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LINEAR MAPPINGS WHICH PRESERVE ACYCLICITY PROPERTIES OF GRAPHS AND DIGRAPHS AND APPLICATIONS TO MATRICES

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We characterize linear mappings which map the set of all graphs (digraphs) with n vertices which contain no circuit (directed circuit) of length greater than or equal to k into or onto itself. We apply these results to characterize linear transformations on $n \times n$ matrices which preserve the above properties of the graph or the diagraph of the matrix.

1. Introduction

The issue of characterizing the linear transformations which map certain classes of square matrices into or onto themselves has been the theme of several papers recently written, e.g. [2–4 and 6]. The "into" problem is, in general, harder than the "onto" one, and it has been solved only under some additional hypothesis, namely, nonsingularity of the transformation or a somewhat weaker condition.

Independently, there is a growing interest in learning the properties of acyclic matrices, i.e., square matrices whose (nondirected) graph contains no cycle except maybe for loops. These matrices, which are a natural generalization of tridiagonal matrices, are studied for example in [5, 7, 1] and the references there.

Motivated by these two research directions, we investigate here the linear transformations which map the acyclic matrices into or onto themselves. In fact, we consider not only acyclic matrices but the general class $\mathcal{M}\mathcal{G}_n^k[\mathcal{M}\mathcal{D}_n^k]$ of all $n \times n$ matrices whose graph (digraph) contains no circuit (directed circuit) of length greater than or equal to k, $k \leq n$. Clearly, the set of all $n \times n$ acyclic matrices is the class $\mathcal{M}\mathcal{G}_n^3$. The flavor of the discussion in this paper is different from that in [2–4 and 6] since, not surprisingly, the "into" problem here turns to be pure graph theoretic. In view of Lemma 6.3 we consider the equivalent problem of characterizing linear mappings on graphs (digraphs), as are defined in the next section, which map the set $\mathcal{G}_n^k[\mathcal{D}_n^k]$ of all graphs (digraphs) with n vertices which contain no circuit (directed circuit) of length greater than or equal to k into itself. Our results are proved under the assumption that the range of the mapping is wide enough to contain every possible edge (arc), except possibly for loops. This assumption is shown, by means of examples, to be necessary.

We now describe our main results in more detail.

Most of the notations and the definitions are given in Section 2. In Section 3 we characterize the linear mappings L for which

$$L(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^k. \tag{1.1}$$

We show in Theorem 3.19 that for $n \geqslant k > 1$ (1.1) holds if and only if L results in just renaming the vertices and possible addition or elimination of loops. The case $n = k = 4$ is shown to be exceptional and is treated in Theorem 3.20. The section is concluded with the discussion of the case $k = 1$. We show in Theorem 3.25 that in this case L results in renaming the vertices and arbitrary mappings of the loops.

Section 4 is devoted to mappings L satisfying

$$L(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^k, \tag{1.2}$$

where $n > k \geqslant 3$. We prove in Theorem 4.18 that L satifises (1.2) if and only if L is a composition of renaming the vertice and/or adding or eliminating loops and/or the transformation which maps any arc (i, j) onto (j, i). This result does not hold for $k = 1, 2$.

The case $n = k$ is discussed in Section 5. As shown in Theorem 5.30, there exists an additional type of transformation for which (1.2) is satisfied. All the theorems of Sections 3, 4, 5 include also the characterization of the mappings L for which equality in (1.1) or (1.2) holds.

The matrix problem is solved in Section 6. The results of the previous sections are translated into terms of matrices to obtain the first part (the "into" case) of Theorems 6.19, 6.21, 6.23, 6.30 and 6.34. The possible types of transformations which play roles in these theorems are: permutation similarity, transposition, Hadamard product with a certain matrix, etc. The second part of the above theorems corresponds to the "onto" problem for which linear algebraic arguments are needed. It is shown that "into" transformations are also "onto" transformations if and only if they are nonsingular. The only exception is the case $\mathcal{M}\mathcal{G}_n^1$ where we need the weaker property of nonsingularity on the subspace of all matrices with zero diagonal entries.

Unlike in the papers mentioned above where the field is assumed to be $\mathbb{R}$ or $\mathbb{C}$, the results of Section 6 hold for any field whose characteristic is either zero or is greater than $n^2 - n$.

2. Notations and definitions

Notation 2.1. We denote

$\langle n \rangle$ = The set $\{1, 2, \ldots, n\}$, where n is a positive integer;

$|\alpha|$ = The cardinality of the set α.

Notation 2.2. For a (nondirected) graph G we denote:

$V(G)$ = The set of vertices of G;
$E(G)$ = The set of edges of G;
$[i, j]$ = An edge between i and j, $i, j \in V(G)$; Observe that $[i, j] = [j, i]$;
$E'(G)$ = The set $\{[i, j] \in E(G): i \neq j\}$, that is the set $E(G)$ without loops;
$|G| = |E'(G)|$.

Definition 2.3. Let G be a graph. A sequence of edges in G which leads from i to j, $[i, p_1], [p_1, p_2], \ldots, [p_{m-1}, p_m], [p_m, j]$, is called a *path* in G between i and j and is denoted by $[i, p_1, p_2, \ldots, p_m, j]$. A path $[i_1, \ldots, i_l]$ in G is said to be a *closed path* if $i_l = i_1$. A closed path $[i_1, \ldots, i_k, i_1]$ is said to be a *circuit* if $i_1, \ldots, i_k$ are distinct. A circuit is said to be of length k, or a *k-circuit*, if it consists of k edges.

Definition 2.4. Two edges in a graph G are said to be *adjacent* if they have exactly one common vertex. Two edges are said to be *separated* if they have no common vertex.

Notation 2.5. Let n and k be positive integers and let $i, j \in \langle n \rangle$. We denote:
$\mathscr{G}_n$ = The set of all graphs with n vertices;
$\mathscr{G}_n^k$ = The subset of $\mathscr{G}_n$ which consists of all graphs with no circuit of length greater than or equal to k;
G_{ij} = The graph in $\mathscr{G}_n$ whose set of edges consists of $[i, j]$;
G_0 = The graph in $\mathscr{G}_n$ with empty set of edges.

Notation 2.6. Let $G_1, G_2 \in \mathscr{G}_n$. We denote:
$G_1 \cup G_2$ = The graph in $\mathscr{G}_n$ whose set of edges is $E(G_1) \cup E(G_2)$;
$G_1 \cap G_2$ = The graph in $\mathscr{G}_n$ whose set of edges is $E(G_1) \cap E(G_2)$.

Definition 2.7. A transformation $L: \mathscr{G}_n \to \mathscr{G}_n$ is said to be a *linear mapping* if

$$L(G_1 \cup G_2) = L(G_1) \cup L(G_2), \quad \forall G_1, G_2 \in \mathscr{G}_n.$$

Observe that in order to define a linear mapping L it is enought to define $L(G_{ij})$ for all $i, j \in \langle n \rangle$.

Definition 2.8. A transformation $L: \mathscr{G}_n \to \mathscr{G}_n$ is said to be a *graph covering* mapping if

$$L\Bigg(\bigcup_{\substack{i,j=1\\ i\leq j}}^{n} G_{ij}\Bigg) \supseteq \bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij},$$

that is every G_{ij}, $i, j \in \langle n \rangle$, $i \neq j$ is contained in the image of some graph under L.

Definition 2.9. A linear mapping $L: \mathscr{G}_n \to \mathscr{G}_n$ is said to be a *vertex permutation* if there exists a permutation σ on $\langle n\rangle$ such that

$$L(G_{ij}) = G_{\sigma(i),\sigma(j)}, \quad \forall i, j \in \langle n\rangle.$$

Observe that performing a vertex permutation on a graph results in just renaming the vertices.

For directed graphs (or digraphs) we similarly have

Notation 2.10. Let D be a digraph. We denote:

$V(D)$ = The set of vertices of D;

$E(D)$ = The set of arcs of D;

(i, j) = An arc from i to j, $i, j \in V(D)$. Observe that $(i, j) = (j, i)$ if and only if $i = j$;

$E'(D)$ = The set $\{(i, j) \in E(D) : i \neq j\}$, that is the set $E(D)$ without loops;

$|D| = |E'(D)|$.

Definition 2.11. Let D be a digraph. A sequence of arcs in D from i to j, $(i, p), (p_1, p_2), \ldots, (p_{m-1}, p_m), (p_m, j)$, is called a *directed path* in D from i to j and is denoted by $(i, p_1, p_2, \ldots, p_m, j)$. A directed path $(i_1, \ldots, i_l)$ in D is said to be a *closed directed path* if $i_l = i_1$. A closed directed path $(i_1, \ldots, i_k, i_1)$ is said to be a *directed circuit* if $i_1, \ldots, i_k$ are distinct. A directed circuit is said to be of length k, or a *directed k-circuit*, if it consists of k arcs.

Definition 2.12. Two arcs (i, j) and (s, t) in a digraph D are said to be *adjacent* if either $j = s$, $i \neq t$ or $i = t$, $j \neq s$, that is these arcs have one common vertex and a same direction. The arcs (i, j) and (s, t) are said to be *separated* if $\{i, j\} \cap \{s, t\} = \emptyset$.

Notation 2.13. Let n and k be positive integers and let $i, j \in \langle n\rangle$. We denote:

$\mathcal{D}_n$ = The set of all digraphs with n vertices;

$\mathcal{D}_n^k$ = the subset of $\mathcal{D}_n$ which consists of all digraphs with no directed circuit of length greater than or equal to k;

D_{ij} = The digraph in $\mathcal{D}_n$ whose set of arcs consists of (i, j);

D_0 = The digraph in $\mathcal{D}_n$ with empty set of arcs.

Notation 2.14. Let $D_1, D_2 \in \mathcal{D}_n$. We denote:

$D_1 \cup D_2$ = The digraph in $\mathcal{D}_n$ whose set of arcs is $E(D_1) \cup E(D_2)$;

$D_1 \cap D_2$ = The digraph in $\mathcal{D}_n$ whose set of arcs is $E(D_1) \cap E(D_2)$.

Definition 2.15. A transformation $L : \mathcal{D}_n \to \mathcal{D}_n$ is said to be a *linear mapping* if

$$L(D_1 \cup D_2) = L(D_1) \cup L(D_2), \quad \forall D_1, D_2 \in \mathcal{D}_n.$$

Observe that in order to define a linear mapping L it is enough to define $L(D_{ij})$ for all $i, j \in \langle n\rangle$.

Definition 2.16. A transformation $L:\mathcal{D}_n \to \mathcal{D}_n$ is said to be a *digraph covering* mapping if

$$L\left(\bigcup_{i,j=1}^{n} D_{ij}\right) \supseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} D_{ij},$$

that is every D_{ij}, $i, j \in \langle n \rangle$, $i \neq j$ is contained in the image of some digraph under L.

Definition 2.17. A linear mapping $L:\mathcal{D}_n \to \mathcal{D}_n$ is said to be a *vertex permutation* if there exists a permutation σ on $\langle n \rangle$ such that

$$L(D_{ij}) = D_{\sigma(i),\sigma(i)}, \quad \forall i, j \in \langle n \rangle.$$

Observe that performing a vertex permutation on a digraph results in just renaming the vertices.

The following notations and definitions involve matrices.

Notation 2.18. Let F be a field, let n be a positive integer and let $i, j \in \langle n \rangle$. We denote:

F^{nn} = The set of all $n \times n$ matrices over F;

E_{ij} = The matrix in F^{nn} all of whose entries are zero except for the one in the ith row and jth column which is 1.

Definition 2.19. Let $A \in F^{nn}$. We define the graph of A $G(A)$ and the digraph of A $D(A)$ by

$V(G(A)) = V(D(A)) = \langle n \rangle$;

$E(G(A)) = \{[i, j],\ i, j \in \langle n \rangle : a_{ij} \neq 0 \text{ or } a_{ji} \neq 0\}$;

$E(D(A)) = \{(i, j),\ i, j \in \langle n \rangle : a_{ij} \neq 0\}$.

Notation 2.20. Let n and k be positive integers. We denote:

$\mathcal{M}\mathcal{G}_n^k = \{A \in F^{nn} : G(A) \in \mathcal{G}_n^k\}$;

$\mathcal{M}\mathcal{D}_n^k = \{A \in F^{nn} : D(A) \in \mathcal{D}_n^k\}$.

Definition 2.21. Let L be a linear transformation $L:F^{nn} \to F^{nn}$. We define a linear mapping on $\mathcal{G}_n$ as well as a linear mapping on $\mathcal{D}_n$ which are associated with L. The linear mapping $L_g:\mathcal{G}_n \to \mathcal{G}_n$ is defined by

$$L_g(G_{ij}) = G(L(E_{ij})) \cup G(L(E_{ji})), \quad i, j \in \langle n \rangle.$$

The linear mapping $L_d:\mathcal{D}_n \to \mathcal{D}_n$ is defined by

$$L_d(D_{ij}) = D(L(E_{ij})), \quad i, j \in \langle n \rangle.$$

Observe that since $G(A + B) \subseteq G(A) \cup G(B)$ where $A, B \in F^{nn}$, it follows from Definitions 2.7 and 2.21 that

$$G(L(A)) \subseteq L_g(G(A)), \quad \forall A \in F^{nn}. \tag{2.22}$$

Similarly, it follows from Definitions 2.15 and 2.21 that

$$D(L(A)) \subseteq L_d(D(A)), \quad \forall A \in F^{nn}. \tag{2.23}$$

Notation 2.24. Let $A, B \in F^{nn}$. We denote by $A \circ B$ the Hadamard product of A and B. That is, the matrix $C = A \circ B$ is defined by

$$c_{ij} = a_{ij}b_{ij}, \quad \forall i, j \in \langle n \rangle.$$

3. Nondirected graphs

In this section we discuss graph covering linear mappings $L: \mathcal{G}_n \to \mathcal{G}_n$. In all our Propositions we assume that k is a positive integer less than or equal to n and that

$$L(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^k. \tag{3.1}$$

We shall show (Example 3.24) that our results are not valid when noncovering mappings are considered. We will also characterize those L for which $L(\mathcal{G}_n^k) = \mathcal{G}_n^k$.

