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Lattice Properties of the $*$-Order for Complex Matrices

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It is shown that the set $\mathbb{C}_{m\times n}$ of complex $m \times n$ matrices forms a lower semilattice under the partial ordering $A \leqslant B$ defined by $A^*A = A^*B$, $AA^* = BA^*$, where A^* denotes the conjugate transpose of A. As a special case of a result for division rings, it is further shown that, over any field F, for $m = n = 2$ and any proper involution $*$ of $F_{2\times 2}$, the corresponding intersections $A \cap B$ all exist.

1. INTRODUCTION

For arbitrary $m \times n$ complex matrices A, B, Magnus Hestenes introduced, in 1961 [10], the concept of $*$-*orthogonality*, defined by the equations $A^*B = 0$, $AB^* = 0$ (also written as $A \perp B$). In the same paper, he defined and discussed the binary relation $A \perp (A - B)$, or equivalently

$$A^*A = A^*B \qquad \text{and} \qquad AA^* = BA^*, \tag{1}$$

in which case he called A a "section" or "direct summand" of B. More recently [8, 9], it has been shown that the relation (1) on the set $\mathbb{C}_{m\times n}$ is in fact a partial ordering, which we shall denote by $A \leqslant B$ and refer to as *the $*$-order* on $\mathbb{C}_{m\times n}$.

Our main result here (see Theorem 2 below) is that $\mathbb{C}_{m\times n}$ is a lower semilattice with respect to this $*$-order. In other words, for arbitrary $A, B \in \mathbb{C}_{m\times n}$, the $*$-*intersection*

$$A \cap B = \max\{C: C \in \mathbb{C}_{m\times n}, C \leqslant A, C \leqslant B\},$$

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where "max" refers to the $*$-order, always exists in $\mathbb{C}_{m\times n}$. Of course other notations such as glb(A, B) or $A \inf B$ could be used for this concept, but we prefer the intersection point of view.

In the course of our arguments, we find (see Proposition 2) a close connection between $*$-intersections and the "parallel sum" concept of Anderson and Duffin [1]; this suggests that matrix $*$-intersections $A \cap B$, whose existence we establish below, may have physical applications in electrical circuit theory. More specifically, Anderson and Duffin showed that the parallel connection of a pair of n-port networks, say, with respective $n \times n$ impedance matrices A, B, has as its matrix the parallel sum $A : B$ of A and B; however, this sum is definable with satisfactory properties only for a relatively narrow class of pairs A, B, e.g., if A, B are both non-negative definite Hermitian, and these restrictions on A, B are satisfied only if the corresponding networks are reciprocal and resistive. The results we develop here may (see Section 2) make it possible, by using $A \cap B$ rather than $A : B$, to extend the work of Anderson and Duffin to more general networks.

Note that, for given A, B in an arbitrary partially ordered set $\mathscr{S}$, the existence of $A \cap B$ $(=K$, say) means, by definition, that, first, $K \in \mathscr{S}$ and K is a lower bound for A, B in $\mathscr{S}$ (i.e., $K \leqslant A$ and $K \leqslant B$), and, second, K is the greatest such element (i.e., every lower bound J for $\mathscr{D} = \{A, B\}$ in $\mathscr{S}$ satisfies $J \leqslant K$). Clearly K is unique when it exists.

More generally, for any given subsets $\mathscr{C}$, $\mathscr{D}$ of $\mathscr{S}$, one may ask whether $\mathscr{D}$ has a greatest lower bound

$$K = \max\{C: C \in \mathscr{C},\ C \leqslant D \text{ for every } D \in \mathscr{D}\}$$

relative to $\mathscr{C}$, i.e., an element $K \in \mathscr{C}$ which is a lower bound for $\mathscr{D}$ and which is the greatest such in $\mathscr{C}$. When $\mathscr{S} = \mathbb{C}_{m\times n}$, relative to the $*$-order, we refer to this constrained optimization problem as the $*$-Max *Problem for* $\mathscr{C}$ *subject to the constraint set* $\mathscr{D}$, and our approach will be to solve the $*$-intersection problem (which is obviously itself of $*$-max type, with $\mathscr{S} = \mathscr{C} = \mathbb{C}_{m\times n}$) by reducing it to certain other $*$-max problems. When a $*$-max K exists (necessarily uniquely), we shall often refer to it as *the* $*$-*greatest* element of $\mathscr{C}$ (subject to the relevant constraints). We use the notation "max" (rather than "sup") to emphasize that all are attained in $\mathscr{C}$.

For the reader unfamiliar with relative greatest lower bounds K, as defined above, it may be helpful to mention two easy but representative examples. To avoid unnecessary complications, we shall consider two concrete sets $\mathscr{S}$ having well-known ordering relations on them (rather than use the $*$-order on matrices):

EXAMPLE 1. Let $\mathscr{S} = \mathbb{R}\backslash\{1\}$ be the set of all real numbers other than 1, and let

$$\mathscr{C} = \{x: x \in \mathscr{S}, x \leqq 0\}, \qquad \mathscr{D} = \{y: y \in \mathscr{S}, y > 1\}.$$

Then, with respect to the standard (total) order on $\mathscr{S}$ regarded as a subset of $\mathbb{R}$, clearly $\mathscr{Q}$ has *no* greatest lower bound in $\mathscr{S}$, but has $K=0$ as greatest lower bound relative to $\mathscr{C}$.

EXAMPLE 2. Let $\mathscr{S}=\{2,3,12,18\}$, $\mathscr{C}=\{2,3\}$, $\mathscr{Q}=\{12,18\}$. Then, on partially ordering $\mathscr{S}$ by divisibility, each of 2, 3 is in $\mathscr{C}$ and is a lower bound for $\mathscr{Q}$ (since $2\mid 12$, $2\mid 18$ and also $3\mid 12$, $3\mid 18$). However, since neither of the pair 2, 3 divides the other, $\mathscr{Q}$ has no *greatest* lower bound relative to $\mathscr{C}$.

Our proof makes use of the familiar *singular value decomposition* of an arbitrary $m\times n$ complex matrix A (see, e.g., [5, p. 244]), i.e.,

$$X^*AY=\begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}=S, \tag{2}$$

say, where

$$X\in\mathbb{C}_{m\times m},\qquad Y\in\mathbb{C}_{n\times n},\qquad S\in\mathbb{C}_{m\times n},$$
$$D=\operatorname{diag}(\sigma_1 I_{r_1},\dots,\sigma_k I_{r_k})\in\mathbb{C}_{r\times r},$$

and the σ_i are the distinct positive singular values of A (i.e., positive eigenvalues of A^*A or AA^*), with X, Y unitary and $r_1+\cdots+r_k=\operatorname{rank}(A)=r$. We note incidentally that, perhaps somewhat surprisingly, this decomposition was among the earliest discoveries in matrix theory (having been described by E. Beltrami in 1873 and by C. Jordan in 1874).

