ELSEVIER

Topology and its Applications 74 (1996) 141–148

TOPOLOGY AND ITS APPLICATIONS

A game on partial orderings [☆]

Sakaé Fuchino [a,*], Sabine Koppelberg [a], Saharon Shelah [b]

[a] *Institut für Mathematik II, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany*
[b] *Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel*
Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

Received 9 May 1995; revised 24 October 1995

Abstract

We study the determinacy of the game $G_\kappa(A)$ introduced in Fuchino, Koppelberg and Shelah (to appear) for uncountable regular κ and several classes of partial orderings A. Among trees or Boolean algebras, we can always find an A such that $G_\kappa(A)$ is undetermined. For the class of linear orders, the existence of such A depends on the size of $\kappa^{<\kappa}$. In particular we obtain a characterization of $\kappa^{<\kappa} = \kappa$ in terms of determinacy of the game $G_\kappa(L)$ for linear orders L.

Keywords: Freese-Nation property; Games; Trees; Linear orders; Boolean algebras

AMS classification: 03E05

We consider in this paper the question whether for every partially ordered set $(A, \leqslant)$, the game $G_\kappa(A)$ described below is determined, i.e., whether one of the players has a winning strategy. Here and in the following, except for the motivation given below, κ is always a regular uncountable cardinal. More precisely we study the question for trees, Boolean algebras and linear orderings. In fact there are trees, respectively Boolean algebras, A of size κ^+ for which $G_\kappa(A)$ is not determined (Propositions 6 and 11); for linear orders, the situation is more complex: if $\kappa^{<\kappa} = \kappa$, then for every linear order L, $G_\kappa(L)$ is determined (Proposition 2); otherwise there is a linear order L of size κ^+ such that $G_\kappa(L)$ is not determined (Proposition 8).

☆ This research was begun when the first author was working at the Hebrew University of Jerusalem. He would like to thank The Israel Academy of Science and Humanities for enabling his stay there. The second author was partially supported by the Deutsche Forschungsgemeinschaft (DFG) Grant No. Ko 490/7-1. The third author would like to thank "Basic Research Foundation" of The Israel Academy of Science and Humanities and the Edmund Landau Center for research in Mathematical Analysis supported by the Minerva Foundation (Germany) for their support.

* Corresponding author.

SSDI 0166-8641(96)00051-X

The motivation for this question comes from the paper [1] which in turn was motivated by [2]. A Boolean algebra A is said to have the Freese–Nation property if there exists a function f which assigns to every $a \in A$ a finite subset $f(a)$ of A such that if $a, b \in A$ satisfy $a \leqslant b$, then $a \leqslant x \leqslant b$ holds for some $x \in f(a) \cap f(b)$. This property is closely related to projectivity; in fact, every projective Boolean algebra has the Freese–Nation property (but not conversely). Heindorf proved that the Freese–Nation property is equivalent to open-generatedness, a notion originally introduced in topology by Ščepin. In [1], it is generalized to from ω to regular cardinals κ and from Boolean algebras to arbitrary partial orderings. This generalization is called κ-Freese–Nation property and the following equivalence was proved: a partial ordering A has the κ-Freese–Nation property iff there is a closed unbounded subset $\mathbb{C}$ of $[A]^\kappa$ such that $C \leqslant_\kappa A$ holds (see below for the definition) for all $C \in \mathbb{C}$ iff in the game $G_\kappa(A)$, Player II has a winning strategy. In fact, in all examples considered in [1], either I or II has a winning strategy.

Let us define the game $G_\kappa(A)$ and some relevant notions for a partial ordering A. $X \subseteq A$ is said to be cofinal (coinitial) in A if, for every $a \in A$, there is some $x \in X$ such that $a \leqslant x$ ($a \geqslant x$). $\operatorname{cf} A$ ($\operatorname{ci} A$ respectively) is the smallest cardinality of a cofinal (coinitial respectively) subset of A.

For $R \subseteq A$ and $a \in A$, we write $R \uparrow a$ for the set $\{x \in R\colon\ a \leqslant x\}$ and $R \downarrow a$ for $\{x \in R\colon\ x \leqslant a\}$. The *type* of a over R is the pair

$$\operatorname{tp}(a, R) = (\operatorname{cf} R \downarrow a, \operatorname{ci} R \uparrow a).$$

$R \subseteq A$ is said to be a κ-subset or a κ-substructure of A, written $R \leqslant_\kappa A$, if for all $a \in A$, the sets $R \downarrow a$ and $R \uparrow a$ have cofinality respectively coinitiality less than κ.