Proposition 3.2. *If $k > 1$, then $E'(L(G_{ii})) = \emptyset$, $1 \leq i \leq n$.*

Proof. Assume that for some $i \in \langle n \rangle$ we have $|L(G_{ii})| \geq 1$, and let $[u, v] \in E'(L(G_{ii}))$. If $k \geq 3$ then we can find $t_1, \ldots, t_{k-2} \in \langle n \rangle$ such that $\alpha = [u, v, t_1, \ldots, t_{k-2}, u]$ is a k-circuit. Since L is a covering mapping there exist $i_1, i_2, \ldots, i_{k-1}, j_1, j_2, \ldots, j_{k-1} \in \langle n \rangle$ such that

$$[v, t_1] \in E(L(G_{i_1j_1})), \qquad [t_{m-1}, t_m] \in E(L(G_{i_mj_m})), \quad m = 2, \ldots, k-2,$$

and

$$[t_{k-2}, u] \in E(L(G_{i_{k-1}j_{k-1}})).$$

The graph $G = G_{ii} \cup (\bigcup_{m=1}^{k-1} G_{i_mj_m})$ is in $\mathcal{G}_n^k$ since $|G| \leq k-1$, but $L(G)$ contains the k-circuit α, in contradiction to (3.1). Hence $|L(G_{ii})| = 0$, $1 \leq i \leq n$. The case $k = 2$ is treated as the case $k = 3$ since $\mathcal{G}_n^2 = \mathcal{G}_n^3$. □

Proposition 3.2 does not hold in general for noncovering mappings as demonstrated by

Example 3.3. Let L be defined by

$$L(G) = G_{12}, \quad \forall G \in \mathcal{G}_n. \tag{3.4}$$

Clearly, (3.4) yields that $L(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^1$ for all $k \in \langle n \rangle$, while $|L(G_{ii})| = 1$, $1 \leq i \leq n$.

Proposition 3.5. *If*

$$L\Big(\bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij}\Big) \supseteq \bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij}, \tag{3.6}$$

then $|L(G_{ij})|=1$ *and* $E'(L(G_{ij}))\cap E'(L(G_{lw}))=\emptyset$ *whenever* $i\neq j$, $l\neq w$ *and* $\{i,j\}\neq\{l,w\}$.

Proof. Assume first that for some $i,j\in\langle n\rangle$, $i\neq j$ we have $|L(G_{ij})|>1$. So let $[u,v],[s,t]\in E(L(G_{ij}))$ where $u\neq v$, $s\neq t$ and $\{u,v\}\neq\{s,t\}$. If $k>3$ or if $k=3$ and $[u,v]$ and $[s,t]$ are adjacent then we can complete $[u,v]$ and $[s,t]$ to a k-circuit by adjoining $k-2$ edges $a_1,\ldots,a_{k-2}$. By (3.6) we can find $i_m,j_m\in\langle n\rangle$, $i_m\neq j_m$, $m=1,\ldots,k-2$ such that $a_m\in E(L(G_{i_mj_m}))$, $m=1,\ldots,k-2$. The graph $G=G_{ij}\cup(\bigcup_{m=1}^{k-2}G_{i_mj_m})$ is in $\mathscr{G}_n^k$ since $|G|\leq k-1$, but $L(G)$ contains a k-circuit, in contradiction to (3.1). If $k=3$ and $[u,v]$ and $[s,t]$ are separated then let

$$U=\{[u,s],[u,t],[v,s],[v,t]\}.$$

If for some $p,q\in\langle n\rangle$, $p\neq q$, we have $|E(L(G_{pq}))\cap U|\geq 2$, then $G=G_{ij}\cup G_{pq}$ contains no circuit at all while $L(G)$ contains a 3-circuit, in contradiction to (3.1). Hence we may assume that

$$|E(L(G_{pq}))\cap U|\leq 1,\quad \forall p,q\in\langle n\rangle,\quad p\neq q. \tag{3.7}$$

By (3.6) let $i_1,j_1\in\langle n\rangle$, $i_1\neq j_1$ be such that $[u,s]\in E(L(G_{i_1j_1}))$. If

$$\{i,j\}\cap\{i_1,j_1\}=\emptyset, \tag{3.8}$$

then let $i_2,j_2\in\langle n\rangle$, $i_2\neq j_2$ be such that $[u,t]\in E(L(G_{i_2j_2}))$. The graph $G=G_{ij}\cup G_{i_1j_1}\cup G_{i_2j_2}$ contains no circuit by (3.8) while $L(G)$ contains the 3-circuit $[u,s,t,u]$, in contradiction to (3.1). So, assume that

$$\{i,j\}\cap\{i_1,j_1\}\neq\emptyset. \tag{3.9}$$

If $\{i,j\}=\{i_1,j_1\}$, then $L(G_{ij})$ contains the two adjacent edges $[u,v]$ and $[u,s]$ and such a case has already been taken care of. If $\{i,j\}\neq\{i_1,j_1\}$, then by (3.9) there exists exactly one edge $[i_3,j_3]$ such that $[i,j]$, $[i_1,j_1]$ and $[i_3,j_3]$ form a 3-circuit. By (3.6) and (3.7) we can find $i_2,j_2\in\langle n\rangle$, $i_2\neq j_2$ such that $\{i_2,j_2\}\neq\{i_3,j_3\}$ and such that

$$E(L(G_{i_2j_2}))\cap(U\backslash\{[u,s]\})\neq\emptyset. \tag{3.10}$$

The graph $G=G_{ij}\cup G_{i_1j_1}\cup G_{i_2j_2}$ contains no circuit while, by (3.10), $L(G)$ contains a 3-circuit, in contradiction to (3.1).

The discussion of the case $k=1$ is essentially the same as of the case $k=3$, observing that no loops are involved along the proof.

As a conclusion, our assumption that $|L(G_{ij})|>1$ turns to be false, so, $|L(G_{ij})|\leq 1$ for all $i,j\in\langle n\rangle$, $i\neq j$. By (3.6) it now follows that $|L(G_{ij})|=1$ and $E'(L(G_{ij}))\cap E'(L(G_{lw}))=\emptyset$ whenever $i\neq j$, $l\neq w$ and $\{i,j\}\neq\{l,w\}$. $\square$

We remark that in the case $k>1$, since L is a covering mapping it follows from Proposition 3.2 that (3.6) holds. Hence, condition (3.6) is needed to be stated explicitly only for the case $k=1$. This remark holds also for the following propositions.

Proposition 3.5 does not hold in general for concovering mappings as shown in the following example.

Example 3.10. Consider the case $n=4$, $k=3$ and let L be the linear mapping defined by $L(G_{12})=G_{12}\cup G_{34}$, $L(G_{13})=G_{13}$, $L(G_{23})=G_{23}$, $L(G_{34})=G_{34}$ and $L(G_{ij})=G_{11}$ for $[i,j]\notin\{[1,2],[1,3],[2,3],[3,4]\}$. Let $G\in\mathcal{G}_4^3$. The only possible 3-circuit in $L(G)$ is $[1,2,3,1]$ which occurs only if G contains the same circuit. Hence $L(\mathcal{G}_4^3)\subseteq\mathcal{G}_4^3$, but $|L(G_{12})|=2$.

Lemma 3.11. *Let $k>1$, let L satisfy* (3.6) *and let $G\in\mathcal{G}_n$. If G contains a k-circuit then $L(G)$ contains a k-circuit.*

Proof. Let S be the set of all graphs in $\mathcal{G}_n$ without loops. By Proposition 3.5 we may define a one-to-one linear mapping $\hat{L}:S\to S$ by

$$\hat{L}(G_{ij})=G_{st},\quad \text{where } [i,j]\in E(L(G_{st})),\quad i\neq j,\quad s\neq t.$$

Observe that

$$|\hat{L}(G)|=|G|,\quad \forall G\in S, \tag{3.12}$$

and

$$L(\hat{L}(G))\supseteq G,\quad \forall G\in S. \tag{3.13}$$

Let $G\in\mathcal{G}_n$ contain a k-circuit. To show that $L(G)$ contains a k-circuit it is enough to assume that G consists of one k-circuit. Since by (3.13) $L(\hat{L}(G))\notin\mathcal{G}_n^k$ it follows that $\hat{L}(G)$ consists of one k-circuit. Since the number of graphs in S which consist of one k-circuit (as well as the cardinality of S) is finite, and since $\hat{L}$ is a one-to-one mapping it follows that $G\in S$ consists of one k-circuit if and only if $\hat{L}(G)$ does. Hence, since G consists of one k-circuit, also $\hat{L}^{-1}(G)$ consists of one k-circuit and by (3.13) $L(G)=L(\hat{L}(\hat{L}^{-1}(G)))\supseteq\hat{L}^{-1}(G)$, which yields that $L(G)$ contains a k-circuit. □

Proposition 3.14. *Let $n>3$, let L satisfy* (3.6) *and let $[i,j]$ and $[i,l]$ be two adjacent edges. Then $E'(L(G_{ij}\cup G_{il}))$ consists of two adjacent edges.*

Proof. By Proposition 3.5 $E(L(G_{ij})\cup L(G_{il}))$ consists of two different edges, say $[s,t]$ and $[u,v]$. We assume that these edges are separated and we shall show a contradiction to (3.1).

Using elementary combinatorial calculations one can obtain that the number of all possible k-circuits through two fixed adjacent edges, provided that $k\geqslant 3$, is $\binom{n-3}{k-3}(k-3)!=(n-3)!/(n-k)!$, and that the number of all possible k-circuits through two fixed separated edges is $\binom{n-4}{k-4}(k-3)!2 = 2(k-3)(n-4)!/(n-k)!$ when $k>3$ and 0 when $k=3$.

We now distinguish between 4 cases:

Case 1. $k > \frac{1}{2}(n+3)$

In this case $k>3$ and the number of k-circuits through $[s, t]$ and $[u, v]$ is greater than the number of k-circuits through $[i, j]$ and $[i, l]$. Hence, in view of Proposition 3.5 we can find $i_1, \ldots, i_{k-2}, j_1, \ldots, j_{k-2} \in \langle n \rangle$ such that the graph $G = G_{ij} \cup G_{il} \cup (\bigcup_{m=1}^{k-2} G_{i_m j_m})$ is in $\mathcal{G}_n^k$ but $L(G)$ contains a k-circuit.

Case 2. $k = \frac{1}{2}(n+3)$

Since $n>3$ we have $3<k<n$ and hence we can consider $(k+1)$-circuits. The number of such circuits through $[s, t]$ and $[u, v]$ is, in this case, greater than the number of those through $[i, j]$ and $[i, l]$. As in case 1 we can find $i_1, \ldots, i_{k-1}, j_1, \ldots, j_{k-1} \in \langle n \rangle$ such that the graph $G = G_{ij} \cup G_{il} \cup (\bigcup_{m=1}^{k-1} G_{i_m j_m})$ is in $\mathcal{G}_n^{k+1}$ while $L(G)$ contains a $(k+1)$-circuit. Since $|L(G)| = k+1$ it follows that $L(G)$ contains no k-circuit. Since $L(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^k$ it follows that G must contain a k-circuit and so does $L(G)$ by Lemma 3.11, which is a contradiction.

Case 3. $3 < k < \frac{1}{2}(n+3)$

In this case the number of k-circuits through $[i, j]$ and $[i, l]$ is greater than the number of those through $[s, t]$ and $[u, v]$. Considering all graphs which consist of one k-circuit through $[i, j]$ and $[i, l]$ we obtain a contradiction to Lemma 3.11.

Case 4. $k \leqslant 3$

Using counting arguments, it follows from Proposition 3.5 that we can find a graph G such that $E(G)$ consists of two separated edges, say $[i_1, j_1]$ and $[i_2, j_2]$, while $E'(L(G))$ consists of two adjacent edges, say $[s, t_1]$ and $[s, t_2]$. By (3.6) let $i_3, j_3 \in \langle n \rangle$, $i_3 \neq j_3$ be such that $[t_1, t_2] \in E(L(G_{i_3 j_3}))$. The graph $G = \bigcup_{m=1}^{3} G_{i_m j_m}$ contains no circuit while $L(G)$ contains the 3-circuit $[s, t_1, t_2, s]$, in contradiction to (3.1).