For an alternative reformulation, if we partition the columns of X, Y conformably with the block diagonal decomposition of D in S, say

$$X=(X_1,\dots,X_{k+1}),\qquad Y=(Y_1,\dots,Y_{k+1}),$$

then, on defining $m\times n$ matrices

$$U_{\sigma_i}=X\operatorname{diag}(0,\dots,0,I_{r_i},0,\dots,0)\,Y^*\qquad (i=1,\dots,k),$$

and $U_\alpha=0$ for every positive real α which is not a singular value of A, we at once obtain the (equivalent) associated unique *Penrose decomposition* of A [14, p. 412, Theorem 4], i.e.,

$$A=\sum_{i=1}^{k}\sigma_i X_i Y_i^*=\sum_{\alpha>0}\alpha U_\alpha, \tag{3.1}$$

where the U_α (nonzero for only finitely many values of α) are pairwise $*$-orthogonal partial isometries, i.e.,

$$U_\alpha\perp U_\beta\qquad \textit{whenever}\quad \alpha\neq\beta, \tag{3.2}$$

and

$$U_\alpha = U_\alpha U_\alpha^* U_\alpha \qquad \textit{for every positive } \alpha. \tag{3.3}$$

Equivalently, one may restate the partial isometry condition (3.3) as $U_\alpha^* = U_\alpha^\dagger$, where $U_\alpha^\dagger$ denotes the Moore–Penrose inverse [14] of U_α.

We shall make free use of the elementary properties of the Moore–Penrose inverse. We shall also need the following basic properties of the $*$-order:

If A has a left or right inverse and $A \leqslant B$, then $A = B$. (4.1)

If E is a projection (i.e. a Hermitian idempotent) and $A \leqslant E$, then A must be a projection. (4.2)

If $B^* = B^\dagger$ and $A \leqslant B$, then $A^* = A^\dagger$. (4.3)

$A \leqslant B$ iff $A^* \leqslant B^*$ iff $A^\dagger \leqslant B^\dagger$ iff $\alpha A \leqslant \alpha B$ (where α denotes any given nonzero scalar). (4.4)

If $A \leqslant B$, then $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$. (4.5)

If E, F are projections, then $E \leqslant F$ iff $\text{range}(E) \subseteq \text{range}(F)$. (4.6)

If E, F are projections, then $E \leqslant F$ iff $I - F \leqslant I - E$. (4.7)

If X, Y are unitary, then $X^*AY \leqslant X^*BY$ iff $A \leqslant B$. (4.8)

Of these, the first five are proved or noted in [9, Proposition 7.7, Theorems 2.1, 2.3, Section 1], while the last three are easily verified.

By (4.1), for $m \leqq n$, all matrices of full row-rank m (or of full column-rank n if $n \leqq m$) are maximal in $\mathbb{C}_{m \times n}$ with respect to the $*$-order. Consequently, if A, B are both of full rank, then their union $A \cup B = \min\{C: A \leqslant C, B \leqslant C\}$ cannot exist unless $A = B$. Thus $\mathbb{C}_{m \times n}$ is *never* a lattice with respect to the $*$-order.

To establish the semilattice property of $\mathbb{C}_{m \times n}$, our strategy will be to show first (in Section 3) that $A \cap B$ exists whenever A is a partial isometry, and then (in Section 4) to reduce the general case to this by means of the Penrose decomposition. Indeed, we shall prove in Theorem 2 that, if $A = \sum_{\alpha > 0} \alpha U_\alpha$, $B = \sum_{\beta > 0} \beta V_\beta$ are the respective decompositions of A, B, then each $U_\alpha \cap V_\alpha$ exists and is a partial isometry, whence $A \cap B$ exists, with the Penrose decomposition

$$A \cap B = \sum_{\alpha > 0} \alpha (U_\alpha \cap V_\alpha).$$

All our matrices will (except in Section 5) be complex, and, for any given matrix A, we use $R(A)$ and $N(A)$ to denote respectively the range space and null space of A. As noted above in (4.2), a *projection* will mean a matrix E satisfying $E^2 = E = E^*$; thus, for us, every projection is a *Hermitian* (in particular, square) matrix. Throughout, we shall reserve the symbols E, F, G to denote only (Hermitian) projections. For any projection $E \in \mathbb{C}_{n \times n}$, and any subspace V of the vector space $\mathbb{C}^n$ of complex column n-vectors, we shall use the terminology "E is the (orthogonal) projection onto V" to mean that $V = R(E)$; of course each of V, E determines the other uniquely, and we shall write $E = P(V)$ to denote this dependency of E on V.

When we have occasion to mention ordinary set-theoretic intersection, we shall use the notation $\bigcap$, in contrast with our $*$-intersection symbol $\frown$; similarly, as above, we use $\leqq$ to denote the standard ordering of the reals, in contrast with $\leqslant$ for the $*$-order relation.

2. $*$-INTERSECTIONS AND $*$-UNIONS OF $n \times n$ PROJECTIONS

Let $\mathscr{P}_n$ denote the set of all projections in $\mathbb{C}_{n \times n}$. When restricted to $\mathscr{P}_n$, obviously the $*$-order $E \leqslant F$ on $\mathbb{C}_{n \times n}$ coincides with the partial order on $\mathscr{P}_n$ given by $E = EF$, or, equivalently, $E = EF = FE$, and it is also well known that this latter relation gives a partial order on the set of idempotents of an arbitrary multiplicative semigroup (see, e.g., [6–9]). Further [6, p. 14, Proposition 7, and p. 229, Proposition 3; 11, p. 244, Lemma 5.3], $\mathscr{P}_n$ is even a *lattice* with respect to the order $E = EF$, or, equivalently, with respect to the $*$-order. Before considering more general cases, it is appropriate to discuss what these remarks imply, for given projections E, F, about the existence of the $*$-intersection $E \frown F = \max\{C: C \leqslant E, C \leqslant F\}$ and the $*$-union $E \smile F = \min\{C: E \leqslant C, F \leqslant C\}$ in the sense of Section 1, where C ranges through the whole of $\mathbb{C}_{n \times n}$, rather than being explicitly restricted to lie in $\mathscr{P}_n$.

