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The chaos game revisited: Yet another, but a trivial proof of the algorithm's correctness

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ABSTRACT

The aim of this work is to explain why the most popular algorithm for approximating IFS fractals, the chaos game, works. Although there are a few proofs of the algorithm's correctness in the relevant literature, the majority of them utilize notions and theorems of measure and ergodic theories. As a result, paradoxically, although the rules of the chaos game are very simple, the logic underlying the algorithm seems to be hard to comprehend for non-mathematicians. In contrast, the proof presented in this work uses only fundamentals of probability and can be understood by anyone interested in fractals.
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1. Preliminaries

An *iterated function system* (IFS) on a complete metric space (X, d) is a finite collection $\{w_1, \dots, w_N\}$, $N > 1$, of contractions $w_i : X \to X$. Given an IFS, one can define the *Hutchinson operator* W that operates on the metric space $(\mathcal{H}(X), h)$:

$$W(E) = \bigcup_{i=1}^{N} w_i(E), \quad E \in \mathcal{H}(X), \tag{1}$$

where $\mathcal{H}(X)$ denotes the collection of the nonempty compact subsets of X, and h is the Hausdorff metric induced by the metric d.

One can show that if (X, d) is complete, then $(\mathcal{H}(X), h)$ is complete as well, and W is a contraction mapping on it [1]. Hence, on the basis of the Banach fixed-point theorem, there is a set $\mathcal{A} \in \mathcal{H}(X)$ that is the unique fixed point of W:

$$\mathcal{A} = W(\mathcal{A}). \tag{2}$$

The fixed point is called the *attractor* of the IFS.

One of the most popular algorithms for generating a point-set approximation of the IFS attractor is the so-called *chaos game* [1]. The algorithm is as follows: Given an IFS $\{w_1, \dots, w_N\}$, associate with each mapping w_i a nonzero probability $p_i \in (0, 1)$, such that $\sum_{i=1}^{N} p_i = 1$, and do the following iteration:

1. Choose any point $x_0 \in X$.
2. Choose a map w_i from $\{w_1, \dots, w_N\}$ at random according to the probabilities p_i.
3. Compute the point $x_{n+1} = w_i(x_n)$.
4. Return to step 2 and repeat the process with x_{n+1} replacing x_n.

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2. The motivation of the work

One can show that, for any $\varepsilon > 0$, there is a natural number M such that the set $\{x_n\}_{n=M}^{\infty}$ generated by the chaos game approximates the IFS attractor $\mathcal{A}$ with accuracy ε with respect to the Hausdorff metric. However, the formal proofs of this fact usually engage notions and theorems of measure and ergodic theories [2–4], which are usually considered an "esoteric" matter for non-mathematicians, e.g. computer scientists. As a result, paradoxically, although the rules of the chaos game are very simple, the reason why the algorithm works seems to be hard to comprehend for many people who utilize it but are not mathematicians. On the other hand, the paper [5] showed that the mechanism that underlies the chaos game can be explained using only fundamentals of probability. Nevertheless, the paper mentioned covers the chaos game for generalizations of the IFS—the hierarchical IFS [6] and the recurrent IFS [7], treating the ordinary IFS as a special case. Since the considerations proceeded in a top-down manner (starting from the chaos game for hierarchical IFS, then for the recurrent IFS, and finally for the IFS), the simplicity of the proof for the ordinary IFS remained "hidden" behind the intricate notation related to the IFS generalizations. It turns out that the proof of the chaos game correctness takes a much simpler form when the IFS is considered separately from its generalizations.

3. Why does the chaos game work?

The answer to the above question is the proof of the following theorem:

Theorem 1. *Let $\{w_1, \ldots, w_N\}$ be an IFS on a complete metric space (X, d). Let $\mathcal{A}$ be the attractor of the IFS. Let $\{x_n\}_{n=0}^{\infty}$ be a set of points generated by the chaos game, where x_0 is any point of X. Then, with probability 1, for any $\varepsilon > 0$, there is a number $M \in \mathbb{N}$ such that $h(\mathcal{A}, \overline{\{x_n\}_{n=M}^{\infty}}) \leq \varepsilon$. That is, the closure of the set $\{x_n\}_{n=M}^{\infty}$ approximates $\mathcal{A}$ with an error not greater than ε with respect to the Hausdorff metric h.*

Proof. The proof consists of two parts. In the first part we assume that the initial point x_0 belongs to the attractor $\mathcal{A}$ and show that the set $\{x_n\}_{n=M}^{\infty}$ is *dense* in $\mathcal{A}$ for any $M \in \mathbb{N}$—thus $\overline{\{x_n\}_{n=M}^{\infty}} = \mathcal{A}$. Then, in the second part, we use that fact to show the correctness of the chaos game in the more general case in which x_0 is any point of X.

Let $x_0 \in \mathcal{A}$. By (2) the IFS mappings transform the attractor into itself, so $\{x_n\}_{n=0}^{\infty} \subset \mathcal{A}$. Let a be any point of $\mathcal{A}$. In order to prove that $\{x_n\}_{n=M}^{\infty}$ is dense in $\mathcal{A}$ for any $M \in \mathbb{N}$, we need to show that for any $\varepsilon > 0$ and any $M \in \mathbb{N}$, the subset $\{x_n\}_{n=M}^{\infty}$ includes, with probability 1, a point $x \in \mathcal{A}$ such that $d(x, a) \leq \varepsilon$. On the basis of (2) we have

$$\mathcal{A} = W^{\circ k}(\mathcal{A}) = \bigcup_{i_1,\ldots,i_k \in \{1,\ldots,N\}} w_{i_1} \circ \cdots \circ w_{i_k}(\mathcal{A}), \quad \text{for any } k \in \mathbb{N}.$$