The contradiction yields that our assumption that two adjacent edges are carried by L onto two separated edges is false and our proposition follows. □

As a consequence of Proposition 3.14 we obtain

Proposition 3.15. *Let* $n>4$ *or* $n=4$ *and* $k \leqslant 3$ *and let* L *satisfy* (3.6). *Then for every* $i \in \langle n \rangle$ *there exists* $i' \in \langle n \rangle$ *such that*

$$E'\Big(L\Big(\bigcup_{\substack{j=1\\ j\neq i}}^{n} G_{ij}\Big)\Big) = \bigcup_{\substack{j=1\\ j\neq i'}}^{n} \{[i', j]\}. \tag{3.16}$$

Proof. Without loss of generality assume that $i=1$. By Proposition 3.14 there exist $i', p, q \in \langle n \rangle$ such that $E'(L(G_{12})) = \{[i', p]\}$ and $E'(L(G_{13})) = \{[i', q]\}$. Assume that for some $j \in \langle n \rangle$, $j \notin \langle 3 \rangle$, we have $E'(L(G_{1j})) = \{[s, t]\}$ where

$i' \neq \{s, t\}$. Then by Proposition 3.1 we necessarily have

$$\{s, t\} = \{p, q\}. \tag{3.17}$$

We now distinguish between two cases.

Case 1. $n > 4$

In this case we can find $l \in \langle n \rangle$, $l \notin \{1, 2, 3, j\}$. Let $E'(L(G_{1l})) = \{[u, v]\}$. If $i' \notin \{u, v\}$ then by Proposition 3.14 $\{u, v\} = \{s, t\}$ and together with (3.17) we have a contradiction to Proposition 3.5. If $i' \in \{u, v\}$, then by Proposition 3.5 $\{u, v\} \cap \{p, q\} = \emptyset$ and we have two adjacent edges $[i, j]$ and $[i, l]$ whose images under L are two separated edges $[s, t]$ and $[u, v]$, in contradiction to Proposition 3.14.

Case 2. $n = 4$ and $k \leqslant 3$

Observe that the graph $G = G_{12} \cup G_{13} \cup G_{1j}$ contains no circuit while $L(G)$ contains the 3-circuit $[i', p, q, i']$, in contradiction to (3.1).

Therefore, our assumption that $i' \notin \{s, t\}$ is false and (3.16) follows. □

Proposition 3.15 does not hold in the case $n = k = 4$ as demonstrated by

Example 3.18. Let L be defined by $L(G_{11}) = G_{11}$, $L(G_{22}) = G_{22}$, $L(G_{33}) = G_{33}$, $L(G_{12}) = G_{12}$, $L(G_{13}) = G_{13}$, $L(G_{14}) = G_{23}$, $L(G_{23}) = G_{14}$, $L(G_{24}) = G_{24}$, $L(G_{34}) = G_{34}$. It is easy to verify that $L(G)$ contains any 4-circuit if and only if G contains the same circuit. Thus $L(\mathscr{G}_4^4) \subseteq \mathscr{G}_4^4$, but $E'(L(G_{12} \cup G_{13} \cup G_{14})) = \{[1, 2], [1, 3], [2, 3]\}$.

We can now state the main theorems of this section.

Theorem 3.19. *Let $n \geqslant k > 1$, except for $n = k = 4$, and let $L: \mathscr{G}_n \to \mathscr{G}_n$ be a graph covering linear mapping. Then $L(\mathscr{G}_n^k) \subseteq \mathscr{G}_n^k$ if and only if L is a composition of one or more of the following types of transformations*:

(1) *Vertex permutation*;

(2) *A linear mapping T satisfying $E'(T(G)) = E'(G)$ for all $G \in \mathscr{G}_n$, that is the transformation T results in possible addition or elimination of loops.*

Furthermore, the following are equivalent:

(i) $L(\mathscr{G}_n^k) = \mathscr{G}_n^k$;

(ii) $L(\mathscr{G}_n^k) \subseteq \mathscr{G}_n^k$,

$$E'(L(G_{ij})) = E(L(G_{ij})), \quad \forall i, j \in \langle n \rangle, \quad i \neq j,$$

and there exists a permutation σ on $\langle n \rangle$ such that

$$L(G_{ii}) = G_{\sigma(i), \sigma(i)}, \quad i = 1, \ldots, n.$$

Proof. The "if" part in the first part of the theorem is obvious observing that addition of loops creates no new circuits but 1-circuits (loops). Conversely, by Proposition 3.14 for every $i \in \langle n \rangle$ there exists $i' \in \langle n \rangle$ such that (3.15) holds. By Propositions 3.5 and 3.14 we have $E'(L(G_{ij})) = \{[i', j']\}$. Hence, by Proposition 3.2 the mapping L results in renaming the vertices $(i \to i')$ and possibly adding or eliminating loops.

To prove the second part of the theorem observe that since $k > 1$ we have

$$G_{ij} \in \mathcal{G}_n^k, \quad \forall i, j \in \langle n \rangle.$$

Thus, in view of the first part of the theorem we have (i) if and only if $L(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^k$, for all $i, j \in \langle n \rangle$, $i \neq j$ there exist $s, t \in \langle n \rangle$, $s \neq t$ such that $L(G_{ij}) = G_{st}$, and for all $i \in \langle n \rangle$ there exists $s \in \langle n \rangle$ such that $L(G_{ss}) = G_{ii}$. By the first part of the theorem, the last three conditions are exactly (ii). □

Theorem 3.20. *For* $n = k = 4$ *Theorem* 3.18. *remains valid if the following transformation is added to the list*:

(3) *The linear mapping* T *defined by*

$$\begin{aligned} &T(G_{11}) = G_{11}, \quad && T(G_{22}) = G_{22}, \quad && T(G_{33}) = G_{33}, \\ &T(G_{12}) = G_{12}, && T(G_{13}) = G_{13}, && T(G_{14}) = G_{23}, \\ &T(G_{23}) = G_{14}, && T(G_{24}) = G_{24}, && T(G_{34}) = G_{34}. \end{aligned} \tag{3.21}$$

Proof. To prove the "if" part in the first part of the theorem observe that as shown in Example 3.18, the linear mapping T defined by (3.21) satisfies $L(\mathcal{G}_4^4) \subseteq \mathcal{G}_4^4$. Conversely assume that $L(\mathcal{G}_4^4) \subseteq \mathcal{G}_4^4$. By Proposition 3.14 we can assume that after an appropriate renaming of the vertices we have $E'(L(G_{12})) = \{[1, 2]\}$ and $E'(L(G_{13})) = \{[1, 3]\}$. By Proposition 3.14 we have either

$$E'(L(G_{14})) = \{[1, 4]\} \tag{3.22}$$

or

$$E'(L(G_{14})) = \{[2, 3]\}. \tag{3.23}$$

It is easy to verify, using Proposition 3.14, that if (3.22) holds then $E'(L(G_{23})) = \{[2, 3]\}$, $E'(L(G_{24})) = \{[2, 4]\}$, $E'(L(G_{34})) = \{[3, 4]\}$, and so L is of type 2. Similarly, if (3.23) holds then

$$E'(L(G_{23})) = \{[1, 4]\}, \quad E'(L(G_{24})) = \{[2, 4]\}, \quad E'(L(G_{34})) = \{[3, 4]\}$$

and L is a composition of transformations of types 2 and 3.

The proof of the second part of the theorem is identical to the proof of the corresponding part in Theorem 3.19. □

Theorems 3.19 and 3.20 do not hold for noncovering mappings as demonstrated by the following example.

Example 3.24. Let T be any linear mapping on $\mathcal{G}_n$ and let H be any element of $\mathcal{G}_n^k$. We define a linear mapping $L: \mathcal{G}_n \rightarrow \mathcal{G}_n$ by

$$L(G) = T(G) \cap H, \quad \forall G \in \mathcal{G}_n.$$

Observe that since $L(G) \subseteq H$ for all $G \in \mathcal{G}_n$ it follows that $L(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^k$, although L is not a composition of the types of transformations specified in Theorems 3.19 and 3.20.

We conclude this section with treatment of the case $k = 1$.

Theorem 3.25. *Let $L: \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a graph covering linear mapping such that*

$$L\Big(\bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij}\Big) \supseteq \bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij}.$$

Then $L(\mathcal{G}_n^1) \subseteq \mathcal{G}_n^1$ if and only if L is a composition of one or more of the following types of transformations:

(1) *Vertex permutation;*

(2) *A linear mapping T satisfying $T(G) = G$ for all loopless graphs $G \in \mathcal{G}_n$. (That is, loops may affect edges under T.)*

In this case we have $L(\mathcal{G}_n^1) = \mathcal{G}_n^1$.

Proof. To prove the "if" part observe that it is enough to check the operation of L on loopless graphs, since a graph which has loops is not in $\mathcal{G}_n^1$. Hence it is clear that a composition of the transformations specified in the statment of the theorem preserve the class $\mathcal{G}_n^1$. Conversely, assume that $L(\mathcal{G}_n^1) \subseteq \mathcal{G}_n^1$ and let $s, t \in \langle n \rangle$, $s \neq t$. By Proposition 3.14, for every $i \in \langle n \rangle$ there exists $i' \in \langle n \rangle$ such that (3.15) holds. By Propositions 3.5 and 3.14 we have $E'(L(G_{lw})) = \{[l', w']\}$. Since $G_{lw} \in \mathcal{G}_n^1$ and since $L(\mathcal{G}_n^1) \subseteq \mathcal{G}_n^1$ it follows that $L(G_{lw})$ contains no loop and hence $L(G_{lw}) = G_{l'w'}$. Thus, after performing the permutation $i \rightarrow i'$ on the vertices we have $L(G) = G$ for loopless graphs G.

Observe that since all the graphs in $\mathcal{G}_n^1$ are loopless it follows from the first part of the theorem that $L(\mathcal{G}_n^1) = \mathcal{G}_n^1$. □

Theorem 3.25 does not hold when condition (3.6) (which appears in the theorem) is omitted, as demonstrated by the following example.

Example 3.26. Let $n = 3$ and let L be defined by

$$L(G_{ij}) = \begin{cases} G_{11} \cup G_{12} \cup G_{13} \cup G_{23}, & [i, j] = [1, 1], \\ G_{12}, & \text{otherwise.} \end{cases}$$

Clearly, for every graph $G \in \mathcal{G}_3^1$ such that $E(G) \neq \emptyset$ we have $L(G) = G_{12}$. Hence, $L(\mathcal{G}_3^1) \subseteq \mathcal{G}_3^1$ although L is not a composition of the types of transformations listed in Theorem 3.25.

4. Digraphs—the case $k<n$

In this section we discuss digraph covering linear mappings $L:\mathscr{D}_n\to\mathscr{D}_n$. In all our propositions we assume that k satisfies $3\leqslant k\leqslant n$ and that

$$L(\mathscr{D}_n^k)\subseteq\mathscr{D}_n^k. \tag{4.1}$$

The case where $L(\mathscr{D}_n^k)=\mathscr{D}_n^k$ will be treated as well.

From Proposition 4.10 and on we assume that $k<n$.

The section will be concluded by showing, using examples, that the main theorem of this section (Theorem 4.18) does not hold in the cases $k=1$ and $k=2$. We shall also show that Theorem 4.18 does not hold for noncovering mappings.

Proposition 4.2. *We have* $E'(L(D_{ii}))=\emptyset$. $1\leqslant i\leqslant n$.

Proof. The proof is essentially the same as the proof of Proposition 3.2 using directed graphs and arcs. □

An example similar to Example 3.4 but using digraphs demonstrates that Proposition 4.2 does not hold in general for noncovering mappings.

Proposition 4.3. *Let* $i,j,l,w\in\langle n\rangle$ *such that* $(i,j)\neq(l,w)$. *Then* $E'(L(D_{ij}))\cap E'(L(D_{lw}))=\emptyset$.

Proof. Assume that there exists $s,t\in\langle n\rangle$ such that

$$(s,t)\in E'(L(D_{ij}))\cap E'(L(D_{lw})). \tag{4.4}$$

Choose $u_1,\ldots,u_{k-2}\in\langle n\rangle$ such that

$$\alpha=(s,t,u_1,\ldots,u_{k-2},s)$$

is a directed k-circuit, and let $i_1,\ldots,i_{k-1},j_1,\ldots,j_{k-1}\in\langle n\rangle$ be such that

$$(t,u_1)\in E(L(D_{i_1j_1})),\qquad (u_{m-1},u_m)\in E(L(D_{i_mj_m})),\quad m=2,\ldots,k-2,$$

and

$$(u_{k-2},s)\in E(L(D_{i_{k-1}j_{k-1}})).$$

Let D be the diagraph $D_{ij}\cup(\bigcup_{m=1}^{k-1}D_{i_mj_m})$. Since $L(D)$ contains the directed k-circuit α, it follows from (4.1) that D must consist of a directed k-circuit. But then, since $(i,j)\neq(l,w)$, the digraph $D'=D_{lw}\cup(\bigcup_{m=1}^{k-1}D_{i_mj_m})$ contains no directed k-circuit, and since $|D'|\leqslant k$ it follows that $D'\in\mathscr{D}_n^k$, while by (4.4) $L(D')$ contains the directed k-circuit α. This is a contradiction to (4.1) and hence the assumption that there exist s and t such that (4.4) holds is false. □

In view of Propositions 4.2 and 4.3 and since L is a covering mapping, we may define a linear mapping $\hat{L}:\mathscr{D}_n\to\mathscr{D}_n$ by $\hat{L}(D_{ij})=D_{st}$, where $(i,j)\in E'(L(D_{st}))$, $i\neq j$, and $\hat{L}(D_{ii})=D_0$, $i=1,\ldots,n$.