By (4.2), the requirement $C \leqslant E$ in itself forces C to lie in $\mathscr{P}_n$, so that obviously

$$\max\{C: C \in \mathbb{C}_{n \times n}, C \leqslant E, C \leqslant F\}$$
$$= \max\{C: C \in \mathscr{P}_n, C \leqslant E, C \leqslant F\}. \tag{5}$$

i.e., the two types of intersection coincide (for projections E, F): in particular (cf. [6, p. 14] again) the $*$-intersection in our sense (i.e., as on the left side of (5)) does indeed exist, and may of course be obtained explicitly from the same formulae (see below) as are known to hold for the intersection as traditionally defined via the right side of (5). Or, more directly, by (4.2) and (4.6), clearly $E \frown F$ exists, and $E \frown F = P([R(E)] \bigcap [R(F)])$; once $E \frown F$ is known to exist, of course this equation can equivalently be written as

$$R(E \frown F) = [R(E)] \bigcap [R(F)],$$

or, again equivalently, as

$$N(E \cap F) = [N(E)] + [N(F)].$$

In view of our results below, it would be of interest to know whether either of these identities holds for arbitrary A, $B \in \mathbb{C}_{m \times n}$ (or at least for partial isometries).

The situation regarding *unions* of projections is slightly less trivial, since the truth of the appropriate analogue of (5) is less self-evident, if only because there is no corresponding analogue of (4.2) which can be invoked. Thus, while the existence of $\min\{C: C \in \mathscr{P}_n, E \leqslant C, F \leqslant C\}$ is well known (see, e.g., [6, p. 14, Proposition 7]) for arbitrary $n \times n$ projections E, F, the existence of $\min\{C: C \in \mathbb{C}_{n \times n}, E \leqslant C, F \leqslant C\}$ must be considered anew on its own merits.

PROPOSITION 1. *If E, F are any $n \times n$ projections, then*

$$E \cup F = \min\{C: C \in \mathbb{C}_{n \times n}, E \leqslant C, F \leqslant C\}$$

exists, and is itself a projection, given explicitly by

$$E \cup F = (E + F)(E + F)^{\dagger} = (E + F)^{\dagger}(E + F). \tag{6}$$

Proof. Since $E + F$ is Hermitian, obviously

$$(E + F)(E + F)^{\dagger} = (E + F)^{\dagger}(E + F), \tag{7}$$

and their common value, G, say, is a projection.

Also (cf. [15, p. 189, (10.1.29)]) $E = EG = GE$, i.e., $E \leqslant G$, and similarly $F \leqslant G$, so that G is, in the sense of the $*$-order, a common upper bound for E and F. Thus it remains only to show that G is the least such upper bound.

To verify this, let C be any common upper bound in $\mathbb{C}_{n \times n}$ for E, F, so that

$$E = CE = EC, \qquad F = CF = FC.$$

Then $CG = C(E + F)(E + F)^{\dagger} = (CE + CF)(E + F)^{\dagger} = (E + F)(E + F)^{\dagger} = G$, and, by (7), similarly $GC = G$, so that $G^*G = G = G^*C = C^*G$ and $GG^* = G = GC^* = CG^*$. Thus, for any common upper bound $C \in \mathbb{C}_{n \times n}$ (i.e., whether or not C is itself a projection), we have $G \leqslant C$, as required. ∎

A result formally analogous to Proposition 1 has been noted by Ben-Israel and Greville [5, p. 198, Lemma 2]; however, those authors did not have the $*$-order available (except on projections), and our Proposition 1 is not a special case of their Lemma 2, which (in view of Proposition 1) amounts to the assertion that

$$R(E \cup F) = R(E) + R(F).$$

COROLLARY 1. *For any $n \times n$ projections E, F, we have*

$$\min\{C: C \in \mathbb{C}_{n \times n}, E \leqslant C, F \leqslant C\}$$
$$= \min\{C: C \in \mathscr{P}_n, E \leqslant C, F \leqslant C\}.$$

In other words, the analogue of (5) for unions does in fact hold, and, for projections E, F, the two alternative definitions of $E \cup F$ agree. Note however that, for non-projections A, $B \in \mathbb{C}_{m \times n}$, there is no corresponding problem of having to reconcile our definition of $*$-intersection or $*$-union with any pre-existing definition, since, even when $m = n$, for general A, B no appropriate subset of $\mathbb{C}_{m \times n}$ suggests itself as a candidate to take over the role of $\mathscr{P}_n$.

Having established that, for projections E, F, both the $*$-intersection $E \cap F$ and the $*$-union $E \cup F$ exist, and are themselves projections, we can easily deduce (cf. [6, p. 14]) the connecting identity

$$E \cup F = I - ((I - E) \cap (I - F)). \tag{8}$$

For, by (5) and Corollary 1, we need only consider projections C, and, by (4.7), such a C will be a common upper bound for E, F iff $I - C$ is a common lower bound for $I - E$, $I - F$, i.e., $E \cup F \leqslant C$ iff $I - C \leqslant (I - E) \cap (I - F)$. Thus (8) follows by applying (4.7) once more.

By (8) and a previous remark, clearly $E \cup F = I - H$, where

$$H = P([R(I - E)] \cap [R(I - F)]) = P([N(E)] \cap [N(F)]);$$

equivalently, $E \cup F = P([(N(E)) \cap (N(F))]^{\perp}) = P(R(E) + R(F))$, which again yields $R(E \cup F) = R(E) + R(F)$.

In any discussion of explicit formulae for $E \cap F$ or $E \cup F$, in view of (8), any result for either leads immediately to a corresponding result for the other, so it becomes a matter of indifference whether to speak in terms in $E \cap F$ or $E \cup F$. Formula (6) is symmetric under $*$ and under the interchange of E with F, in contrast with the earlier formulae

$$E \cup F = E + (I - E)[(I - E)F]^{\dagger},$$
$$E \cap F = E - E[E(I - F)]^{\dagger}$$

of Kaplansky [11, p. 244, Lemma 5.3], which, by the same symmetries, yield three further equivalent formulae for $E \cup F$, and also three for $E \cap F$.

While one may combine Proposition 1 with (8) to obtain a new formula for $E \cap F$, having the same intrinsic symmetry as that in (6), we note next an even simpler symmetric expression for $E \cap F$. We recall from [1; 15, pp. 188–189, Theorem 10.1.8(a)] that, for any $n \times n$ projections E, F, one has

$$E(E+F)^{\dagger}F = F(E+F)^{\dagger}E,$$

this matrix being called the *parallel sum* of E, F and denoted $E : F$.