The game $G_\kappa(A)$ is played on A as follows. Players I and II alternately choose terms of an increasing chain of subsets x_α and y_α of A for $\alpha < \kappa$ (i.e., I chooses x_0, II chooses y_0, I chooses x_1, II chooses y_1, etc.) such that x_α and y_α have size less than κ, $x_\alpha \subseteq y_\alpha$ and $\bigcup_{\nu<\alpha} y_\nu \subseteq x_\alpha$. In the end of a play, II wins iff the result $R = \bigcup_{\alpha<\kappa} x_\alpha = \bigcup_{\alpha<\kappa} y_\alpha$ of the play is a κ-subset of A; I wins otherwise.

Note that in this game, Player II has a winning strategy for any partial ordering A of size at most κ: she plays so that every element of A is gradually captured in one of the y_α's.

The main body of the paper is organized as follows. In 5 we define a tree $T = T(S)$ depending on a subset S of $\lambda = \kappa^+$. If neither S nor $\lambda \setminus S$ are in the ideal I_λ defined in 3 then T is not determined (Proposition 6). From T we define a linear order L_T in 7. and a Boolean algebra B_T in 10. such that $G_\kappa(L_T)$ and $G_\kappa(B_T)$ are not determined (Propositions 8 and 11). The construction of L_T requires the extra assumption $\kappa^{<\kappa} > \kappa$ — cf. Proposition 2.

Let us start with an easy example.

Example 1. If κ^+ (or $(\kappa^+)^*$, the reverse order type of κ^+) embeds into A, then Player I has a winning strategy in $G_\kappa(A)$: assume, for simplicity, that $\kappa^+ \subseteq A$. We define a partial function f from A into κ^+ by letting $f(a)$ for $a \in A$ be the least $\alpha \in \kappa^+$ such that $a \leqslant \alpha$, if such an α exists. Clearly f is order preserving and satisfies $f(a) = a$ for $a \in \kappa^+$. Player I wins by assuring that the result R of a play satisfies

(a) $R \cap \kappa^+$ has cofinality κ,

(b) if $a \in R$ and $f(a)$ exists, then $f(a) \in R$.

The following proposition shows that the assumption $\kappa^{<\kappa} > \kappa$ in 6 and 7 cannot be dispensed with.

Proposition 2. *Assume that $\kappa^{<\kappa} = \kappa$. If $(L, <_L)$ is a linear order of cardinality $> \kappa$, then Player I has a winning strategy in $G_\kappa(L)$. Hence the game $G_\kappa(L)$ is determined for any linear order L whenever $\kappa^{<\kappa} = \kappa$.*

Proof. Let χ be sufficiently large. $\mathcal{H}(\chi)$ denotes the set of all sets which are hereditarily of size less than χ. We show:

Claim. *Suppose M is an elementary submodel of $(\mathcal{H}(\chi), \in)$ such that $(L, <_L) \in M$ and ${}^{<\kappa}M \subseteq M$. Then for any $d \in L \setminus M$, either $L \cap M \downarrow d$ has cofinality $\geqslant \kappa$ or $L \cap M \uparrow d$ has coinitiality $\geqslant \kappa$.*

Proof of Claim. Otherwise, some $d \in L \setminus M$ fills a gap (X, Y) in $L \cap M$ such that $|X|, |Y| < \kappa$ and (X, Y) is unfilled inside $L \cap M$. But $(X, Y) \in M$ by ${}^{>\kappa}M \subseteq M$ and $M \prec \mathcal{H}(\chi)$, a contradiction. □

Now Player I wins in $G_\kappa(L)$ by choosing an increasing sequence M_α, $\alpha < \kappa$, of elementary submodels of $\mathcal{H}(\chi)$ along with his moves x_α, $\alpha < \kappa$, such that $(L, <_L) \in M_0$, $x_\alpha \subseteq M_\alpha$, ${}^{<\kappa}M_\alpha \subseteq M_\alpha$, $|M_\alpha| = \kappa$ and $\bigcup_{\alpha<\kappa} x_\alpha = M \cap L$ where $M = \bigcup_{\alpha<\kappa} M_\alpha$. Such a choice is possible because of our assumption $\kappa^{<\kappa} = \kappa$. The result of the game $L \cap M$ is not a κ-subset of L, by Claim above. □