Observe that

$$L(\hat{L}(D))\supseteq D,\quad \forall D\in\mathscr{D}_n, \tag{4.5}$$

and

$$|\hat{L}(D)|\leqslant|D|,\quad \forall D\in\mathscr{D}_n.$$

Since (4.1) holds it follows from (4.5) that

$$D\notin\mathscr{D}_n^k\Rightarrow\hat{L}(D)\notin\mathscr{D}_n^k. \tag{4.6}$$

Proposition 4.7. *Let $i,j\in\langle n\rangle$, $i\neq j$. Then $E'(L(D_{ij}))$ does not contain two adjacent arcs.*

Proof. Assume that (s,t) and (u,v) are two adjacent arcs in $E'(L(D_{ij}))$ (namely, either $t=u$ or $s=v$). Let D be a digraph which consists of a directed k-circuit through (s,t) and (u,v). Since $\hat{L}(D_{st})=\hat{L}(D_{uv})=D_{ij}$ it follows that $|\hat{L}(D)|\leqslant k-1$ and hence $\hat{L}(D)\in\mathscr{D}_n^k$. Since $D\notin\mathscr{D}_n^k$ we have a contradiction to (4.6). Therefore, $E'(L(D_{ij}))$ does not contain two adjacent arcs. □

Proposition 4.8. *Let $i,j\in\langle n\rangle$, $i\neq j$. Then $E'(L(D_{ij}))$ does not contain two separated arcs.*

Proof. Assume that (s,t) and (u,v) are two separated arcs in $E'(L(D_{ij}))$. We distinguish between two cases:

Case 1. $k>3$

In this case we can find a directed k-circuit α which contains the arcs (s,t) and (u,v). Let D be the directed graph which consists of the circuit α. Since $\hat{L}(D_{st})=\hat{L}(D_{uv})=D_{ij}$ it follows that $|\hat{L}(D)|\leqslant k-1$ and hence $\hat{L}(D)\in\mathscr{D}_n^k$. Since $D\notin\mathscr{D}_n^k$ we have a contradiction to (4.6).

Case 2. $k=3$

Let D be the directed graph which consists of the directed 4-circuit (s,t,u,v,s). Since $\hat{L}(D_{st})=\hat{L}(D_{uv})=D_{ij}$ it follows from (4.6) that $\hat{L}(D)$ must consist of a directed 3-circuit through (i,j). Without loss of generality we may assume that

$$\hat{L}(D_{vs})=D_{jl}\quad\text{and}\quad\hat{L}(D_{tu})=D_{li}$$

for some $l\in\langle n\rangle$, $l\neq i,j$. Considering the digraph which consists of the directed 3-circuit (s,t,u,s), we obtain by (4.6) that $\hat{L}(D_{us})=D_{jl}$.

Similarly, considering the directed 3-circuits (u,v,s,u), (t,u,v,t) and (v,s,t,v) we show that

$$\hat{L}(D_{su})=D_{li},\qquad\hat{L}(D_{vt})=D_{jl},\quad\text{and}\quad\hat{L}(D_{tv})=D_{li}.$$

Consider now the digraph which consists of the directed 4-circuit (u, s, v, t, u). By (4.6) we now obtain that

$$\hat{L}(D_{sv}) = D_{ij}.$$

Considering (u, s, v, u) it follows that $\hat{L}(D_{vu}) = D_{li}$.

We have shown that $E'(L(D_{li}))$ contains the two adjacent arcs (t, v) and (v, u), in contradiction to Proposition 4.7.

Therefore, out assumption that there are two separated edges contained in $E'(L(D_{ij}))$ is false, so our claim is proved. □

Lemma 4.9. *Let $i, j, l, m \in \langle n \rangle$, $i \neq j$, $l \neq m$, be such that the arcs (i, j) and (l, m) are neither separated nor adjacent. Then the set $S = E'(L(D_{ij})) \cup E'(L(D_{lm}))$ does not contain adjacent arcs. If, furthermore, $k > 3$, then S does not contain also separated arcs.*

Proof. Assume that S contains two arcs which are either adjacent or (only in the case that $k > 3$) separated. Observe that we can find a directed k-circuit α through these two arcs. Let D be the graph which consists of the circuit α. We have $|\hat{L}(D)| \leq k$, but since $E'(\hat{L}(D))$ contains the arcs (i, j) and (l, m) which are nonseparated as well as nonadjacent, it follows that $\hat{L}(D)$ contains no directed k-circuit. Hence, $\hat{L}(D) \in \mathscr{D}_n^k$, and since $D \notin \mathscr{D}_n^k$ we have a contradiction to (4.6). □

For $i, j \in \langle n \rangle$ we define the set

$$S_{ij} = \bigcup_{(l,w) \in E'(L(D_{ij}))} \{l, w\}.$$

Proposition 4.10. *If $k < n$, then $|S_{ij}| \leq 2$ for all $i, j \in \langle n \rangle$, $i \neq j$.*

Proof. Assume that $|S_{ij}| > 2$ for some $i, j \in \langle n \rangle$, $i \neq j$. By Propositions 4.7 and 4.8 it follows that $E'(L(D_{ij}))$ contains two arcs which have one common vertex and opposite directions. Without loss of generality we may assume that (s, u), $(s, v) \in E'(L(D_{ij}))$ (the case of (u, s), $(v, s) \in E'(L(D_{ij}))$ is treated in the same way). Since $k < n$ we can choose $t_1, \ldots, t_{k-2} \in \langle n \rangle$ such that $s, u, v, t_1, \ldots, t_{k-2}$ are distinct.

We now distinguish between two cases.

Case 1. $k > 3$

Let D be the digraph which consists of the directed k-circuit $\alpha = (s, u, t_1, \ldots, t_{k-2}, s)$. By (4.6), the digraph $\hat{L}(D)$ consists of a directed k-circuit $\beta = (i, j, p_1, \ldots, p_{k-2}, i)$. The arc (v, u) is either separated or adjacent to each arc of α except for (s, u). Let $\hat{L}(D_{vu}) = D_{lm}$. By Lemma 4.9 it follows that (l, m) is either separated or adjacent to each arc of β except maybe for (i, j). Since

(v, u) is adjacent to (s, v) and since $\hat{L}(D_{sv}) = D_{ij}$ it follows by Lemma 4.9 that (l, m) and (i, j) are also either separated or adjacent. Hence, (l, m) is either separated or adjacent to each arc of β. In this case (l, m) cannot be adjacent to any of these arcs, since if, for example, we had $l = j$, then the arcs (j, p_1) and (l, m) would be nonadjacent as well as nonseparated. Therefore, (l, m) is separated from all arcs of β. Let D' be the digraph which consists of the directed k-circuit $(v, u, t_1, \ldots, t_{k-2}, v)$. Observe that (l, m), which is an arc in $\hat{L}(D')$, is separated from at least $k-2$ other arcs of $\hat{L}(D')$ (those which are also arcs of $\hat{L}(D)$). Since $|\hat{L}(D')| \leq k$, it means that $\hat{L}(D')$ does not consist of a directed k-circuit and thus $\hat{L}(D') \in \mathscr{D}_n^k$. Recall that $D' \notin \mathscr{D}_n^k$, so we have a contradiction to (4.6).

Case 2. $k = 3$

Let D be the digraph which consists of the directed 3-circuit (s, u, t_1, s). By (4.6), $\hat{L}(D)$ consists of a directed 3-circuit through (i, j), say (i, j, l, i). Without loss of generality assume that $\hat{L}(D_{ut_1}) = D_{jl}$ and $\hat{L}(D_{t_1 s}) = D_{li}$. Considering the directed 3-circuit (s, v, t_1, s) we obtain, using similar arguments, that

$$\hat{L}(D_{vt_1}) = D_{jl}.$$

Let $\hat{L}(D_{vu}) = D_{pq}$. Considering the directed graph which consists of (v, u, t_1, v) we show, using (4.6), that (p, q) and (j, l) are adjacent arcs as they are two arcs in a directed 3-circuit. Considering (v, u, s, v) we similarly show that (p, q) and (i, j) are also adjacent. Hence necessarily $(p, q) = (l, i)$. It now follows that $E'(L(D_{li}))$ contains the two separated arcs (t_1, s) and (v, u), which is a contradiction to Proposition 4.8.

As a conclusion, the contradictions we got yield that our assumption that $|S_{ij}| > 2$ was false, so $|S_{ij}| \leq 2$. □

The meaning of Proposition 4.10 is that if $(s, t) \in E'(L(D_{ij}))$ for some $i, j \in \langle n \rangle$, $i \neq j$, then the only other possible arc in $E'(L(D_{ij}))$ is (t, s). Therefore, the following is an immediate consequence of Proposition 4.10.

Corollary 4.11. *Let $k < n$. If $D \in \mathscr{D}_n$ consists of one directed k-circuit $(i_1, i_2, \ldots, i_k, i_1)$, then the only other possible directed k-circuit in $L(\hat{L}(D))$, if any, is $(i_k, i_{k-1}, \ldots, i_1, i_k)$.*

Proposition 4.12. *Let $k < n$ and let $D \in \mathscr{D}_n$. If $E'(D)$ consists of two adjacent arcs, then $E'(\hat{L}(D))$ consists of two adjacent arcs.*

Proof. Let $E'(D) = \{(u, v), (v, w)\}$. If $k = 3$ then let D' be the digraph which consists of the directed 3-circuit (u, v, w, u). By (4.6) $\hat{L}(D')$ consists of a directed 3-circuit. Therefore, the two arcs in $E'(\hat{L}(D))$ are part of a directed 3-circuit and as such they are adjacent. If $k > 3$, then by Lemma 4.9, $E'(\hat{L}(D))$ consists of either separated arcs or adjacent arcs. Assume that $E'(L(D))$ consists of the two separated arcs (i, j) and (s, t). By combinatorial calculations the cardinality of the set S_1 of all possible directed k-circuits through (u, v) and (v, w) is $n_1 = \binom{n-3}{k-3}(k-3)!$, while the cardinality of the set S_2 of all possible directed k-circuits through (i, j) and (s, t) is $n_2 = \binom{n-4}{k-4}(k-3)!$ Since $k < n$ we have

$$n_1 > n_2. \tag{4.13}$$

For all $\alpha \in S_1$ define D_α to be the digraph which consists of α. Let $\alpha, \beta \in S_1$, $\alpha \neq \beta$. Since both α and β pass through (u, v) and (v, w) it follows from Corollary 4.11 that $\hat{L}(D_\alpha) \neq \hat{L}(D_\beta)$. Therefore, by (4.13) there exists $\alpha \in S_1$ such that $\hat{L}(D_\alpha)$ does not contain an element of S_2. Since $E'(\hat{L}(D_\alpha))$ contains (i, j) and (s, t) and since $|\hat{L}(D_\alpha)| = k$ it follows that $\hat{L}(D_\alpha) \in \mathcal{D}_n^k$ in contradiction to (4.6). Thus $E'(L(D))$ does not consist of two separated arcs so it must consist of two adjacent arcs. □

As consequences of Proposition 4.12 we obtain the following two propositions.

Proposition 4.14. *If $k < n$, then $|L(D_{ij})| = 1$ for all $i, j \in \langle n \rangle$, $i \neq j$.*

Proof. Assume that for some $i, j \in \langle n \rangle$, $i \neq j$ we have $|L(D_{ij})| > 1$. By Proposition 4.10 we have $E'(L(D_{ij})) = \{(u, v), (v, u)\}$ for some $u, v \in \langle n \rangle$, $u \neq v$. Since $3 \leqslant k < n$ we have $n \geqslant 4$ so we can choose $s, t \in \langle n \rangle$ such that u, v, s, and t are distinct. By Proposition 4.12, the set $E'(\hat{L}(D_{vs}))$ consists of one arc which is adjacent to (i, j). Without loss of generality we may assume that $\hat{L}(D_{vs}) = D_{jl}$. Again, by Proposition 4.12, the set $E'(\hat{L}(D_{su}))$ must consist of an arc which is adjacent to (j, l) as well as to (i, j). Hence, necessarily $\hat{L}(D_{su}) = D_{li}$. Similarly, since (u, t) is adjacent to (s, u) as well as to (v, u) it follows that the only arc contained in $\hat{L}(D_{ut})$ is adjacent to (i, j) as well as to (l, i) and hence $\hat{L}(D_{ut}) = D_{jl}$. It now follows that $E'(L(D_{jl}))$ contains the two separated arcs (v, s) and (u, t), a contradiction to Proposition 4.8. Therefore, $|L(D_{ij})| \leqslant 1$ for all $i, j \in \langle n \rangle$, $i \neq j$. Since L is given to be a digraph covering mapping it follows that $|L(D_{ij})| = 1$ for all $i, j \in \langle n \rangle$, $i \neq j$. □

Corollary 4.15. *Let $k < n$ and let $D \in \mathcal{D}_n$. If $E'(D)$ consists of two adjacent arcs, then $E'(L(D))$ consists of two adjacent arcs.*

Proof. The claim follows from Propositions 4.12 and 4.14 using counting arguments. □

Proposition 4.16. *Let $k < n$, let $i, j \in \langle n \rangle$, $i \neq j$, and let $u, v \in \langle n \rangle$ be such that $E'(L(D_{ij})) = \{(u, v)\}$. Then $E'(L(D_{ji})) = \{(v, u)\}$.*

Proof. Let $E'(L(D_{ji})) = \{(s, t\}$ and assume that $(s, t) \neq (v, u)$. We distinguish between two cases:

Case 1. The arcs (s, t) and (u, v) are nonseparated

Here by Lemma 4.9 it follows that (s, t) and (u, v) are also nonadjacent. Therefore, without loss of generality assume that $t = v$. Since $3 \leq k < n$ we have $n \geq 4$ so let $w \in \langle n \rangle$ be such $w \neq u, v, s$. By Proposition 4.12, the set $E'(\hat{L}(D_{vw}))$ consists of an arc which is adjacent to (i, j) as well as to (j, i). Clearly, no such an arc exists.