PROPOSITION 2. *If E, F are any $n \times n$ projections, then*

$$E \cap F = 2(E : F).$$

Proof. Writing $2(E : F) = J$ for brevity, we have

$$EJ = 2E^2(E+F)^{\dagger}F = 2E(E+F)^{\dagger}F = J.$$

Also [15, p. 189, Theorem 10.1.8(h)], J is a projection. Hence $J \leqslant E$, and similarly $J \leqslant F$, i.e., J is a common lower bound.

Conversely, if C is any common lower bound for E, F, then, by (4.2), C is necessarily a projection, so that $CE = C = CF$. Also [15, p. 189, (10.1.30)], $(E+F)(E+F)^{\dagger}F = F$, whence

$$CJ = 2CE(E+F)^{\dagger}F = C(E+F)(E+F)^{\dagger}F = CF = C,$$

i.e., $C \leqslant J$, as required. ∎

Given the fact (noted earlier in this section) that $E \cap F = P([R(E)] \cap [R(F)])$, our Proposition 2 may be regarded as a restatement of [1, p. 581, Theorem 8] (or [15, p. 189, Theorem 10.1.8(h)]). However, our point of view here (cf. also [2–4, 13]) differs from that of [1, 16], in that Proposition 2 refers to the matrices E, F only as such (i.e., without reference to their action on vectors).

While the parallel sum, as presented in [1; 15, p. 188–192], does apply to certain pairs of $n \times n$ (and $m \times n$) matrices other than projections, those writers did not arrive at any definition of $A : B$ having satisfactory properties for arbitrary $n \times n$ matrices A, B, but only for certain ("parallel summable") $m \times n$ pairs. Indeed, specifically, in terms of the $*$-order, their criterion for A, B to be parallel summable may be stated as

$$AA^{\dagger} \leqslant (A+B)(A+B)^{\dagger}, \qquad A^{\dagger}A \leqslant (A+B)^{\dagger}(A+B),$$

or, equivalently, as

$$BB^{\dagger} \leqslant (A+B)(A+B)^{\dagger}, \qquad B^{\dagger}B \leqslant (A+B)^{\dagger}(A+B).$$

In view of Proposition 2 and the fact (to be proved in Section 4 below) that $A \cap B$ exists for *all* complex $m \times n$ matrices A, B, the $*$-intersection operation provides, for arbitrary $m \times n$ complex matrices, at least a formal generalization of $2(E:F)$ as applied to $n \times n$ projections. However, there exist parallel summable pairs A, B for which $A \cap B \neq 2(A:B)$; for example, for scalar matrices $A = \alpha I$, $B = \beta I$, one has $A \cap B = 0$ whenever $\alpha \neq \beta$, while $A:B = (\alpha\beta/(\alpha+\beta))I$ whenever $\alpha + \beta \neq 0$. Nevertheless, these observations do not exclude the possibility that there is some generalization of the identity $E \cap F = 2(E:F)$ yielding $A \cap B$ explicitly in terms of A, B and (say) the unary operations $*$, $\dagger$, together with the binary operations of multiplication, ordinary addition, and parallel addition, valid whenever the parallel summands are parallel summable.

As a further remark concerning projections, of course, in $\mathbb{C}_{n \times n}$, the case where $A - B$ is invertible shows that every matrix C with $C \leqslant A$, $C \leqslant B$ may be a projection without either A or B being a projection.

3. Existence of $A \cap B$ when A is a Partial Isometry

We shall require the following:

Lemma 1. *Let $A, B \in \mathbb{C}_{m \times n}$, and suppose that B has the form $B = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}$. Then, if $A \leqslant B$, we must have, conformably, $A = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ with $K \leqslant L$ (and, of course, conversely).*

Proof. This follows easily from the definition (1) and the fact that $RR^* + SS^* = 0$ only if R, S are both zero. ∎

We shall also need the following $*$-max principle which, though almost obvious, seems worth isolating formally:

Lemma 2. *Let $r, s, m_1, \ldots, m_s$ be arbitrary positive integers, let $B_i \in \mathbb{C}_{m_i \times r}$ $(i = 1, \ldots, s)$ be given matrices, and define*

$$\mathscr{E} = \{E \colon E \in \mathscr{P}_r, B_i E = 0 \ (i = 1, \ldots, s)\}.$$

Then the set $\mathscr{E}$ contains a (unique) $*$-greatest element, namely, $P(\bigcap_{i=1}^{s} N(B_i))$.

Proof. For any $E \in \mathscr{P}_n$, we have, for each i, that $B_i E = 0$ iff $R(E) \subseteq N(B_i)$. Hence $E \in \mathscr{E}$ iff $R(E) \subseteq \bigcap_{i=1}^{s} N(B_i) = V$, say, where of course $V = R(P(V))$; by (4.6), it follows that $E \in \mathscr{E}$ iff $E \leqslant P(V)$, as stated. ∎

In particular, when $s=1$, of course $P(N(B)) = I - B^{\dagger}B$. More generally, for $s > 1$, if we write $E_i = I - B_i^{\dagger}B_i$ ($i = 1,\ldots, s$), then, by Proposition 2 and our previous observation that $E \cap F = P([R(E)] \bigcap [R(F)])$ for projections E, F, we have

$$P\left(\bigcap_{i=1}^{s} [N(B_i)]\right) = P\left(\bigcap_{i=1}^{s} [R(E_i)]\right) = 2(E_1 : [2(E_2 : \cdots E_s)\cdots]).$$

For a partial isometry A, i.e., any complex matrix satisfying $AA^*A = A$, the singular value decomposition (2) takes the simple form

$$X^*AY = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where X, Y are unitary and $r = \operatorname{rank}(A)$; and it is this fact which enables us to prove that $A \cap B$ exists for every $B \in \mathbb{C}_{m\times n}$

LEMMA 3. *Let $A, B \in \mathbb{C}_{m\times n}$, let A be a partial isometry, say with singular value decomposition*

$$X^*AY = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

and write, conformably,

$$X^*BY = \begin{pmatrix} L & M \\ S & T \end{pmatrix}.$$

Then $A \cap B$ exists, and in fact

$$A \cap B = X \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} Y^*,$$

where

$$Q = P([N(I-L)] \bigcap [N(I-L^*)] \bigcap [N(M^*)] \bigcap [N(S)]).$$

Proof. By (4.8), we have

$$A \cap B = X\left[\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} L & M \\ S & T \end{pmatrix}\right] Y^*,$$

provided that the intersection on the right exists. In other words, the intersection problem for the given pair A, B reduces to the corresponding problem for the pair $\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} L & M \\ S & T \end{smallmatrix}\right)$. Also, given any $C \in \mathbb{C}_{m\times n}$, by Lemma 1 and

(4.2), we have $C \leqslant \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ iff $C = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ for some $E \in \mathscr{P}_r$. Thus $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} L & M \\ S & T \end{pmatrix}$ (if it exists) is the $*$-greatest matrix (if such exists) of the form $C = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ subject to the constraints $E \in \mathscr{P}_r$, $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} L & M \\ S & T \end{pmatrix}$.

