17] to the classical Bernoulli matrix and to the generalized Pascal matrix, we can also establish a connection between the hypercomplex matrices that we have just defined.

Theorem 3.2. *Let z_1, z_2 elements of $\mathcal{H}^n$.*
Then

$$\mathcal{B}(z_1, z_2) = \mathcal{P}(z_1, z_2)\mathcal{B}.$$

Proof. It is clear that

$$\mathcal{B}_{ij}^{sr}(z_1, z_2) = (\mathcal{P}(z_1, z_2)\mathcal{B})_{ij}^{sr} = 0,$$

if $i < j \vee s < r$.

Now, supposing $i \geq j \wedge s \geq r$ and writing $i = j + l$ and $s = r + m$, $l, m \geq 0$,

$$\begin{aligned}
(\mathcal{P}(z_1, z_2)\mathcal{B})_{ij}^{sr} &= \sum_{k_1=0}^{l}\sum_{k_2=0}^{m}\mathcal{P}_{j+l,j+k_1}^{r+m,r+k_2}(z_1, z_2)\mathcal{B}_{j+k_1,j}^{r+k_2,r} \\
&= \binom{j+l}{j}\binom{r+m}{r}\sum_{k_1=0}^{l}\sum_{k_2=0}^{m}\binom{l}{k_1}\binom{m}{k_2} z_1^{l-k_1}\times z_2^{m-k_2}\mathcal{B}_{k_1,k_2} \\
&= \binom{i}{j}\binom{s}{r}\sum_{k_1=0}^{i-j}\sum_{k_2=0}^{s-r}\binom{i-j}{k_1}\binom{s-r}{k_2}\mathcal{B}_{k_1,k_2} z_1^{i-j-k_1}\times z_2^{s-r-k_2}.
\end{aligned}$$

Applying (12) leads to

$$\begin{aligned}(\mathcal{P}(z_1,z_2)\mathcal{B})_{ij}^{sr} &= \binom{i}{j}\binom{s}{r}\mathcal{B}_{i-j,s-r}(z_1,z_2)\\ &= \mathcal{B}_{ij}^{sr}(z_1,z_2). \quad \square\end{aligned}$$

Taking $i, j, r, s = 0, 1, 2$, the matrices $\mathcal{P}(z_1, z_2)$ and $\mathcal{B}$ are

$$\mathcal{P}(z_1,z_2) = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ z_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ z_1^2 & 2z_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline z_2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ z_1\times z_2 & z_2 & 0 & z_1 & 1 & 0 & 0 & 0 & 0\\ z_1^2\times z_2 & 2z_1\times z_2 & z_2 & z_1^2 & 2z_1 & 1 & 0 & 0 & 0\\ \hline z_2^2 & 0 & 0 & 2z_2 & 0 & 0 & 1 & 0 & 0\\ z_1\times z_2^2 & z_2^2 & 0 & 2z_1\times z_2 & 2z_2 & 0 & z_1 & 1 & 0\\ z_1^2\times z_2^2 & 2z_1\times z_2^2 & z_2^2 & 2z_1^2\times z_2 & 4z_1\times z_2 & 2z_2 & z_1^2 & 2z_1 & 1 \end{array}\right],$$

and

$$\mathcal{B} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1/6 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline -1/2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 1/6 & -1/2 & 0 & -1/2 & 1 & 0 & 0 & 0 & 0\\ 0 & 1/3 & -1/2 & 1/6 & -1 & 1 & 0 & 0 & 0\\ \hline 1/6 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1/6 & 0 & 1/3 & -1 & 0 & -1/2 & 1 & 0\\ -1/30 & 0 & 1/6 & 0 & 2/3 & -1 & 1/6 & -1 & 1 \end{array}\right],$$

respectively, which illustrates the last result.

In this paper we have considered generalized Bernoulli polynomials in the context of Clifford Analysis. Using a similar approach we discussed in [11] corresponding generalized Euler polynomials. Following the ideas expressed in [16] for the classical Bernoulli and Euler polynomials, we also achieved relationships between the two types of generalized polynomials as well.

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