Case 2. The arcs (s, t) and (u, v) are separated

(In view of Lemma 4.9 such a situation is possible only in the case $k = 3$.) By Proposition 4.12, the set $E'(\hat{L}(D_{tu}))$ consists of an arc which is adjacent to (i, j) as well as to (j, i) which, as observed above, is impossible.

We have shown that our assumption that $(s, t) \neq (v, u)$ leads to a contradiction, hence $(s, t) = (v, u)$. □

Proposition 4.17. *Let $k < n$. Then there exists $i \in \langle n \rangle$ such that the set $S = E'(L(\bigcup_{j=2}^{n} D_{1j}))$ is either*

$$\bigcup_{\substack{j=1\\ j\neq i}}^{n} \{(i, j)\} \quad or \quad \bigcup_{\substack{j=1\\ i\neq j}}^{n} \{(j, i)\}.$$

Proof. Observe that as in Propositions 4.14 and 4.16 we have $n \geq 4$. Let $E'(L(D_{12})) = \{(s, t)\}$ and let $E'(L(D_{31})) = \{(u, v)\}$. By Corollary 4.15, the arcs (s, t) and (u, v) are adjacent. We have two possibilities:

(1) $s = v$. By Proposition 4.16 we have $E'(L(D_{13})) = \{(s, u)\}$. Let $l \in \langle n \rangle$, $l > 3$, and let $E'(L(D_{l1})) = \{(p, q)\}$. Since $(l, 1)$ is adjacent to $(1, 2)$ as well as to (1.3), it follows from Corollary 4.15 that $q = s$. By Proposition 4.16 we have $E'(L(D_{1l})) = \{(s, p)\}$. It now follows, by Proposition 4.3, that the set S is $\bigcup_{j=1, j\neq s}^{n} \{(s, j)\}$.

(2) $t = u$. Similar to Case 1 we show that in this case the set S is $\bigcup_{j=1, j\neq s}^{n} \{(j, s)\}$. □

We are now ready for stating the main theorem of this section.

Theorem 4.18. *Let $n > k \geq 3$ and let $L: \mathscr{D}_n \to \mathscr{D}_n$ be a digraph covering linear mapping. Then $L(\mathscr{D}_n^k) \subseteq \mathscr{D}_n^k$ if and only if L is a composition of one or more of the following types of transformations*:

(1) *Vertex permutation*;

(2) *The linear mapping T satisfying $T(D_{ij}) = D_{ji}$ for all $i, j \in \langle n \rangle$*;

(3) *A linear mapping T satisfying $E'(T(D)) = E'(D)$ for all $D \in \mathscr{D}_n$, that is the mapping T results in possible addition or elimination of loops.*

Furthermore, the following are equivalent

(i) $L(\mathscr{D}_n^k) = \mathscr{D}_n^k$;

(ii) $L(\mathscr{D}_n^k) \subseteq \mathscr{D}_n^k$,

$$E'(L(D_{ij})) = E(L(D_{ij})), \quad \forall i, j \in \langle n \rangle, \quad i \neq j,$$

and there exists a permutation σ on $\langle n \rangle$ such that

$$L(D_{ii}) = L(D_{\sigma(i),\sigma(i)}), \quad i = 1, \ldots, n.$$

Proof. The "if" part of the first part of the theorem is obvious. Conversely, by Proposition 4.17, after an appropriate permutation of the vertices we have either

$$E'(L(D_{1j})) = \{(1, j)\}, \quad j = 2, \ldots, n, \tag{4.19}$$

or

$$E'(L(D_{1j})) = \{(j, 1)\}, \quad j = 2, \ldots, n. \tag{4.20}$$

Assume that (4.19) holds. By Proposition 4.16

$$E'(L(D_{j1})) = \{(j, 1)\}, \quad j = 2, \ldots, n. \tag{4.21}$$

Let $i, j \in \langle n \rangle$, $i, j > 1$, $i \neq j$. Since (i, j) is adjacent to $(1, i)$ as well as to $(j, 1)$, it follows from (4.19), (4.21) and from Corollary 4.15 that $E'(L(D_{ij})) = \{(i, j)\}$. Similarly we show that if (4.20) holds, then $E'(L(D_{ij})) = \{(j, i)\}$. Hence, the mapping L is a composition of the types of transformations listed in the theorem.

The proof of the second part of the theorem is essentially the same as the proof of the corresponding part is in Theorem 3.19. □

Theorem 4.18 does not hold in the case $k = 2$ as demonstrated by the following example.

Example 4.22. Let L be defined by

$$L(D_{12}) = \bigcup_{\substack{i,j=1\\ i<j}}^{n} D_{ij}, \qquad L(D_{21}) = \bigcup_{\substack{i,j=1\\ i>j}}^{n} D_{ij},$$

and

$$L(D_{ij}) = D_0 \quad \{i, j\} \neq \{1, 2\}.$$

Observe that if $L(D)$ contains any directed circuit then $D_{12} \cup D_{21} \subseteq D$. Hence L satisfies even $L(\mathscr{D}_n^2) \subseteq \mathscr{D}_n^1$, although L is not a composition of transformations of the types specified in Theorem 4.18.

It is easy to verify that if L is a digraph covering linear mapping such that $L(\mathscr{D}_n^2) \subseteq \mathscr{D}_n^2$ then, as in Proposition 4.2, we have

$$E'(L(D_{ii})) = \emptyset, \quad i = 1, \ldots, n, \tag{4.23}$$

and also

$$(s, t) \in E'(L(D_{ij})) \Leftrightarrow (t, s) \in E'(L(D_{ji})), \quad s, t \in \langle n \rangle, \quad s \neq t. \tag{4.24}$$

Furthermore, since the digraph $D = \bigcup_{i,j=1;\, i<j}^{n} D_{ij}$ contains no directed circuit it follows that $L(D)$ contains no directed circuit of length greater than 1. As is well known, after an appropriate permutation of the vertices we have

$$E'\Bigg(L\Bigg(\bigcup_{\substack{i,j=1\\ i<j}}^{n} D_{ij}\Bigg)\Bigg) \subseteq \bigcup_{\substack{i,j=1\\ i<j}}^{n} \{(i, j)\}. \tag{4.25}$$

However, Conditions (4.23), (4.24) and (4.25) are not sufficient for having $L(\mathscr{D}_n^2) \subseteq \mathscr{D}_n^2$ as demonstrated by the following example.

Example 4.26. Let $n = 3$ and let L be defined by

$$L(D_{12}) = D_{12}, \qquad L(D_{13}) = D_{23}, \qquad L(D_{23}) = D_{13},$$
$$L(D_{21}) = D_{21}, \qquad L(D_{31}) = D_{32}, \qquad L(D_{32}) = D_{31},$$

and

$$L(D_{ii}) = D_{ii}, \quad i = 1, 2, 3.$$

Let D be the digraph $D_{12} \cup D_{13} \cup D_{32}$. Observe that D contains no directed circuit while $L(D)$ contains the directed 3-circuit $(1, 2, 3, 1)$. So even $L(\mathscr{D}_n^1) \not\subseteq \mathscr{D}_n^2$ although, as can easily be verified, (4.23), (4.24) and (4.25) are satisfied.

Example 4.22 shows that Theorem 4.18 does not hold also in the case $k = 1$. It is easy to verify that if we assume that

$$L\Bigg(\bigcup_{\substack{i,j=1\\ i\neq j}}^{n} D_{ij}\Bigg) = \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} D_{ij}. \tag{4.27}$$

then $L(\mathscr{D}_n^1) \subseteq \mathscr{D}_n^1$ implies that

$$(s, t) \in E'(L(D_{ij})) \Leftrightarrow (t, s) \in E'(L(D_{ji})), \quad \text{whenever } i \neq j,\ s \neq t. \tag{4.28}$$

As in the case $k = 2$, after an appropriate permutation of the vertices we have

$$L\Bigg(\bigcup_{\substack{i,j=1\\ i<j}}^{n} D_{ij}\Bigg) \subseteq \bigcup_{\substack{i,j=1\\ i<j}}^{n} D_{ij}. \tag{4.29}$$

However, Conditions (4.27), (4.28) and (4.29) are not sufficient for having $L(\mathscr{D}_n^1) \subseteq \mathscr{D}_n^1$ as demonstrated by Example 4.26.

Finally, we use an example which is similar to Example 3.24 in order to show that Theorem 4.18 does not hold for noncovering mappings.

Example 4.30. Let T be any linear mapping on $\mathscr{D}_n$ and let H be any element of $\mathscr{D}_n^k$. We define a linear mapping $L: \mathscr{D}_n \to \mathscr{D}_n$ by

$$L(D) = T(D) \cap H, \quad \forall D \in \mathscr{D}_n.$$

The mapping L is not a composition of the types of transformations listed in Theorem 4.18 but clearly $L(\mathscr{D}_n^k) \subseteq \mathscr{D}_n^k$.

5. Digraphs—the case $k = n$

The discussion of this section is devoted to digraph covering linear mappings $L: \mathscr{D}_n \to \mathscr{D}_n$ satisfying

$$L(\mathscr{D}_n^n) \subseteq \mathscr{D}_n^n. \tag{5.1}$$

In all our propositions we assume that $n \geq 4$. The cases $n = 1$, $n = 2$ and $n = 3$ will be discussed after the proof of Theorem 5.30. As in the previous sections we shall also describe those L for which $L(\mathscr{D}_n^n) = \mathscr{D}_n^n$.

Recall that the discussion in the previous section through Lemma 4.11 holds also for the case $k = n$.

Proposition 5.2. *If for some $i, j \in \langle n \rangle$, $i \neq j$ we have $|S_{ij}| > 2$, then there exist n arcs $(s_1, t_1), \ldots, (s_n, t_n)$ which form a directed n-circuit such that either*

$$E'(L(D_{s_i t_i})) = \bigcup_{\substack{j=1 \\ j \neq i}}^{n} \{(i, j)\}, \quad i = 1, \ldots, n \tag{5.3}$$

or

$$E'(L(D_{s_i t_i})) = \bigcup_{\substack{j=1 \\ j \neq i}}^{n} \{(j, i)\}, \quad i = 1, \ldots, n. \tag{5.4}$$

Proof. Assume that $|S_{ij}| > 2$ for some $i, j \in \langle n \rangle$, $i \neq j$. by Proposition 4.7 and 4.8 we may assume that after an appropriate permutation of the vertices we have either

$$\{(1, 2), (1, 3)\} \subseteq E'(L(D_{ij})) \tag{5.5}$$

or

$$\{(2, 1), (3, 1)\} \subseteq E'(L(D_{ij})). \tag{5.6}$$

Consider the case that (5.5) holds and let D be the digraph which consists of the directed n-circuit $(1, 2, \ldots, n, 1)$. Let

$$E'(\hat{L}(D_{l,l+1})) = \{(s_l, t_l)\}, \quad l = 2, \ldots, n-1, \tag{5.7}$$

and

$$E'(\hat{L}(D_{n1})) = \{(s_n, t_n)\}. \tag{5.8}$$

By (4.6), the n arcs (i, j), (s_2, t_2), $(s_3, t_3), \ldots, (s_n, t_n)$ form a directed n-circuit α. Let D_m, $m = 3, \ldots, n$ be the digraph consists of the directed n-circuit $(1, m, m+1, \ldots, n, 2, \ldots, m-1, 1)$. Observe that by (5.5) and (5.7), the set $E'(\hat{L}(D_3))$ contains the $n-2$ arcs (i, j), $(s_3, t_3), \ldots, (s_{n-1}, t_{n-1})$. It is easy to verify that the only directed n-circuit through these $n-2$ arcs is α. Since, by (4.6), $\hat{L}(D_3)$ consists of a directed n-circuit, that circuit must be α and hence

$$E'(\hat{L}(D_{n2} \cup D_{21})) = \{(s_2, t_2), (s_n, t_n)\}. \tag{5.9}$$

If

$$E'(\hat{L}(D_{n2})) = \{(s_2, t_2)\}, \tag{5.10}$$

then by (5.7) and (5.10), the set $E'(L(D_{s_2 t_2}))$ contains the arcs $(2, 3)$ and $(n, 2)$ which, since $n \geq 4$, are adjacent. This contradicts Proposition 4.7. Hence, (5.10) is false by (5.9) we have

$$E'(\hat{L}(D_{n2})) = \{(s_n, t_n)\}, \tag{5.11}$$

$$E'(\hat{L}(D_{21})) = \{(s_2, t_2)\}. \tag{5.12}$$

Let $4 \leq m \leq n$. By (5.7) and (5.11), the set $E'(\hat{L}(D_m))$ contains the arcs

$$(s_m, t_m), \ldots, (s_{n-1}, t_{n-1}), (s_n, t_n), (s_2, t_2), \ldots, (s_{m-2}, t_{m-2}).$$