Moreover w_i are contractive mappings, so given $\varepsilon > 0$, and taking

$$k \geq \left\lceil \frac{\log \varepsilon - \log \operatorname{diam}(\mathcal{A})}{\log \max\limits_{i=1,\ldots,N} \operatorname{Lip}(w_i)} \right\rceil$$

the attractor $\mathcal{A}$ can be regarded as a finite union of N^k subsets $w_{i_1} \circ \cdots \circ w_{i_k}(\mathcal{A})$ whose diameters $\operatorname{diam}(w_{i_1} \circ \cdots \circ w_{i_k}(\mathcal{A})) \leq \varepsilon$, where $\operatorname{Lip}(w_i) \in [0, 1)$ denotes the Lipschitz constant of the mapping w_i. Hence, the point a belongs to at least one of the sets of the attractor decomposition, say $a \in w_{i_1} \circ \cdots \circ w_{i_k}(\mathcal{A})$. Thus, if, starting from any mth iteration of the chaos game, the sequence $w_{i_k}, \ldots, w_{i_1}$ of the IFS mappings gets chosen in the k successive iterations of the algorithm, then the point $x_{m+k-1} \in \{x_n\}_{n=0}^{\infty}$ will belong to the set $w_{i_1} \circ \cdots \circ w_{i_k}(\mathcal{A})$ and, hence, $d(x_{m+k-1}, a) \leq \varepsilon$. Now we show that, with probability 1, the sequence will occur infinitely many times as the chaos game proceeds.

Let B_m denote the event that the mappings $w_{i_k}, \ldots, w_{i_1}$ get chosen successively in the mth, $(m+1)$th, $\ldots$, $(m+k-1)$th iteration of the chaos game respectively. Since, in successive iterations of the algorithm the IFS mappings are chosen independently, the probability for the event B_m to occur is

$$P(B_m) = \prod_{j=1}^{k} p_{i_j} > 0. \tag{3}$$

Obviously the events B_m, $m = 1, 2, \ldots$, are not independent. Nevertheless, the events B_{km}, $m = 1, 2, \ldots$, are independent and, thus, the occurrence of the events from the subsequence $\{B_{km}\}_{m=1}^{\infty}$ can be regarded in terms of infinite Bernoulli trials with probabilities p for success specified by (3). On the basis of Borel's law of large numbers we have that with probability 1,

$$\lim_{m\to\infty} \frac{S_m}{m} = p,$$

where S_m denotes the number of successes in the first m trials. Hence, with probability 1, $S_m \to \infty$ as $m \to \infty$. Therefore, during the infinite iteration of the chaos game, infinitely many events from the subsequence $\{B_{km}\}_{m=1}^{\infty}$ will occur, and all the more so from the sequence $\{B_m\}_{m=1}^{\infty}$.

On the basis of the above, for any $M \in \mathbb{N}$, during the infinite iteration of the chaos game, infinitely many events from $\{B_m\}_{m=M}^{\infty}$ will occur with probability 1 as well. The point a has been specified as any point of the attractor $\mathcal{A}$. Therefore, assuming that the initial point $x_0 \in \mathcal{A}$, we obtain that for any point $a \in \mathcal{A}$, any $M \in \mathbb{N}$, and every $\varepsilon > 0$, the set of points $\{x_n\}_{n=M}^{\infty}$ generated by the chaos game includes, with probability 1, a point lying within a distance not greater than ε from a. Since $\{x_n\}_{n=M}^{\infty} \subset \mathcal{A}$, it follows that, with probability 1, the set $\{x_n\}_{n=M}^{\infty}$ is dense in $\mathcal{A}$.

Now we go to the second part of the proof in which we take the initial point x_0 to be any point of the space X. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be sets of points which are generated concurrently by the same realization of the chaos game for the initial points $x_0 \in X$ and $y_0 \in \mathcal{A}$, respectively. In other words, $x_{n+1} = w_i(x_n)$ and $y_{n+1} = w_i(y_n)$, where w_i is the IFS mapping chosen in the nth iteration of the algorithm. Since the IFS mappings are contractive, we obtain that the distance between the points x_m and y_m in the mth iteration satisfies

$$\begin{aligned} d(x_m, y_m) &= d(w_{i_m} \circ \cdots \circ w_{i_1}(x_0), w_{i_m} \circ \cdots \circ w_{i_1}(x_0)) \\ &\leq \mathrm{Lip}(w_{i_m} \circ \cdots \circ w_{i_1}) d(x_0, y_0) \\ &\leq \prod_{j=1}^{m} \mathrm{Lip}(w_{i_j}) d(x_0, y_0) \end{aligned}$$

and, moreover, for every $n > m$,

$$d(x_n, y_n) \leq \mathrm{Lip}(W)^{n-m} d(x_m, y_m),$$

where $\mathrm{Lip}(W) = \max_{i=1,\ldots,N} \mathrm{Lip}(w_i) < 1$. It follows that for any $\varepsilon > 0$, there is an $M \in \mathbb{N}$ such that the distance $d(x_n, y_n) \leq \varepsilon$ for every $n \geq M$. Hence, the Hausdorff distance between the closures of the subsets $\{x_n\}_{n=M}^{\infty}$ and $\{y_n\}_{n=M}^{\infty}$ satisfies

$$h(\overline{\{x_n\}_{n=M}^{\infty}}, \overline{\{y_n\}_{n=M}^{\infty}}) \leq \varepsilon.$$

But $y_0 \in \mathcal{A}$. So on the basis of the first part of the proof, $\overline{\{y_n\}_{n=M}^{\infty}} = \mathcal{A}$ with probability 1. This completes the proof. □

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