Construction 3 (of the ideal I_λ). For the rest of the paper, fix $\lambda = \kappa^+$ (where κ was a regular uncountable cardinal). Let us first recall the definition and some properties of the ideal I_λ on λ introduced by Shelah, see, e.g., [4, Chapter VIII]. Fix a sufficiently large cardinal $\chi > \lambda$; we work in the structure $(\mathcal{H}(\chi), \in, <^*)$ where $<^*$ is some fixed well-ordering of $\mathcal{H}(\chi)$. For $x \in \mathcal{H}(\chi)$ and $\gamma < \lambda$, call $(M_i)_{i<\kappa}$ an x-approximation of γ if:

(1) $M_i \prec (\mathcal{H}(\chi), \in, <^*)$, $|M_i| < \kappa$,

(2) $x, \lambda \in M_0$,

(3) $(M_i)_{i<\kappa}$ is a continuously increasing chain,

(4) $(M_i)_{i\leqslant j} \in M_{j+1}$ for all $j < \kappa$,

(5) $M = \bigcup_{i<\kappa} M_i$ satisfies $M \cap \lambda = \gamma$.

For $x \in \mathcal{H}(\chi)$, put $C_x = \{\gamma \in \lambda\colon$ there is an x-approximation of $\gamma\}$ and define I_λ by

$$I_\lambda = \{A \subseteq \lambda\colon\ A \cap C_x = \emptyset,\ \text{for some } x \in \mathcal{H}(\chi)\}.$$

Then I_λ is a λ-complete proper ideal containing all singletons and

$$N = \{\gamma \in \lambda\colon\ \mathrm{cf}\,\gamma = \kappa\} \in {I_\lambda}^*$$

(i.e., $\lambda \setminus N \in I_\lambda$). By Ulam's Theorem (cf. [3, 27.8]), every $A \subseteq \lambda$ not in I_λ can be represented as the disjoint union $A = A_1 \cup A_2$ where $A_1, A_2 \notin I_\lambda$.

Game 4 ($G_\kappa(T)$ for a tree T). Assume that $(T, <_T)$ is a tree of height $\kappa + 1$. We call $Y \subseteq T$ a *subtree of* T if for all $y \in Y$: if $x <_T y$ then $x \in Y$. Y is *closed in* T if the following holds: if $x \in T$ is in the κth level and all predecessors of x are in Y, then $x \in Y$.

In $G_\kappa(T)$ each of the players can ensure that the result Y of a play will be a subtree of T. In this case Player II wins (i.e., $Y \leqslant_\kappa T$) iff Y is closed in T.

Construction 5 (of the tree $T = T(S)$). Recall that $\lambda = \kappa^+$ and $N = \{\gamma \in \lambda\colon \operatorname{cf}\gamma = \kappa\}$. For each subset S of N, we construct a tree $T = T(S)$; in fact, we shall show that if $T = T(S)$ where $S \subseteq N$ and S, $N \setminus S \notin I_\lambda$, then none of the players has a winning strategy.

Assume $S \subseteq N$. For each $\gamma \in S$, fix a function $f_\gamma : \kappa \to \gamma$ such that range f_γ is cofinal in γ. Let

$$T = T(S) = \{f_\gamma \restriction \alpha\colon \gamma \in S,\ \alpha \leqslant \kappa\},$$

a tree under set-theoretic inclusion. Clearly T has height $\kappa + 1$ if S is nonempty, $\{f_\gamma\colon \gamma \in S\}$ is the κth level of T, and $|T| = \lambda$ if $|S| = \lambda$.

Proposition 6. *Let $T = T(S)$ for $S \subseteq N$.*

(a) *If $S \notin I_\lambda$, then Player II has no winning strategy in $G_\kappa(T)$.*

(b) *If $N \setminus S \notin I_\lambda$, then Player I has no winning strategy in $G_\kappa(T)$.*

Thus if both S and $N \setminus S$ are not in I_λ, then the game $G_\kappa(T)$ is undetermined.