Again, it is easy to verify that the only directed n-circuit through these $n-2$ arcs is α. Since, by (4.6), $\hat{L}(D_m)$ consists of a directed n-circuit, that circuit must be α and hence

$$E'(\hat{L}(D_{1m} \cup D_{m-1,1})) = \{(i, j), (s_{m-1}, t_{m-1})\}. \tag{5.13}$$

If

$$E'(\hat{L}(D_{m-1,1})) = \{(i, j)\}, \tag{5.14}$$

then by (5.5) and (5.14) the set $E'(L(D_{ij}))$ contains the arcs $(m-1, 1)$ and $(1, 2)$ which, since $m \geq 4$, are adjacent. This is a contradiction to Proposition 4.7. Hence, (5.14) is false and by (5.13) we have

$$(1, m) \in E'(L(D_{ij})) \tag{5.15}$$

and

$$\hat{L}(D_{m-1,1}) = \{(s_{m-1}, t_{m-1})\}. \tag{5.16}$$

Since (5.15) holds for all m, $4 \leq m \leq n$ it follows by (5.5) that

$$E'(L(D_{ij})) \supseteq \bigcup_{p=2}^{n} \{(1, p)\}. \tag{5.17}$$

Also, by (5.7), (5.8), (5.11), (5.12) and (5.15) we have

$$\begin{aligned} &\{(l, 1), (l, l+1)\} \subseteq E'(L(D_{s_l t_l})), \quad l = 2, \ldots, n-1 \\ &\{(n, 1), (n, 2)\} \subseteq E'(L(D_{s_n t_n})). \end{aligned} \tag{5.18}$$

In a similar way to our proof that (5.5) implies (5.17) we prove that (5.18) implies

$$E'(L(D_{s_l t_l})) \supseteq \bigcup_{\substack{p=1 \\ p \neq l}}^{n} \{(l, p)\}. \tag{5.19}$$

By Proposition 4.3 equality holds in (5.17) and (5.19). If we let $s_1 = i$ and $t_1 = j$ then we have (5.3).

In the case that (5.6) holds we prove in essentially the same way that (5.4) holds. □

Proposition 5.20 *If*

$$|S_{ij}| \leq 2 \quad \text{for all } i, j \in \langle n \rangle, i \neq j, \tag{5.21}$$

then

$$|L(D_{ij})| = 1 \quad \text{for all } i, j \in \langle n \rangle, i \neq j. \tag{5.22}$$

Proof. Assume that $|L(D_{ij})| > 1$ for some $i, j \in \langle n \rangle, i \neq j$. By (5.21) we have $E'(L(D_{ij})) = \{(u, v), (v, u)\}$ for some $u, v \in \langle n \rangle$, $u \neq v$. Let $\{t_1, \ldots, t_{n-2}\} =$

$\langle n\rangle\backslash\{u, v\}$, and let D and D' be the digraphs which consist of the directed n-circuits $(u, v, t_1, \ldots, t_{n-2}, u)$ and $(v, u, t_1, \ldots, t_{n-2}, v)$ respectively. By (4.6) the digraphs $\hat{L}(D)$ and $\hat{L}(D')$ consist of a single directed n-circuit each, say α and β respectively. Let

$$\hat{L}(D_{ut_1}) = D_{rs} \tag{5.23}$$

and

$$\hat{L}(D_{t_{n-2}v}) = D_{pq}. \tag{5.24}$$

Since $\hat{L}(D_{uv}) = \hat{L}(D_{vu})$ it follows that α and β have at least $n-2$ common arcs. Since α and β are both directed n-circuits it follows that $\alpha = \beta$ and so

$$E'(\hat{L}(D_{vt_1} \cup D_{t_{n-2}u})) = \{(r, s), (p, q)\}. \tag{5.25}$$

The fact that $n \geqslant 4$ yields that $t_{n-2} \neq t_1$ and hence the arcs (t_{n-2}, v) and (v, t_1) are adjacent. By Proposition 4.7 and (5.24) it follows that $(v, t_1) \notin E'(L(D_{pq}))$ and therefore it follows from (5.25) that

$$(v, t_1) \in E'(L(D_{rs})). \tag{5.26}$$

By (5.23) and (5.26) we now have $|S_{rs}| \geqslant 3$ in contradiction to (5.21). We thus conclude that $|L(D_{ij})| \leqslant 1$ for all $i, j \in \langle n\rangle$, $i \neq j$. Since L is a digraph covering mapping, (5.22) follows. □

Proposition 5.27. *Given that* (5.22) *holds let* $i, j \in \langle n\rangle$, $i \neq j$, *and let* $u, v \in \langle n\rangle$ *be such that* $E'(L(D_{ij})) = \{(u, v)\}$. *Then* $E'(L(D_{ji})) = \{(v, u)\}$.

Proof. Assume that $\hat{L}(D_{vu}) = D_{st}$. Since (u, v) and (v, u) are neither adjacent nor separated it follows from Lemma 4.9 and (5.22), using counting arguments, that (s, t) and (i, j) are neither adjacent nor separated. If $(s, t) \neq (j, i)$, then it follows that either $s = j$ or $t = i$. In each of these cases, given that $n \geqslant 4$, we can find $p \in \langle n\rangle$, $p \neq i, j, s, t$. The arc (s, p) in the case that $s = j$ or the arc (p, t) in the case that $t = i$ is neither separated nor adjacent to both (s, t) and (j, i). By Lemma 4.9, $E'(L(D_{sp}))$ (in case $s = j$) or $E'(L(D_{pt}))$ (in case $t = i$) consists of an arc which is neither separated nor adjacent to both (u, v) and (v, u), but no such an arc exists (except for (u, v) and (v, u) themselves). Hence $(s, t) = (j, i)$ and our claim is proved. □

As a consequence we have

Proposition 5.28. *Assume that* (5.22) *holds and let* $D \in \mathcal{D}_n$. *If* $E'(D)$ *consists of two adjacent arcs, then* $E'(L(D))$ *consists of two adjacent arcs.*

Proof. Let $i, j, l \in \langle n\rangle$ be distinct, and let $E'(L(D_{ij})) = \{(s, t)\}$ and $E'(L(D_{jl})) = \{(u, v)\}$. By Proposition 5.27 we have $E'(L(D_{lj})) = \{(v, u)\}$ and by Lemma 4.9 and Proposition 5.27 we have either $s = v$ and $t \neq u$ or $t = u$ and $s \neq v$. In each case it follows that the arcs (s, t) and (u, v) are adjacent. □

Proposition 5.29. *If* (5.22) *holds, then there exists* $i \in \langle n \rangle$ *such that the set* $E'(L(\bigcup_{j=2}^{n} D_{1j}))$ *is either*

$$\bigcup_{\substack{j=1\\ j\neq i}}^{n} \{(i, j)\} \quad or \quad \bigcup_{\substack{j=1\\ i\neq j}}^{n} \{(j, i)\}.$$

Proof. The proof is the same as of Proposition 4.17, using Proposition 5.27 instead of Proposition 4.16 and Proposition 5.28 instead of Corollary 4.15. □

Theorem 5.30. *Let* $n \geq 4$ *and let* $L: \mathcal{D}_n \to \mathcal{D}_n$ *be a digraph covering linear mapping. Then* $L(\mathcal{D}_n^n) \subseteq \mathcal{D}_n^n$ *if and only if L is a composition of one or more of the following types of transformations*:

(1) *Vertex permutation*;

(2) *The linear mapping T satisfying* $T(D_{ij}) = D_{ji}$ *for all* $i, j \in \langle n \rangle$;

(3) *A linear mapping T satisfying* $E'(T(D)) = E'(D)$ *for all* $D \in \mathcal{D}_n$;

(4) *A linear mapping T defined as follows*: *Let* $s_1, \ldots, s_n, t_1, \ldots, t_n \in \langle n \rangle$ *be such that the arcs* $(s_1, t_1), \ldots, (s_n, t_n)$ *form a directed n-circuit and let S be the set which consists of these n arcs. Then T is defined by*

$$T(D_{s_i t_i}) = \bigcup_{\substack{j=1\\ j\neq i}}^{n} D_{ij}, \quad i = 1, \ldots, n, \qquad T(D_{ii}) = D_{ii}, \quad i \in \langle n \rangle,$$

and

$$T(D_{ij}) = D_0, \quad i, j \in \langle n \rangle, \quad (i, j) \notin S, \quad i \neq j.$$

Furthermore, the following are equivalent:

(i) $L(\mathcal{D}_n^n) = \mathcal{D}_n^n$;

(ii) $L(\mathcal{D}_n^n) \subseteq \mathcal{D}_n^n$;

$$E'(L(D_{ij})) = E(L(D_{ij})), \quad \forall i, j \in \langle n \rangle, \quad i \neq j,$$

and there exists a permutation σ *on* $\langle n \rangle$ *such that*

$$L(D_{ii}) = D_{\sigma(i),\sigma(i)}, \quad i = 1, \ldots, n.$$

Proof. It is easy to check that mappings T of the types 1, 2 and 3 satisfy $T(\mathcal{D}_n^n) \subseteq \mathcal{D}_n^n$. Let T be a mapping of the type 4, and assume that $T(D)$ contains a directed n-circuit for some $D \in \mathcal{D}_n$. Observe that for every $i \in \langle n \rangle$ the set $E'(T(D))$ must contain an arc (i, j), where $i \neq j$. Hence, $(s_i, t_i) \in E'(D)$ for all $i \in \langle n \rangle$, so D contains a directed n-circuit. Therefore $T(D) \notin \mathcal{D}_n^n \Rightarrow D \notin \mathcal{D}_n^n$, or $T(\mathcal{D}_n^n) \subseteq \mathcal{D}_n^n$.

Conversely, if (5.21) holds, then by Proposition 5.20 (5.22) holds and our proof follows as the proof of Theorem 4.18, using Propositions 5.28 and 5.29 instead of Corollary 4.15 and Proposition 4.17. If (5.21) does not hold then by Proposition 5.2 we have either (5.3), in which case L is a composition of transformations of types 3 and 4, or (5.4) in which case L is a composition of transformation of types 2, 3, and 4.

The proof of the second part of the theorem is essentially the same as the proof of the second part of Theorem 3.19. □

Theorem 5.30 does not hold for noncovering mappings as demonstrated by Example 4.30.

We conclude the section with the investigation of the cases where $n \leqslant 3$.

The case $k = n = 1$ is trivial. It is easy to verify that in the case $k = n = 2$ Proposition 4.2 is valid. Hence we clearly have either $E'(L(D_{12})) = \{(1, 2)\}$, $E'(L(D_{21})) = \{(2, 1)\}$ or $E'(L(D_{12})) = \{(2, 1)\}$, $E'(L(D_{21}) = \{(1, 2)\}$. Observe that where $n = 2$, a mapping of the type 4 is, either of the type 3 or a composition of transformations of types 2 and 3. Therefore, Theorem 5.30 (in fact Theorem 4.18) holds also in the case $k = n = 2$.

In the remaining case, the case $k = n = 3$, we have two possible directed 3-circuits $\alpha = (1, 2, 3, 1)$ and $\beta = (1, 3, 2, 1)$. In view of Propositions 4.2 and 4.3 we may define the mapping $\hat{L}$ and we have (4.6). Let D and D' be the digraphs which consist of α and β respectively. By (4.6) we have either $\hat{L}(D) = D$ or $\hat{L}(D) = D'$, and independently we have either $\hat{L}(D') = D$ or $\hat{L}(D') = D'$. So, there are 144 different possibilities for $\hat{L}$, and hence there are exactly 144 different digraph covering linear mappings L (up to adding or eliminating loops) satisfying $L(\mathscr{D}_3^3) \subseteq \mathscr{D}_3^3$. We conclude the section with two examples of such mappings.

Example 5.31. Let L be defined by

$$L(D_{12}) = D_{12};\quad L(D_{23}) = D_{23},\quad L(D_{31}) = D_{31},\quad L(D_{21}) = D_{21};$$
$$D(D_{13}) = D_{32},\quad L(D_{32}) = D_{13},\quad L(D_{ii}) = D_{ii},\quad i = 1, 2, 3.$$

Example 5.32. Let L be defined by

$$L(D_{ij}) = \begin{cases} D_{12} \cup D_{21}, & (i, j) = (1, 2), \\ D_{23} \cup D_{13}, & (i, j) = (2, 3), \\ D_{32} \cup D_{31}, & (i, j) = (3, 1), \\ D_0, & \text{otherwise}. \end{cases}$$

6. Applications to matrices

Let F be a field with characteristic p.

Consider the set $\mathscr{M}\mathscr{G}_n^k[\mathscr{M}\mathscr{D}_n^k]$ of all matrices in F^{nn} whose graph (digraph) contains no circuit (directed circuit) of length greater than or equal to k. In this section we apply the results of the previous sections to characterizing linear transformations $L: F^{nn} \to F^{nn}$ for which $L(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k$ or $L(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k$. Our results are obtained for $p = 0$ or $p > n^2 - n$, under certain restrictions on L induced by the covering requirements in our previous results.