But Lemma 2 guarantees the existence, and provides the value, of this $*$-greatest C. For, by (1), clearly

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} L & M \\ S & T \end{pmatrix} \qquad \text{iff } LE = E = EL,\ EM = 0,\ SE = 0,$$

i.e., iff $(I - L)E = 0 = (I - L^*)E$, $M^*E = 0$, $SE = 0$, whence the result follows. ∎

As above, one may alternatively express

$$Q = 2(E_1 : [2(E_2 : [2(E_3 : E_4)])]),$$

where $E_i = I - W_i^\dagger W_i$ and W_i takes the values $I - L$, $I - L^*$, M^*, S. It should be noted also that, in the $*$-max problem for C just solved as part of the proof of Lemma 3, the value of the $*$-greatest C genuinely depends on r, and indeed different r's can even yield non-comparable C's. For example, for 3×4 matrices A, B, on taking the case where $X = I_3$, $Y = I_4$,

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

we find

$$C = \frac{1}{2} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{when } r = 2,$$

while

$$C = \frac{1}{3} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \qquad \text{when } r = 3.$$

For square matrices A, B, the results and arguments above become somewhat simpler, and in this case (still with A being a partial isometry) $A \cap B$ has an algebraic representation (see Corollary 4) which could be verified directly.

PROPOSITION 3. *Let A, B be arbitrary $n \times n$ complex matrices. Then the* *set* $\{E: E \in \mathscr{P}_n,\ E \leqslant A,\ E \leqslant B\}$ *contains a (unique)* $*$*-greatest element, namely,*

$$P([N(I-A)] \cap [N(I-A^*)] \cap [N(I-B)] \cap [N(I-B^*)]).$$

Proof. We have $E \leqslant A$ iff $E = EA = AE$, i.e., iff $(I-A)E = 0 = (I-A^*)E$, and similarly for $E \leqslant B$, so this result follows at once from Lemma 2. ∎

COROLLARY 2. *Let* A *be any* $n \times n$ *matrix. Then the set* $\{E: E \in \mathscr{P}_n,\ E \leqslant A\}$ *contains a* $*$*-greatest element, namely,* $2(F:G)$, *where* F, G *denote the values, at* $W = I - A$, *of the functions*

$$F(W) = I - W^\dagger W, \qquad G(W) = I - WW^\dagger.$$

Proof. Take $B = I$ in Proposition 3, and apply Proposition 2. ∎

In analogy with Corollary 2, it is natural to ask whether, given any A, there must be a $*$-greatest partial isometry R satisfying $R \leqslant A$; we answer this $*$-max question in Corollary 6 below.

COROLLARY 3. *For any* $A \in \mathbb{C}_{n \times n}$ *and any* $E \in \mathscr{P}_n$, *we have*

$$A \cap E = 2(E: [2(F:G)]) = E \cap F \cap G,$$

where $F = F(I-A)$ *and* $G = G(I-A)$ *are as in Corollary* 2.

Proof. Use (4.2), Corollary 2, and Proposition 2. ∎

COROLLARY 4. *Given any* $n \times n$ *partial isometry* A, *say, with singular value decomposition* $X^*AY = E$, *then, for any* $B \in \mathbb{C}_{n \times n}$, *we have*

$$A \cap B = 2[A: (2[(XFY^*): (XGY^*)])],$$

where the parallel summands on the right are necessarily parallel summable, and where now

$$F = I - (I - X^*BY)^\dagger (I - X^*BY),$$
$$G = I - (I - X^*BY)(I - X^*BY)^\dagger.$$

Proof. Use (4.8) and Corollary 3. ∎

It seems appropriate to regard Corollary 3 as providing a "fully explicit" form for $A \cap E$, in contrast to Corollary 4, where the expression for $A \cap B$ involves the unitary factors X, Y. We have found (but will not detail here) fully explicit formulae for $A \cap B$ in various special cases, e.g., when one or both of A, B is a partial isometry. However, for general A, B, we have not been able to find any expression for $A \cap B$ fully explicit in the same sense that Corollary 3 is (i.e., in terms of $*$, $\dagger$, etc., but without involving anything like X or Y). Thus our arguments above throw little light on whether $*$-intersections must always exist in a general regular proper $*$-ring.

4. Existence of $A \cap B$ for Arbitrary A, B

To step up to the general case from the case (just solved in Section 3) where A is a partial isometry, we shall need the following general result, which shows how closely the $*$-order on $\mathbb{C}_{m\times n}$ interrelates with the Penrose decompositions of the matrices involved:

THEOREM 1. *Given any $A, B \in \mathbb{C}_{m\times n}$, say with Penrose decompositions*

$$A = \sum \alpha U_\alpha, \qquad B = \sum \beta V_\beta,$$

then

(i) $A \leqslant B$ *under the $*$-order*

iff

(ii) $U_\alpha \leqslant V_\alpha$ *for every (positive real) α.*

Moreover, whenever (i) *or* (ii) *holds, then, for all α, β with $\alpha \neq \beta$, we have*

(iii) $U_\alpha \perp V_\beta$, *i.e.*, $U_\alpha^* V_\beta = 0$ *and* $U_\alpha V_\beta^* = 0$.