Proof. (a) Suppose that σ is a strategy for Player II; we show that it is not a winning strategy. Let $x = (\sigma, (f_\gamma)_{\gamma \in S})$. Since $S \notin I_\lambda$, there is a $\delta \in S \cap C_x$; let $(M_i)_{i<\kappa}$ be an x-approximation of δ. In a game in which Player II plays according to σ, Player I can ensure that the result $Y \subseteq T$ of the play will be the subtree

$$Y = \{f_\gamma \restriction \alpha\colon \gamma \in S \cap \delta,\ \alpha \leqslant \kappa\}.$$

More precisely, in the ith move, Player I may take a subset x_i of $T \cap M_{i+1}$ so that all elements of Y are gradually captured. Furthermore, using the well-ordering $<^*$, Player I can ensure that each of his moves x_i is definable so that $(x_j, y_k)_{j \leqslant i,\, k<i}$ and hence also the next move $\sigma((x_j, y_k)_{j \leqslant i,\, k<i})$ by Player II will be an element of M_{i+1}.

Now $\delta \in S$ and thus f_δ witnesses that Y is not closed in T, i.e., Player I wins.

The proof of (b) is similar to (a). If Player I plays according to a strategy τ, Player II can assure that the result $Y \subseteq T$ has the form $Y = \{f_\gamma \restriction \alpha\colon \gamma \in S \cap \delta,\ \alpha \leqslant \kappa\}$ for some $\delta \in N \setminus S$. Thus Y is closed in T and Player II wins. □

Construction 7 (of the linear order L_T). Assume that $\kappa^{<\kappa} > \kappa$; let $(T, <_T)$ be any tree of height $\kappa + 1$ and size $\lambda = \kappa^+$. We shall construct a linear order $L = L_T$ of size λ. Moreover, we shall define for each $Y \subseteq T$ a subset L_Y of L such that $|L_Y| = |Y|$ holds for infinite Y and in the game $G_\kappa(L)$, each player can ensure that the result R has the form L_Y for Y a subtree of T.

Let us first note that there exists a linear order I of size λ without any sequences (i.e., increasing or decreasing sequences) of type κ. This holds because our assumption $\kappa^{<\kappa} \geqslant \kappa^+ = \lambda$ implies that $\lambda \leqslant 2^\mu$, for some $\mu < \kappa$, and the lexicographic ordering on ${}^\mu 2$ has no sequence of type μ^+ (cf. [3, 29.4]), hence no sequence of type κ. It follows that, letting I be any subordering of ${}^\mu 2$ of cardinality λ, every subset of I has cofinality and coinitiality less than κ.

The following notation concerning the tree $(T, <_T)$ will be used in the rest of 7 and in 8: for $\alpha \leqslant \kappa$, $\mathrm{lev}_\alpha T$ is the αth level of T. For $t \in T$, $\mathrm{pred}\, t$ is the set of predecessors of t in T and $\mathrm{ht}\, t$ is the height of t. For $\alpha \leqslant \mathrm{ht}\, t$, $\mathrm{pr}_\alpha t$, the projection of t to level α, is the unique predecessor of t in the αth level. Call $x, y \in T$ equivalent and write $x \sim y$ if $\mathrm{pred}\, x = \mathrm{pred}\, y$ and let $\overline{x}$ be the equivalence class of x. For each equivalence class $\overline{x}$, since $|\overline{x}| \leqslant \lambda$, we can fix a linear order $\leqslant_{\overline{x}}$ on $\overline{x}$ without any sequences of type κ.

The linear order we construct is a sort of squashing of T with respect to $\leqslant_{\overline{x}}$, $x \in T$: we put $L = \{a_t, b_t \colon t \in T\}$ where the elements a_t, b_t, $t \in T$, are all pairwise distinct. The linear order $<_L$ on L is defined as follows: we will have $a_t <_L b_t$ for all $t \in T$. Now assume $x, y \in T$. If $x <_T y$, then we put $a_x <_L a_y <_L b_y <_L b_x$. If x and y are incomparable in T, let $\alpha \leqslant \kappa$ be minimal such that $\mathrm{pr}_\alpha x \neq \mathrm{pr}_\alpha y$; thus $\mathrm{pr}_\alpha x \sim \mathrm{pr}_\alpha y$. Then if $\mathrm{pr}_\alpha x <_{\overline{\mathrm{pr}_\alpha x}} \mathrm{pr}_\alpha y$, we let $a_x <_L b_x <_L a_y <_L b_y$. Finally, for $Y \subseteq T$ let $L_Y = \{a_t, b_t \colon t \in Y\}$.