Definition 6.1. A linear transformation $L: F^{nn} \to F^{nn}$ is said to be a *covering* transformation if for all $i, j \in \{n\}$, $i \neq j$ there exist $s, t \in \langle n \rangle$ such that $(L(E_{st}))_{ij} \neq 0$ or $(L(E_{st}))_{ji} \neq 0$. The transformation L is said to be a *strongly covering* transformation if for all $i, j \in \langle n \rangle$, $i \neq j$ there exist $s, t \in \langle n \rangle$ such that $(L(E_{st}))_{ij} \neq 0$.

Remark 6.2. Observe that if L is nonsingular then $L(F^{nn}) = F^{nn}$. Hence it follows that every nonsingular linear transformation on F^{nn} is a strongly covering transformation.

Lemma 6.3. (i) *The transformation L is a covering transformation if and only if L_g is a graph covering mapping*;

(ii) *The transformation L is a strongly covering transformation if and only if L_d is a digraph covering mapping.*

Proof. The observations of our lemma follow from Definitions 2.8, 2.16, 2.21 and 6.1. □

The following lemma is pure algebraic.

Lemma 6.4. *Let $a_1, \ldots, a_m, b_1, \ldots, b_m \in F$ such that for every $i \in \langle m \rangle$ we have $a_i \neq 0$ or $b_i \neq 0$. If either $p = 0$ or $p > m$, then there exists $c \in F$ such that $a_i + cb_i \neq 0$, $i = 1, \ldots, m$.*

Proof. If $b_i = 0$, then, since $a_i \neq 0$, for every $x \in F$ we have $a_i + xb_i \neq 0$. If $b_i \neq 0$, then, since $x = a_i b_i^{-1}$ is the unique solution of the equation $a_i + xb_i = 0$, it follows that the cardinality of the set $S = \{x \in F : a_i + xb_i = 0 \text{ for some } i \in \langle m \rangle\}$ is at most m. Since either $p = 0$ or $p > m$, we can find a number $c \in F$ which is not in S, so we have $a_i + cb_i \neq 0$, $i = 1, \ldots, m$. □

Lemma 6.4 is used in proving the following proposition.

Proposition 6.5. *Let $n \geq 2$. If either $p = 0$ or $p > n^2 - n$, then for every $G \in \mathcal{G}_n$ there exists a matrix $A \in F^{nn}$ such that $G(A) = G$ and*

$$E'(G(L(A))) = E'(L_g(G)). \tag{6.6}$$

Proof. We prove our assertion by induction on the number $h = |E(G)|$. If $h = 1$, then let $E(G) = \{[s, t]\}$. We set $B = L(E_{st})$ and $C = L(E_{ts})$. By Lemma 6.4 we can find $x \in F$ such that $b_{ij} + xc_{ij} \neq 0$ whenever $b_{ij} \neq 0$ or $c_{ij} \neq 0$, $i, j \in \langle n \rangle$, $i \neq j$. Hence the matrix $A = E_{st} + xE_{ts}$ satisfies $G(A) = G = G_{st}$ and

$$E'(G(L(A))) = E'(G(L(E_{st})) \cup G(L(E_{ts}))) = E'(L_g(G)).$$

Assume that our claim holds for $h < m$ and let $h = m$. Let $[s, t] \in E(G)$, and let G' be the graph obtained from G by eliminating the edge $[s, t]$. By the inductive assumption there exists a matrix $A' \in F^{nn}$ such that $G(A') = G'$ and $E'(G(L(A'))) = E'(L_g(G'))$.

Let $B = (L(A')$, $C = L(E_{st})$ and $K = L(E_{ts})$. By Lemma 6.4 we can find $x \in F$ such that $r_{ij} = b_{ij} + xc_{ij} \neq 0$ whenever $b_{ij} \neq 0$ or $c_{ij} \neq 0$, $i, j \in \langle n \rangle$, $i \neq j$. Using Lemma 6.4 again we now find $y \in F$ such that $r_{ij} + yk_{ij} \neq 0$ whenever $r_{ij} \neq 0$ or $k_{ij} \neq 0$, $i, j \in \langle n \rangle$, $i \neq j$. The matrix $A = A' + xE_{st} + yE_{ts}$ is the required matrix. □

The same result holds for digraphs.

Proposition 6.7. *Let $n \geq 2$. If either $p = 0$ or $p > n^2 - n$, then for every $D \in \mathcal{D}_n$ there exists a matrix $A \in F^{nn}$ such that $D(A) = D$ and*

$$E'(D(L(A))) = E'(L_d(D)). \tag{6.8}$$

Proof. The lines of the proof are the same as those of the proof of Proposition 6.5, using induction on $|E(D)|$. The proof here is even somewhat simpler because we have to consider only E_{st} and not E_{ts} whose digraph is different. □

Corollary 6.9. *Let $n \geq 2$. If either $p = 0$ or $p > n^2 - n$, then we have*

$$L(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k \tag{6.10}$$

if and only if

$$L_g(\mathcal{G}_n^k) \subseteq \mathcal{G}_n^k. \tag{6.11}$$

Proof. Assume that (6.10) holds and let

$$G \in \mathcal{G}_n^k. \tag{6.12}$$

Consider first the case $k > 1$. By Proposition 6.5 there exists $A \in F^{nn}$ such that (6.6) holds. By (6.12) we have $A \in \mathcal{M}\mathcal{G}_n^k$, and hence it follows from (6.10) that $G(L(A)) \in \mathcal{G}_n^k$. Since $k > 1$ it follows from (6.6) that $L_g(G) \in \mathcal{G}_n^k$.

In the case $k = 1$ we use the same proof to show that $L_g(G)$ contains no circuit of length greater than 1. To see that $L_g(G)$ contains no loop observe that by (6.10) $L(E_{ij})$ has zero diagonal elements whenever $i \neq j$. Hence, since G contains no loop it follows from Definition 2.21 that $L_g(G)$ contains no loop. Therefore $L_g(G) \in \mathcal{G}_n^1$.

Conversely, assume that (6.11) holds and let $A \in \mathcal{M}\mathcal{G}_n^k$. Since $G(A) \in \mathcal{G}_n^k$ it follows from (2.22) and (6.11) that $L(A) \in \mathcal{M}\mathcal{G}_n^k$. □

Corollary 6.13. *Let $n \geq 2$. If either $p = 0$ or $p > n^2 - n$, then we have $L(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k$ if and only if $L_d(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^k$.*

Proof. The proof is essentially the same as the proof of Corollary 6.9, using Proposition 6.7 instead of Proposition 6.5 and (2.23) instead of (2.22). □

Lemma 6.14. *We have*

$$L_g(G) = G, \quad \forall G \in \mathcal{G}_n \tag{6.15}$$

if and only if L satisfies

$$L(E_{ii}) = b_{ii}E_{ii}, \quad b_{ii} \neq 0, \quad i \in \langle n \rangle, \tag{6.16}$$

and

$$L(E_{ij}) = b_{ij}E_{ij} + c_{ij}E_{ji}, \quad i, j \in \langle n \rangle, \quad i \neq j, \tag{6.17}$$

where at least one of b_{ij}, b_{ji}, c_{ij} and c_{ji} is nonzero for all $i, j \in \langle n \rangle$, $i < j$.

Proof. By Definition 2.7, (6.15) holds if and only if

$$L_g(G_{ij}) = G_{ij}, \quad \forall i, j \in \langle n \rangle. \tag{6.18}$$

In view of Definition 2.21, (6.18) holds if and only if (6.16) and (6.17) hold with at least one of b_{ij}, b_{ji}, c_{ij} and c_{ji} is nonzero for all $i, j \in \langle n \rangle$, $i < j$. □

In view of Corollaries 6.9 and 6.13, the first part of the following theorems are translations of the results of the previous sections into terms of matrices.

Theorem 6.19. *Let $n \geq k > 1$, except for $n = k = 4$, let F be a field with characteristic p where either $p = 0$ or $p > n^2 - n$, and let L be a covering linear transformation on F^{nn}. Then $L(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k$ if and only if L is a composition of one or more of the following types of transformations:*

(1) *$A \to P^{\mathrm{T}}AP$, in which P is a permutation matrix (permutation similarity).*

(2) *$A \to A + F$, where F is a diagonal matrix whose diagonal entries are linear combinations of the entries of A.*

(3) *A linear transformation T satisfying*

$$T(E_{ii}) = b_{ii}E_{ii}, \quad b_{ii} \neq 0, \quad i \in \langle n \rangle,$$

and

$$T(E_{ij}) = b_{ij}E_{ij} + c_{ij}E_{ji}, \quad i, j \in \langle n \rangle, \quad i \neq j,$$

where at least one of b_{ij}, b_{ji}, c_{ij} and c_{ji} is nonzero for all $i, j \in \langle n \rangle$, $i < j$.

Furthermore, the following are equivalent:

(i) *$L(\mathcal{M}\mathcal{G}_n^k) = \mathcal{M}\mathcal{G}_n^k$;*

(ii) *L is nonsingular and $L(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k$.*

Proof. By Lemma 6.3, L is a covering transformation if and only if L_g is a graph covering mapping, and by Corollary 6.9, (6.10) and (6.11) are equivalent. Therefore, we have $L(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k$ if and only if L_g is a composition of mappings of the types listed in Theorem 3.19. Observe that L_g is a vertex permutation if and only if after a permutation similarity we have $L_g(G) = G$ for all $G \in \mathcal{G}_n$. Therefore it follows, using Lemma 6.14, that L_g is a vertex permutation if and only if L is a composition of the types 1 and 3 in our theorem. Similarly L_g is of type 2 in Theorem 3.19 if and only if L is a composition of types 2 and 3 in our theorem.

We now prove the equivalence statement in the second part of the theorem.

(i) $\Rightarrow$ (ii). Since $k > 1$, it follows that

$$E_{ij} \in \mathcal{M}\mathcal{G}_n^k, \quad \forall i, j \in \langle n \rangle.$$

Therefore, $L(\mathcal{M}\mathcal{G}_n^k) = \mathcal{M}\mathcal{G}_n^k$ implies that $L(F^{nn}) = F^{nn}$ which is possible only if L is nonsingular.

(ii) $\Rightarrow$ (i). Assume that

$$L(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k. \tag{6.20}$$

Observe that permutation similarity is a nonsingular transformation which clearly maps $\mathcal{M}\mathcal{G}_n^k$ onto itself. Hence, it view of the first part of the theorem it is enough to consider a nonsingular L which is a composition of transformations of types 2 and 3. Consider the $n^2 \times n^2$ matrix $\tilde{L}$ which represents L with respect to the following basis of F^{nn}

$$\{E_{11}, E_{22}, \ldots, E_{nn}, E_{12}, E_{21}, E_{13}, E_{31}, \ldots, E_{n-1,n}, E_{n,n-1}\}.$$

We partition $\tilde{L}$ as

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}.$$

where $\tilde{L}_{11}$ is $n \times n$. It is easy to verify that L is a composition of types 2 and 3 if and only if $\tilde{L}_{21} = 0$ and $\tilde{L}_{22}$ is a block diagonal matrix with nonzero 2×2 blocks along the diagonal. Since L is nonsingular if follows that $\tilde{L}$ is invertible. Partition $\tilde{L}^{-1}$ in accordance to the partitioning of $\tilde{L}$. Observe that $\tilde{L}_{21}^{-1} = 0$ and that $\tilde{L}_{22}^{-1}$ is a block diagonal matrix with nonzero 2×2 blocks along the diagonal. Since $\tilde{L}^{-1} = (\widetilde{L^{-1}})$ it now follows that L^{-1} too is a composition of types 2 and 3 and hence $L^{-1}(\mathcal{M}\mathcal{G}_n^k) \subseteq \mathcal{M}\mathcal{G}_n^k$. Together with (6.20) we have (i). □

Theorem 6.21. *For* $n = k = 4$ *Theorem* 6.19 *remains valid if the following transformation is added to the list*:

(4) *The linear transformation* T *defined by*

$$T(E_{14}) = E_{23}, \quad T(E_{41}) = E_{32}, \quad T(E_{23}) = E_{14}, \quad T(E_{32}) = E_{41},$$

and

$$T(E_{ij}) = E_{ij}, \quad \forall i, j \in \langle 4 \rangle, \quad \{i, j\} \neq \{1, 4\}, \{2, 3\}.$$

Proof. The first part of the theorem (as stated in Theorem 6.19) follows from Theorem 3.20 and Corollary 6.9, observing that L_g is of type 3 in Theorem 3.20 if and only if L is a composition of the transformation 4 and a transformation of the type 3 in Theorem 6.19. As to the second part of the theorem (the equivalence statement) observe that in view of Theorem 6.19 and its proof, all we have to show is that the nonsingular transformation T defined in 4 satisfies $T(\mathcal{MG}_4^4) = \mathcal{MG}_4^4$. Observe that $T^{-1} = T$ and hence $T^{-1}(\mathcal{MG}_4^4) \subseteq \mathcal{MG}_4^4$. Since $T(\mathcal{MG}_4^4) \subseteq \mathcal{MG}_4^4$ it follows that $T(\mathcal{MG}_4^4) = \mathcal{MG}_4^4$. □

Theorems 6.19 and 6.21 do not hold for noncovering transformations as demonstrated by the following example, based on Example 3.24.