Proof. (i) *implies* (iii). For any positive α, we have, by (3.2),

$$U_\alpha(A^*B) = U_\alpha \left(\sum \gamma U_\gamma^*\right) B = \alpha U_\alpha U_\alpha^* B,$$

and in particular $U_\alpha(A^*A) = \alpha U_\alpha U_\alpha^*(\sum \gamma U_\gamma) = \alpha^2 U_\alpha U_\alpha^* U_\alpha = \alpha^2 U_\alpha$ by (3.3). Hence, if (i) holds, then $\alpha^2 U_\alpha = \alpha U_\alpha U_\alpha^* B$, i.e.,

$$\alpha U_\alpha = U_\alpha U_\alpha^* B. \tag{9}$$

Also $B V_\beta^* = (\sum \gamma V_\gamma) V_\beta^* = \beta V_\beta V_\beta^*$, whence, by (9), for all positive α, β,

$$\alpha U_\alpha V_\beta^* = \beta U_\alpha U_\alpha^* V_\beta V_\beta^*, \tag{10}$$

and, by left–right symmetry (even though A, B play different roles), similarly

$$\alpha U_\alpha^* V_\beta = \beta U_\alpha^* U_\alpha V_\beta^* V_\beta. \tag{11}$$

Now (10) gives

$$\alpha U_\alpha^* U_\alpha V_\beta^* V_\beta = U_\alpha^*(\beta U_\alpha U_\alpha^* V_\beta V_\beta^*) V_\beta = \beta U_\alpha^* V_\beta,$$

which, together with (11), yields $(\alpha^2 - \beta^2) U_\alpha^* V_\beta = 0$, and (since α, β are both positive) consequently $U_\alpha^* V_\beta = 0$ whenever $\alpha \neq \beta$. That $U_\alpha V_\beta^* = 0$ then follows by symmetry.

(i) *implies* (ii). Assuming (i), we may, in view of our argument so far, also make use of (9) and (iii). Thus, for any α, we have

$$\alpha U_\alpha^* U_\alpha = U_\alpha^*(U_\alpha U_\alpha^* B) = U_\alpha^* B = U_\alpha^* \sum \beta V_\beta = \alpha U_\alpha^* V_\alpha,$$

i.e., $U_\alpha^* U_\alpha = U_\alpha^* V_\alpha$, and similarly $U_\alpha U_\alpha^* = V_\alpha U_\alpha^*$, i.e., $U_\alpha \leqslant V_\alpha$.

(ii) *implies* (iii). If (ii) holds, then

$$U_\alpha^* V_\beta = U_\alpha^*(U_\alpha U_\alpha^*) V_\beta = U_\alpha^*(U_\alpha V_\alpha^*) V_\beta = U_\alpha^* U_\alpha (V_\alpha^* V_\beta) = 0,$$

whenever $\alpha \neq \beta$, and similarly $U_\alpha V_\beta^* = 0$.

(ii) *implies* (i). Assuming (ii) and also (as we now may) using (iii), we have

$$A^*B = \left(\sum \alpha U_\alpha^*\right)\left(\sum \beta V_\beta\right) = \sum \alpha^2 U_\alpha^* V_\alpha = \sum \alpha^2 U_\alpha^* U_\alpha$$

$$= \left(\sum \gamma U_\gamma^*\right)\left(\sum \alpha U_\alpha\right) = A^*A,$$

and similarly $BA^* = AA^*$, i.e., $A \leqslant B$. ∎

COROLLARY 5. *If $A = \sum \alpha U_\alpha$ and B are as above, then $A \leqslant B$ iff $\alpha U_\alpha \leqslant B$ for all α.*

One might expect that Corollary 5 (which of course follows immediately from Theorem 1) would in itself suffice to complete our main argument; however, somewhat surprisingly, it seems that the more detailed result stated in Theorem 1 is essential.

THEOREM 2. *Given any $A, B \in \mathbb{C}_{m\times n}$, say with Penrose decompositions*

$$A = \sum \alpha U_\alpha, \qquad B = \sum \beta V_\beta,$$

then $A \cap B$ exists, and has Penrose decomposition

$$A \cap B = \sum \gamma (U_\gamma \cap V_\gamma),$$

where $U_\gamma \cap V_\gamma$ exists for all γ (and may be computed from Lemma 3).

Proof. Of course the existence of each $W_\gamma = U_\gamma \cap V_\gamma$ is guaranteed by Lemma 3, and, by (4.3), each W_γ is a partial isometry. Thus we have only to show that $D = \sum \gamma W_\gamma$ is the Penrose decomposition of D (i.e., that $W_\alpha \perp W_\beta$ whenever $\alpha \neq \beta$), and that $D = A \cap B$.

Now, since $W_\gamma \leqslant U_\gamma$, we have $W_\gamma W_\gamma^* = W_\gamma U_\gamma^* = U_\gamma W_\gamma^*$ for all γ, so that

$$W_\alpha^* W_\beta = W_\alpha^* W_\alpha W_\alpha^* \cdot W_\beta W_\beta^* W_\beta = W_\alpha^* \cdot W_\alpha U_\alpha^* \cdot U_\beta W_\beta^* \cdot W_\beta$$
$$= W_\alpha^* W_\alpha (U_\alpha^* U_\beta) W_\beta^* W_\beta = 0$$

by (3.2), and similarly $W_\alpha W_\beta^* = 0$, i.e., $W_\alpha \perp W_\beta$.

Finally, given any $C \in \mathbb{C}_{m \times n}$, say with Penrose decomposition $C = \sum \gamma T_\gamma$, then, by Theorem 1, we have $C \leqslant A$, B iff $T_\gamma \leqslant U_\gamma$, V_γ, or, equivalently, $T_\gamma \leqslant W_\gamma$, for all γ; hence, by Theorem 1 again, in fact $C \leqslant A$, B iff $C \leqslant D$, as required. ∎

Now that existence is established, (4.4) has the following immediate consequence:

PROPOSITION 4. *For arbitrary $A, B \in \mathbb{C}_{m \times n}$, we have*

$$A^* \cap B^* = (A \cap B)^*, \qquad A^\dagger \cap B^\dagger = (A \cap B)^\dagger,$$

and

$$(\alpha A) \cap (\alpha B) = \alpha (A \cap B) \qquad \textit{for every scalar } \alpha.$$

Various useful facts follow in turn from Proposition 4: for example, *if A, B are both Hermitian, then so is $A \cap B$.*

By Theorems 1 and 2, we see that, for any given A, $B \in \mathbb{C}_{m \times n}$, the set $\{R: R \in \mathbb{C}_{m \times n}, R^\dagger = R^*, \ R \leqslant A, \ R \leqslant B\}$ always contains a $*$-greatest element, namely, $U_1 \cap V_1$. In the special case $A = B$, a simpler version of this holds:

COROLLARY 6. *For every $A \in \mathbb{C}_{m \times n}$, the set $\{R: R \in \mathbb{C}_{m \times n}, \ R^\dagger = R^*, R \leqslant A\}$ contains a $*$-greatest element, namely,*

$$U_1 = A[I_n - (I_n - A^*A)^\dagger (I_n - A^*A)]$$
$$= [I_m - (I_m - AA^*)(I_m - AA^*)^\dagger] A.$$

Here the formulae for U_1 are as given by Penrose [14, p. 412].