Proposition 8. *If Y is a subtree of T, then $L_Y \leqslant_\kappa L_T$ iff Y is closed in T. In particular, if $G_\kappa(T)$ is undetermined, then so is $G_\kappa(L_T)$.*

From Propositions 2, 6 and 8 (plus the observation in 7 that $\kappa^{<\kappa} > \kappa$ implies the existence of a linear order of size λ without sequences of type κ), we obtain the following equivalences to the condition $\kappa^{<\kappa} = \kappa$.

Corollary 9. *Let κ be a regular uncountable cardinal.*

(a) *If $\kappa^{<\kappa} > \kappa$, then there is a linear order L of cardinality $\lambda = \kappa^+$ such that $G_\kappa(L)$ is undetermined.*

(b) *The following are equivalent:*

(1) $\kappa^{<\kappa} = \kappa$;

(2) *in every linear order of cardinality $> \kappa$, there is an increasing or a decreasing sequence of order type κ;*

(3) *$G_\kappa(L)$ is determined for every linear order L.*

Let us explain how the second assertion of Proposition 8 follows from the first one: each of the players in $G_\kappa(L_T)$ (say II, playing against some strategy τ of Player I) can ensure that the result of the play is $R = L_Y$, for some subtree Y of T. Playing simultaneously on T as in the proof of Proposition 6, II can ensure that Y is closed in T. Thus $R = L_Y$ is a κ-substructure of L_T and she wins.

Proof of Proposition 8. Suppose first that Y is not closed. Pick some t in the highest level K of T such that $t \notin Y$ but $\operatorname{pred} t \subseteq Y$. Then $\{a_y\colon y \in \operatorname{pred} t\}$ is an increasing sequence of type κ, and it is a cofinal subset of $L_Y \downarrow l$ where $l = a_t$. Thus L_Y is not a κ-subset of L.

Now assume that Y is closed in T. Fix $l \in L \setminus L_Y$. We have to analyze the cofinality of $L_Y \downarrow l$ and the coinitiality of $L_Y \uparrow l$; by symmetry, we will consider only $\operatorname{cf}(L_Y \downarrow l)$. Now let $l = a_t$ or $l = b_t$ for some $t \in T \setminus Y$; since Y is a subtree of T, a_t and b_t realize the same cut in L_Y. Thus we assume that $l = a_t$.

We may also assume that $\operatorname{ht} t < \kappa$ and $\operatorname{pred} t \subseteq Y$. For this, consider the least element t^* of $\operatorname{pred} t \setminus Y$. Now $\operatorname{ht} t^* < \kappa$ since Y is a closed subtree of T; moreover, a_t and a_{t^*} realize the same cut in L_Y. Thus we consider t^* instead of t.

To prove $\operatorname{cf}(L_Y \downarrow l) < \kappa$, consider the following subsets of L respectively Y: let

$$N = \{a_x\colon x <_T t\};$$

thus N is a subset of $L_Y \downarrow l$ of size less than κ. Next, put $\gamma = \operatorname{ht} t$ and

$$Y' = \{z \in Y\colon z \in \operatorname{lev}_\gamma T,\ z \sim t,\ z <_{\bar{t}} t\}.$$

Y' is included in the $\sim$-equivalence class of t, thus it has a cofinal subset Y'' of size less than κ. We put

$$N' = \{b_z\colon z \in Y''\},$$

again a subset of $L_Y \downarrow l$ of size less than κ.

We prove that $N \cup N'$ is cofinal in $L_Y \downarrow l$. For, let $x \in L_Y$ and $x <_L l$, say $x \in \{a_y, b_y\}$ where $y \in Y$. Consider the relative position of t and y in T. It is impossible that $t <_T y$, since Y is a subtree of T and $t \notin Y$.