Example 6.22. Let T be any linear transformation on F^{nn} and let H be any matrix in $\mathcal{MG}_n^k$. The linear transformation L on F^{nn} defined by

$$L(A) = T(A) \circ H, \quad \forall A \in F^{nn},$$

satisfies $L(\mathcal{MG}_n^k) \subseteq \mathcal{MG}_n^k$, but L is not a composition of transformations of the types described in Theorems 6.19 and 6.21.

Theorem 6.23. *Let F be a field with characteristic p where either $p = 0$ or $p > n^2 - n$, and let L be a linear transformation on F^{nn} such that for all $i, j \in \langle n \rangle$, $i \neq j$ there exist $s, t \in \langle n \rangle$, $s \neq t$ satisfying $(L(E_{st}))_{ij} \neq 0$ or $L((E_{st}))_{ji} \neq 0$. Then $L(\mathcal{MG}_n^1) \subseteq \mathcal{MG}_n^1$ if and only if L is a composition of one or more of the following types of transformations*:

(1) *Permutation similarity*;

(2) $A \rightarrow A + F$, *where F is a matrix whose entries are linear combinations of the diagonal entries of A*;

(3) *A linear transformation T satisfying*

$$T(E_{ii}) = b_{ii}E_{ii}, \quad b_{ii} \neq 0, \quad i \in \langle n \rangle,$$

and

$$T(E_{ij}) = b_{ij}E_{ij} + c_{ij}E_{ji}, \quad i, j \in \langle n \rangle, \quad i \neq j,$$

where at least one of b_{ij}, b_{ji}, c_{ij} and c_{ji} is nonzero for all $i, j \in \langle n \rangle$, $i < j$.

Furthermore, the following are equivalent

(i) $L(\mathcal{MG}_n^1) = \mathcal{MG}_n^1$.

(ii) $L(\mathcal{MG}_n^1) \subseteq \mathcal{MG}_n^1$ *and*

$$N(L) \cap \{A \in F^{nn}: a_{ii} = 0, \forall i \in \langle n \rangle\} = \{0\},$$

where $N(L)$ denotes the null space (kernel) of L.

Proof. The proof of the first part of the theorem is similar to the proof of Theorem 6.19, using Theorem 3.25 and observing that L satisfies the covering condition of the theorem if and only if L_g satisfies

$$L_g\Big(\bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij}\Big) \supseteq \bigcup_{\substack{i,j=1\\ i<j}}^{n} G_{ij}.$$

In order to prove the second part of the theorem we denote

$$S=\{A\in F^{nn}\colon a_{ii}=0,\ \forall i\in\langle n\rangle\}$$

and we define $\bar{L}$ to be the restriction of L to the subspace S. Observe that if

$$L(\mathcal{M}\mathcal{G}_n^1)\subseteq \mathcal{M}\mathcal{G}_n^1, \tag{6.24}$$

then

$$(L(E_{ij}))_{ll}=0,\quad \forall i,j,l\in\langle n\rangle,\quad i\neq j.$$

Therefore, if (6.24) holds then $l(S)\subseteq S$, so $\bar{L}$ can be considered as a linear transformation on S. Moreover, since $\mathcal{M}\mathcal{G}_n^1\subseteq S$ it follows that

$$L(\mathcal{M}\mathcal{G}_n^1)=\mathcal{M}\mathcal{G}_n^1 \Leftrightarrow \bar{L}(\mathcal{M}\mathcal{G}_n^1)=\mathcal{M}\mathcal{G}_n^1. \tag{6.25}$$

We now prove the equivalence statement.

(i) ⇒ (ii). Since

$$E_{ij}\in\mathcal{M}\mathcal{G}_n^1,\quad \forall i,j\in\langle n\rangle,\quad i\neq j,$$

it follows from (i) that $\bar{L}(S)=S$ so $\bar{L}$ is nonsingular which implies $N(L)\cap S=\{0\}$.

(ii) ⇒ (i). As in the proof of the corresponding part of Theorem 6.19 we may assume that L is a composition of transformations of types 2 and 3. It is easy to verify that L is such if and only if $\bar{L}_{12}=0$ and $\bar{L}_{22}$ is a block diagonal matrix with nonzero 2×2 blocks along the diagonal. Define a linear transformation T on F^{nn} by

$$T(E_{ii})=E_{ii},\quad i\in\langle n\rangle,\qquad T(A)=L(A),\quad \forall A\in S. \tag{6.26}$$

Observe that $\bar{T}_{12}=0$ and $\bar{T}_{22}=\bar{L}_{22}$. Hence T is composition of types 2 and 3 and by the first part of the theorem

$$T(\mathcal{M}\mathcal{G}_n^1)\subseteq\mathcal{M}\mathcal{G}_n^1. \tag{6.27}$$

Furthermore, by (6.26) if $T(A)=0$, then $A\in S$, but then it follows from (ii) that $A=0$. Therefore T is nonsingular. Since $\bar{T}_{12}^{-1}=0$ and $\bar{T}_{22}^{-1}$ is a block diagonal matrix with nonzero 2×2 blocks along the diagonal, it follows that

$$T^{-1}(\mathcal{M}\mathcal{G}_n^1)\subseteq\mathcal{M}\mathcal{G}_n^1. \tag{6.28}$$

By (6.27) and (6.28) we have $T(\mathcal{M}\mathcal{G}_n^1)=\mathcal{M}\mathcal{G}_n^1$, and by (6.25) (with respect to T) we obtain

$$\bar{L}(\mathcal{M}\mathcal{G}_n^1)=\bar{T}(\mathcal{M}\mathcal{G}_n^1)=\mathcal{M}\mathcal{G}_n^1.$$

By (6.25) (with respect to L) we now have $L(\mathcal{M}\mathcal{G}_n^1)=\mathcal{M}\mathcal{G}_n^1$. □

Theorem 6.23 does not hold without the covering requirement on L as demonstrated by the following example, based on Example 3.26.

Example 6.29. Let $n = 3$ and let L be defined by

$$L(E_{ij}) = \begin{cases} E_{11} + E_{12} + E_{13} + E_{23}, & i = j = 1 \\ E_{12}, & \text{otherwise}. \end{cases}$$

Observe that $L(\mathcal{M}\mathcal{G}_3^1) \subseteq \mathcal{M}\mathcal{G}_3^1$, but although L is a covering transformation L is not a composition of transformations of the types in Theorem 6.23. This is because L does not fulfill the special covering requirement of Theorem 6.23.

Theorem 6.30. *Let $n > k \geq 3$, let F be a field with characteristic p where either $p = 0$ or $p > n^2 - n$, and let L be a strongly covering linear transformation on F^{nn}. Then $L(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k$ if and only if L is a composition of one or more of the following types of transformations*:

(1) *Permutation similarity*;

(2) $A \to A^T$ (transposition);

(3) $A \to A + F$, *where F is a diagonal matrix whose diagonal entries are linear combinations of the entries of A*;

(4) $A \to K \circ A$, *where $K \in F^{nn}$ and all the entries of K are nonzero.*

Furthermore, the following are equivalent:

(i) $L(\mathcal{M}\mathcal{D}_n^k) = \mathcal{M}\mathcal{D}_n^k$;

(ii) $L(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k$ *and L is nonsingular*;

(iii) $L(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k$ *and*

$$N(L) \cap \{A \in F^{nn} : a_{ij} = 0, \forall i, j \in \langle n \rangle, i \neq j\} = \{0\}.$$

Proof. By Lemma 6.3, L is a strongly covering transformation if and only if L_d is a digraph covering mapping, and by Corollary 6.13 we have

$$L(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k \tag{6.31}$$

if and only if $L_d(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^k$. Therefore, (6.31) holds if and only if L_d is a composition of transformations of the types listed in Theorem 4.18. Since we have

$$L_d(D) = D, \quad \forall D \in \mathcal{D}_n \tag{6.32}$$

if and only if

$$L_d(D_{ij}) = D_{ij}, \quad \forall i, j \in \langle n \rangle,$$

it follows from Definition 2.21 that (6.32) holds if and only if $L(A) = K \circ A$, where $K \in F^{nn}$ and all the entries of k are nonzero. Observe that L_d is a vertex permutation if and only if after a permutation similarity we have (6.32). Therefore, L_d is a vertex permutation if and only if L is a composition of the types 1 and 4 in our theorem. Similarly, L_d is of type 2 [type 3] in Theorem 4.18 if and only if L is a composition of types 2 and 4 [types 3 and 4] in our theorem.

We now prove the equivalence statement the second part of the theorem.

(i) ⇒ (ii). Since $k > 1$, it follows that

$$E_{ij} \in \forall \mathcal{D}_n^k, \quad \forall i, j \in \langle n \rangle.$$

Therefore, statement (i) implies that $L(F^{nn}) = F^{nn}$ which is possible only if L is nonsingular.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). Assume that (iii) holds. Observe that permutation similarity as well as transposition are nonsingular transformations which map $\mathcal{M}\mathcal{D}_n^k$ onto itself. Hence, in view of the first part of the theorem it is enough to consider a transformation L which is a combination of transformations of types 3 and 4. Clearly, L is such a transformation if and only if $\tilde{L}_{21} = 0$ and $\tilde{L}_{22}$ is a nonsingular diagonal matrix. Since $\tilde{L}_{21} = 0$ it follows from (iii) that $\tilde{L}_{11}$ is nonsingular, and hence $\tilde{L}$ is invertible. Observe that $\tilde{L}_{21}^{-1} = 0$ and $\tilde{L}_{22}^{-1}$ is a nonsingular diagonal matrix. Thus, L^{-1} is also a composition of transformations of types 3 and 4 and so $L^{-1}(\mathcal{M}\mathcal{D}_n^k) \subseteq \mathcal{M}\mathcal{D}_n^k$. Together with (iii) we now obtain (i). □

Theorem 6.30 does not hold in the case that $k < 3$ as demonstrated by the following example, based on Example 4.22.

Example 6.33. Let L be the strongly covering transformation defined by

$$L(E_{12}) = \sum_{\substack{i,j=1\\ i<j}}^{n} E_{ij}, \qquad L(E_{21}) = \sum_{\substack{i,j=1\\ i>j}}^{n} E_{ij},$$

and

$$L(E_{ij}) = 0, \quad \{i, j\} \neq \{1, 2\}.$$

Clearly, $L(\mathcal{M}\mathcal{D}_n^2) \subseteq \mathcal{M}\mathcal{D}_n^1$, although L is not a composition of transformations of the types specified in Theorem 6.30.

In order to show that Theorem 6.30 does not hold for transformations which are not strongly covering one can use Example 6.22, where H is chosen to be in $\mathcal{M}\mathcal{D}_n^k$ (instead of $\mathcal{M}\mathcal{G}_n^k$).

Theorem 6.34. *For $n = k \geq 4$ Theorem* 6.30 *remains valid if the following type of transformation is added to the list*:

(5) *A linear transformation T defined as follows*:

For any permutation σ on $\langle n \rangle$ we have

$$T(E_{i,i+1}) = \sum_{\substack{j=1\\ j\neq\sigma(i)}}^{n} E_{\sigma(i),j}, \quad i = 1, \ldots, n-1,$$

$$T(E_{n1}) = \sum_{\substack{j=1\\ j\neq\sigma(n)}}^{n} E_{\sigma(n),j}, \qquad T(E_{ii}) = E_{ii}, \quad \forall i \in \langle n \rangle,$$

and

$$T(E_{ij})=0, \quad \forall ij \in \langle n \rangle, \quad (i,j) \notin \{1, 2, (2, 3), \\ \dots, (n-1, n), (n, 1)\}, \quad i \neq j.$$

Proof. Observe that after an appropriate vertex permutation the set $\{(s_1, t_1), \dots, (s_n, t_n)\}$ in theorem 5.30 becomes $\{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$. Hence, the mapping L_d is of type 4 in Theorem 5.30 if and only if the transformation L is a composition of transformations of types 1, 4 (in Theorem 6.30) and 5 in our theorem. The rest of the proof of the first part of the theorem follows as in Theorem 6.30, using Theorem 5.30 instead of Theorem 4.18.

As to the second part of the theorem, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) follow as in Theorem 6.30. To prove (iii)$\Rightarrow$(i) observe that any composition of transformations of types 1–5 which contains a transformation of type 5 vanishes at diagonal matrices, and thus such a transformation does not satisfy (iii). Therefore, if L satisfies (iii), then it is a composition of transformations of types 1–4 and we prove the implication (iii)$\Rightarrow$(i) as in Theorem 6.30. □

Theorem 6.34 does not hold for transformations which are not strongly covering as demonstrated by Example 6.22 (choosing $H \in \mathcal{MD}_n^k$).

As observed in Section 5, Theorem 4.18 holds also in the case $n=k=2$, and hence Theorem 6.30 holds in that case. For $n=k=3$, in spirit of the corresponding discussion in Section 5, we have 144 different strongly covering linear transformations L satisfying $L(\mathcal{MD}_3^3) \subseteq \mathcal{MD}_3^3$, besides transformations of the types 3 and 4 in Theorem 6.30.

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