It was noted in Section 1 that, for trivial reasons, $A \cup B$ in general fails to exist. In connection with Theorem 2, it is worth observing the contrast between the identity $A \cap B = \sum \gamma (U_\gamma \cap V_\gamma)$ and the corresponding expression $\sum \gamma (U_\gamma \cup V_\gamma)$ for unions: specifically, the summands $X_\gamma = U_\gamma \cup V_\gamma$ may themselves not exist, and, even if they do, may not be partial isometries (or $*$-orthogonal), so that, in general, $\sum \gamma X_\gamma$ is not a Penrose decomposition.

5. Matrices over Other Fields and Division Rings

In our discussions above we have used the familiar properties of the concrete operation $*$ of conjugate transposition as applied to complex matrices. If we now wish to consider similar questions for matrices over other fields, or over division rings, we could either (a) replace $*$ by ordinary transposition, or (b) try to work with an involution $*$ subject only to appropriate axioms rather than being explicitly specified. However, in view of the crucial role played by "properness," on which, e.g., both the transitivity and the antisymmetry of the $*$-order depend [9, Sect. 1], each of these alternatives has its own attendant problems, in that, even for the ring $F_{n \times n}$ of $n \times n$ matrices over a field F, (a) the involution of transposition may not be proper on $F_{n \times n}$ (consider, e.g., $n = 2$ with $F = \mathbb{Z}_2$, or indeed $F = \mathbb{C}$), while in practice it seems that, for (b), one may need an explicit description of the proper involutions on $F_{n \times n}$ (see [9, Sects. 5, 9]). Nevertheless, when $m = n = 2$, even for an arbitrary division ring D of finite dimension over its center, we can prove the existence of $*$-intersections in the ring $R = D_{2 \times 2}$ with respect to the $*$-order of *any* proper involution $*$ of R.

In the remainder of this section, dealing with (non-commutative) division rings D will involve us in some minor technicalities which would be unnecessary if D were a field; the reader interested only in fields need not concern himself with the distinction between left and right vector spaces, nor with the proof of (13).

An *involution* $A \to A^*$ of any ring R will mean any map from R to itself satisfying

$$(A^*)^* = A, \qquad (AB)^* = B^*A^*, \qquad (A + B)^* = A^* + B^*$$

for all $A, B \in R$; such an involution $*$ is called *proper* iff $A^*A = 0$ implies $A = 0$. It is well known (see, e.g., [9, Theorem 5.4]) that every involution of $R = D_{2 \times 2}$ is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = P^{-1} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} P, \tag{12}$$

which we shall also write as $A^* = P^{-1}\bar{A}^t P$, where $d \to \bar{d}$ is a suitable involution of D, and where P (which depends only on $*$) is an invertible matrix in R. It should be noted that, for general invertible $P \in R$, the map $A \to A^* = P^{-1}\bar{A}^t P$ may not satisfy $(A^*)^* = A$; however, our arguments below do not require any consideration of how P must be further restricted to achieve this.

THEOREM 3. *Let D be any division ring of finite dimension over its center, and let $*$ be any given proper involution of $R = D_{2\times 2}$. Then the $*$-order* (1) *is a partial order on the set R, and, with respect to this $*$-order, the $*$-intersection $A \cap B$ exists for all $A, B \in R$.*

Proof. It is well known that R is von Neumann regular (see, e.g., [12, p. 124, Corollary 7.5]) and hence that every matrix $C \in R$ has a (unique) Moore–Penrose inverse $C^{\dagger}$ (see, e.g., [9, Theorem 4.1]). That (1) remains a partial order in the present context is easily verified (cf. [9]); similarly, given properness, we may still make use of (4.1),..., (4.8) as needed. Let V denote the right D-space consisting of all column 2-vectors over D, and, given any vector $\mathbf{x} \in V$, define $X = (\mathbf{x}, \mathbf{0}) \in R$. Then for any $A \in R$, if $\mathbf{x} \in N(A^*A)$, we have $A^*AX = (A^*A\mathbf{x}, \mathbf{0}) = 0$, whence $(AX)^*AX = X^*A^*AX = 0$, so that $AX = 0$ by properness, i.e., $\mathbf{x} \in N(A)$. Thus in fact $N(A^*A) = N(A)$; in particular, the ranks $r(A^*A)$, $r(A)$ must coincide. Similarly, by considering row vectors, we find that $r(A^*A) = r(A^*)$, and so

$$r(A^*) = r(A) \textit{ for every } A \in R. \tag{13}$$

Note next that, if $C \leqslant A$, then, by (4.5), $R(C) = (CC^{\dagger})\,CV = (AC^{\dagger})\,CV \subseteq R(A)$, while, if also $R(C) = R(A)$, then, for each $\mathbf{y} \in V$, we have $A\mathbf{y} = C\mathbf{z}$ for some $\mathbf{z} \in V$, which yields $A\mathbf{y} = CC^{\dagger}C\mathbf{z} = CC^{\dagger} \cdot A\mathbf{y} = C(C^{\dagger}A)\mathbf{y} = C(C^{\dagger}C)\mathbf{y}$ by (4.5) again, i.e., $A = C$. Thus

$$C \leqslant A,\ C \neq A \textit{ implies } r(C) < r(A). \tag{14}$$

In proving the existence of $A \cap B$ for arbitrary A, $B \in R$, obviously we may assume that $A \neq B$, and indeed that neither $A \leqslant B$ nor $B \leqslant A$ holds. Let $C \leqslant A$, B. Then $AC^* = CC^* = BC^*$, so that $(A - B)\,C^* = 0$; hence, if $A - B$ were of rank 2, it would follow that $A \cap B$ exists (with value zero). Thus we may assume that $r(A - B) = 1$, so that, by (13), there exist nonzero vectors $\mathbf{u}$, $\mathbf{v} \in N$ such that

$$N(A^* - B^*) = \mathbf{u}D, \qquad N(A - B) = \mathbf{v}D. \tag{15}$$

We shall now show, using (14) and (15), that the set $\mathscr{H} = \{C\colon C \in R, C \leqslant A, C \leqslant B\}$ must consist either of zero alone, or of the zero matrix together with just *one* other matrix (of rank 1), so that, in either case, of course $\mathscr{H}$ has a $*$-greatest element, i.e., $A \cap B$ exists.