If $y <_T t$, then $a_y <_L a_t = l <_L b_t <_L b_y$ holds, hence $x = a_y \in N$. Otherwise, let α be minimal such that $\operatorname{pr}_\alpha y \neq \operatorname{pr}_\alpha t$; thus $\alpha \leqslant \gamma$.

If $\alpha < \gamma$, then let $z = \operatorname{pr}_\alpha t$; it follows that $x \leqslant_L b_y <_L a_z \in N$. Otherwise $\alpha = \gamma$, $\operatorname{pr}_\alpha y \sim t$ and hence $\operatorname{pr}_\alpha y \in Y'$. Take $z \in Y''$ such that $\operatorname{pr}_\alpha y \leqslant_{\bar{t}} z$; then $x \leqslant b_y \leqslant b_z \in N'$. □

Construction 10 (of the Boolean algebra B_T). Let $(T, <_T)$ be any tree of height $\kappa + 1$ and size λ. We shall construct a Boolean algebra B_T of size λ. Moreover, we shall define for $Y \subseteq T$ a subalgebra B_Y of B_T such that $|B_Y| = |Y|$ holds for infinite Y. In the game $G_\kappa(B_T)$, each player can ensure that the result R has the form B_Y for Y a subtree of T.

In fact, we define B_T to be the Boolean algebra generated by a set $\{x_t\colon t \in T\}$ freely except that $s \leqslant_T t$ implies $x_s \leqslant x_t$. More precisely, let $\operatorname{Fr}(x_t\colon t \in T)$ be the free Boolean algebra over $\{x_t\colon t \in T\}$, let B_T be the quotient algebra $\operatorname{Fr}(x_t\colon t \in T)/K$ where K is the ideal of $\operatorname{Fr}(x_t\colon t \in T)$ generated by $\{x_s \cdot -x_t\colon s \leqslant_T t\}$ and let $\pi : \operatorname{Fr}(x_t\colon t \in T) \to B_T$ be the canonical homomorphism. We write x_t ($\in B_T$) for $\pi(x_t)$, since π is one-one on the generators x_t (see the proof of 11 below). For $Y \subseteq T$, we define B_Y to be the subalgebra of B_T generated by $\{x_t\colon t \in Y\}$.

Proposition 11. *If Y is a subtree of T, then $B_Y \leqslant_\kappa B_T$ iff Y is closed in T. In particular, if $G_\kappa(T)$ is undetermined, then so is $G_\kappa(B_T)$.*

Proof. Similarly to the remark after Corollary 9 it is easy to see that the second assertion follows from the first. For the first assertion we start with a normal form lemma on the generators of B_T.

Step 1. Let $w \subseteq T$ be finite and assume $f: w \to 2$. Then the elementary product

$$q_f = \prod_{f(t)=1} x_t \cdot \prod_{f(t)=0} -x_t$$

is nonzero in B_T iff f is monotone, i.e., $s \leqslant_T t$ in w implies $f(s) \leqslant f(t)$. This follows immediately from the definition of the ideal K of $\mathrm{Fr}(x_t\colon\ t \in T)$ in 9.

Step 2. If $Y \subseteq T$ is not closed, then B_Y is not a κ-subalgebra of B_T.

To see this, fix an element t in the highest (i.e., κth) level of T such that $t \notin Y$ but all predecessors of t in T are in Y and consider the ideal $I = B_Y \downarrow x_t$ of B_Y. The set $J = \{x_s\colon\ s <_T t\}$ is a chain of order type κ included in I; we show that J generates I as an ideal. Thus suppose $x \in I$ with the aim of finding some $s <_T t$ such that $x \leqslant x_s$. We may assume that x is a nonzero elementary product q_f where $f: w \to 2$. By $q_f \leqslant x_t$ and Step 1, it follows that f is monotone but $f \cup \{(t, 0)\}$ is not. Hence there is some $s \in w$ such that $s <_T t$ and $f(s) = 1$; thus $x = q_f \leqslant x_s$.

Step 3. The following remark simplifies Step 4: assume B is a Boolean algebra, A a subalgebra and M, N are finite subsets of B such that for all $m \in M$ and $n \in N$, there is an element α of A separating m and n, i.e., we have $m \leqslant \alpha$ and $n \leqslant -\alpha$ or $n \leqslant \alpha$ and $m \leqslant -\alpha$. Then there is an $a \in A$ separating $\sum M$ and $\sum N$: simply let

$$a = \prod_{n\in N} \sum_{m\in M} a_{mn}$$

where $a_{mn} \in A$ is such that $m \leqslant a_{mn}$ and $n \leqslant -a_{mn}$.

Step 4. If Y is a closed subtree of T, then $B_Y \leqslant_\kappa B_T$.

For the proof, fix an element b of B_T and consider the ideal

$$I = \{x \in B_Y\colon\ x \cdot b = 0\}$$

of B_Y. We shall find $Z \subseteq T$ such that $|Z| < \kappa$ and each element of I is separated from b by an element of B_Z; since $|B_Z| < \kappa$, this shows that I is generated by less than κ elements.

Fix a finite subset of T generating b, say

$$b \in \langle x_{s_1}, \dots, x_{s_n}, x_{t_1}, \dots, x_{t_m} \rangle,$$

where every s_i is in Y and every t_j is in $T \setminus Y$. We put

$$Z = \{s_1, \dots, s_n\} \cup \bigcup \{\operatorname{pred} t_j \cap Y\colon\ 1 \leqslant j \leqslant m\},$$

where, for $t \in T$, $\operatorname{pred} t$ is the set of predecessors of t in the tree $(T, <_T)$. Z has size less than κ since Y is closed and a subtree of T.

Now let $x \in I$ with the aim of finding an element of B_Z which separates x and b. By Step 3, we may assume that both b and x are elementary products over the generators of B_T, say

$$b = q_h, \quad h:\{s_1, \ldots, s_n, t_1, \ldots, t_m\} \to 2,$$

$$x = q_f, \quad f: w \to 2,\ w \subseteq Y,$$

where h and f are monotone. Define

$$h' = h \restriction \{s_1, \ldots, s_n\}, \qquad f' = f \restriction (w \cap Z);$$

we show that either $q_{h'}$ or $q_{f'}$ separate x and b.

Case 1. $f \cup h'$ is not a function or not monotone. Then $b \leqslant q_{h'}$ and $x \cdot q_{h'} = 0$.

Note that if Case 1 does not hold, then also $f \cup h$ is a function: otherwise, let $r \in w \cap \{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ be such that $f(r) \neq h(r)$. Then $r \in Y$ and thus $r = s_i$ for some i, hence $r \in \operatorname{dom} f \cap \operatorname{dom} h'$. Note also that, since $x \cdot b = 0$, $f \cup h$ cannot be monotone. Hence the remaining case is the following.

Case 2. $f \cup h'$ is a monotone function and $f \cup h$ is a function but not monotone. In this case, there are $r, u \in T$ such that $r <_T u$ and $f(r) = 1$, $h(u) = 0$. For otherwise, we have $r <_T u$ satisfying $h(r) = 1$, $f(u) = 0$. It follows that $u \in w \subseteq Y$, $r \in Y$ since Y is a subtree of T, and $r \in \operatorname{dom} h'$, contradicting the fact that $f \cup h'$ is monotone.

Now $r \in w \subseteq Y$ and $u \in \{s_1, \ldots, s_n, t_1, \ldots, t_m\}$. In fact, $u = t_j$ for some j, since $u = s_i$ would imply that $u \in \operatorname{dom} h'$, but $f \cup h'$ was monotone. But then $r \in \operatorname{pred} t_j \cap Y \subseteq Z$, $r \in \operatorname{dom} f'$, and $f' \cup h$ is not monotone. Thus $b \cdot q_{f'} = q_h \cdot q_{f'} = 0$ and $x = q_f \leqslant q_{f'}$ show that $q_{f'}$ separates x and b. $\square$

References

[1] S. Fuchino, S. Koppelberg and S. Shelah, Partial orderings with the weak Freese–Nation property, Ann. Pure Applied Logic, to appear.

[2] L. Heindorf and L.B. Shapiro, Nearly Projective Boolean Algebras, Lecture Notes in Mathematics 1596 (Springer, Berlin, 1994).

[3] T. Jech, Set Theory (Academic Press, New York, 1978).

[4] S. Shelah, Cardinal Arithmetic (Oxford Univ. Press, Oxford, 1994).