By (14), any nonzero $C \in \mathscr{H}$ has rank 1, and is thus of the form $C = \mathbf{p}\bar{\mathbf{q}}'$

for appropriate nonzero $\mathbf{p}, \mathbf{q} \in V$. Completing each of $\mathbf{p}, \mathbf{q}$ to a 2×2 matrix by adjoining an extra column of zeros, we may equivalently rewrite C as

$$C = (\mathbf{p}, \mathbf{0})\overline{(\mathbf{q}, \mathbf{0})'},$$

so that, by (12),

$$C^* = (\overline{(\mathbf{q}, \mathbf{0})'})^*(\mathbf{p}, \mathbf{0})^* = P^{-1}(\mathbf{q}, \mathbf{0})P \cdot P^{-1}\begin{pmatrix}\bar{\mathbf{p}}' \\ \mathbf{0}'\end{pmatrix} P = P^{-1}\mathbf{q}\bar{\mathbf{p}}'P,$$

and consequently $CC^* = (\mathbf{p}\bar{\mathbf{q}}')(P^{-1}\mathbf{q}\bar{\mathbf{p}}'P) = \mathbf{p}\alpha\bar{\mathbf{p}}'P$, where $\alpha = \bar{\mathbf{q}}'P^{-1}\mathbf{q} \in D$. Also, since $C \in \mathscr{X}$, we have $C \leqslant A$, so that

$$\mathbf{p}\alpha\bar{\mathbf{p}}'P = CC^* = AC^* = AP^{-1}\mathbf{q}\bar{\mathbf{p}}'P,$$

where we may cancel the invertible matrix P and the nonzero row-vector $\bar{\mathbf{p}}'$. Thus in fact $\mathbf{p}\alpha = AP^{-1}\mathbf{q}$.

If A were singular, then, by our assumption that $A \leqslant B$ is false, we should have $C = 0$ by (14), i.e., C would be uniquely determined, as required. Thus we may assume A to be invertible, so that $P^{-1}\mathbf{q} = A^{-1}\mathbf{p}\alpha$, whence $0 \neq \alpha = \bar{\mathbf{q}}'(P^{-1}\mathbf{q}) = \bar{\mathbf{q}}'A^{-1}\mathbf{p}\alpha$, i.e., $\bar{\mathbf{q}}'A^{-1}\mathbf{p} = 1$. Also $A^*C = C^*C = B^*C$, i.e., $(A^* - B^*)\mathbf{p}\bar{\mathbf{q}}' = 0$, which gives $(A^* - B^*)\mathbf{p} = \mathbf{0}$. Hence, by (15), we have $\mathbf{p} = \mathbf{u}\beta$ for suitable (nonzero) $\beta \in D$, and similarly $P^{-1}\mathbf{q} = \mathbf{v}\gamma$ for some $\gamma \in D$, so that

$$1 = \bar{\mathbf{q}}'A^{-1}\mathbf{p} = \overline{(P\mathbf{v}\gamma)'}A^{-1}(\mathbf{u}\beta) = \bar{\gamma}(\bar{\mathbf{v}}'\bar{P}'A^{-1}\mathbf{u})\beta.$$

But this yields $\beta\bar{\gamma} = (\bar{\mathbf{v}}'\bar{P}'A^{-1}\mathbf{u})^{-1}$, and consequently

$$C = \mathbf{p}\bar{\mathbf{q}}' = (\mathbf{u}\beta)\overline{(P\mathbf{v}\gamma)'} = \mathbf{u}(\beta\bar{\gamma})\overline{(P\mathbf{v})'} = \mathbf{u}(\bar{\mathbf{v}}'\bar{P}'A^{-1}\mathbf{u})^{-1}\overline{(P\mathbf{v})'}$$

is uniquely determined. ∎

Under the hypotheses of Theorem 3, (13) follows immediately from (12); we have included the details of the lengthier argument above so as to emphasize that (13), as also (14), holds for *every* division ring D and for every proper involution $*$ of $R = D_{n \times n}$ $(n = 1, 2, \ldots)$. The assumption $n = 2$ and the restriction on D are used only in the later parts of the argument.

For a field F, if we apply Theorem 3 in the case where $d \to \bar{d}$ is the identity map on F and where $P = I_2$, obviously the corresponding involution on $F_{2 \times 2}$ is ordinary transposition, which clearly is proper iff $a^2 + c^2 = 0$ implies $a = c = 0$, or, equivalently, iff $x^2 + 1 = 0$ has no solution x in F. Thus, for example, transposition is proper on $F_{2 \times 2}$ whenever $F = \mathbb{Z}_p$ with $p \equiv 3$

(modulo 4), in which case Theorem 3 guarantees that, with respect to the $*$-order (1) induced by transposition, $A \cap B$ exists for all $A, B \in F_{2\times 2}$.

Returning to division rings D, we note that Theorem 3 applies, for example, to the "standard" involution of the ring of 2×2 matrices with real (or rational) quaternion entries. We conjecture that Theorem 3 holds also for all proper involutions of $D_{n\times n}$ ($n = 2, 3,...$). Regarding this possible generalization of Theorem 3, we note that the argument of the proof above amounts to using a "weak D-version" of ordinary (complex) singular value decomposition theory. For $A^*C = C^*C$ yields $A^*\mathbf{p} = C^*\mathbf{p} = (P^{-1}\mathbf{q})(\bar{\mathbf{p}}^tP\mathbf{p})$, and similarly $A(P^{-1}\mathbf{q}) = \mathbf{p}(\bar{\mathbf{q}}^tP^{-1}\mathbf{q})$, so that

$$AA^*\mathbf{p} = \mathbf{p}(\bar{\mathbf{q}}^tP^{-1}\mathbf{q})(\bar{\mathbf{p}}^tP\mathbf{p}), \qquad A^*A(P^{-1}\mathbf{q}) = (P^{-1}\mathbf{q})(\bar{\mathbf{p}}^tP\mathbf{p})(\bar{\mathbf{q}}^tP^{-1}\mathbf{q}).$$

Thus the $\mathbf{p}$, $P^{-1}\mathbf{q}$ which appear in the proof above are eigenvectors of AA^* and A^*A, respectively, as are used, in the standard complex theory, to construct the unitary transforming matrices X, Y in (2). However, we have not been able to take advantage of this analogy to extend Theorem 3 to $D_{n\times n}$ (or even $F_{n\times n}$).

Another natural objective would be to prove or disprove the lower semilattice property for arbitrary regular proper $*$-rings.

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