A Completeness Problem for Pattern Generation in Tessellation Automata

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Received July 1, 1969

This report deals with the question of whether or not, for a given tessellation automaton, there exists a finite pattern that cannot evolve from a given primitive pattern no matter what sequence of environmental input transformations are applied. This is closely related to Moore's Garden-of-Eden problem. We begin dealing with this question for the simplest nontrivial tessellation automata, namely, one-dimensional binary scope-\(n\) tessellation automata. We show that any finite pattern can evolve from the primitive pattern if the neighborhood scope is four or more. We show that there are finite patterns that cannot evolve from the primitive pattern for the scope-two case. Although some partial results are presented, the question is still open for the scope-three case. Some results for more general tessellation automata are also discussed.

I. INTRODUCTION

The tessellation automaton is a formalization of the concept of an infinite regular array of finite-state machines each directly connected to a finite number of other machines in a uniform way throughout the array. Each machine can synchronously change its state at discrete time steps as a function of the states of the machines from which it can directly receive information. This function can change from time step to time step, but will be identical for each machine in the array at any given time step. The simultaneous action of these "local" functions will define "global" functions which will act on the entire array changing "patterns" of machine states in the array to other patterns.
Automata of this form are receiving more attention not only as a medium for theoretical studies of pattern recognition [1]–[3], biological modeling [4]–[10], and evolution processes [3], [11], and [12] but also as a foundation for a theory of logical design based on integrated circuits [13] and [14].

A systematic theory of the structure of tessellation automata is being developed and will be reported elsewhere [15]–[18]. Here we begin an attack on a fundamental problem concerning the behavior of such automata, namely: For an arbitrarily given tessellation automaton, are there state patterns of the array that can never be assumed by the array no matter what sequence of global transformations is applied to a certain canonical starting pattern? We shall refer to this as the completeness problem for tessellation automata. A slight variation of this would be a generalization of Moore's Garden-of-Eden Problem [19] and [20]: Are there state patterns for the array that can never be assumed except at time 0? Note that any global transformation is allowed here, whereas Moore restricted himself to a single global transformation applied to a (two-dimensional) array repeatedly.

The main results of this report will be concerned with the completeness problem for a class of one-dimensional tessellation automata. Along the way, we shall obtain many further results, some relating to the generalization of Moore's problem.

II. THE GENERAL TESSELLATION AUTOMATON

Before defining the specific class of tessellation automata that will be studied in this report, it seems useful for purposes of perspective and orientation to look, in this section and the next, at the completeness problem for arbitrary tessellation automata.

Our formal definition of the (general d-dimensional) tessellation automaton (TA) takes the form of a quadruple

$$(A, E^d, X, I),$$

where $A$ is a finite nonempty set called the state alphabet, and can be thought of as the set of states that each of the machines in the array can assume. $E^d$, called the tessellation array, is the set of all $d$-tuples of integers. The elements of $E^d$ will be referred to as cells and can be thought of as names of the machines in the array.\(^1\) For example, an element $i$ in $E^3$ can be visualized as the name of the machine situated at the lattice point in 3-space indicated by the triple, $i$. The superscript $d$ will be referred to as the tessellation dimension. $X$, called the neighborhood index of the TA, is an $n$ tuple of distinct $d$ tuples of integers and is used in the formal development to define the neighbors of any cell, i.e., those cells from which the cell will directly

\(^1\) We use the notation $E^d$ rather than the usual $Z^d$ whenever the interpretation at hand is the set of names of cells.
receive information. If $X = (\xi_1, \ldots, \xi_n)$ and $i \in E^d$, then $N(i)$, called the neighborhood of cell $i$, is the $n$ tuple $(i + \xi_1, \ldots, i + \xi_n)$, where $i + \xi_k, 1 \leq k \leq n$, is the componentwise sum of the $d$ tuples. Each component of $N(i)$ is called a neighbor of cell $i$.

Note that since $X$ defines the neighbors of each cell in $E^d$, the "relative positions" of the neighbors of any cell can be thought of as being the same throughout the array.

Before we can define $I$, we need some preliminary notions. Let $c$, which we shall refer to as an (array) configuration, be an arbitrary mapping from $E^d$ into $A$. The image of $i \in E^d$ under $c$, $c(i)$, will be referred to as the state of cell $i$ in configuration $c$. By the state of the neighborhood of cell $i$ in configuration $c$ we mean the $n$ tuple in $A^n$ denoted by $cN(i)$ and defined to be $(c(i + \xi_1), \ldots, c(i + \xi_n))$, where $X = (\xi_1, \ldots, \xi_n)$. The state of any cell $i$ at time $t + 1$ can be thought of as determined solely by the state of its neighborhood at time $t$ according to a function $\sigma : A^n \rightarrow A$, and each cell will use the same function at the same time step. Any such mapping will be referred to as a local transformation. Given a configuration $c$ "at time $t"$, the application of $\sigma : A^n \rightarrow A$ to the state of the neighborhood of cell $i$ determining $c'(i)$, for all $i$, will determine a "global" or parallel transformation $\tau$ which will produce configuration $c'$ at "time $t + 1". More precisely, for each $\sigma : A^n \rightarrow A$ there is a uniquely determined mapping $\tau : C \rightarrow C$, where $C$ is the set of all array configurations for the given TA. $\tau$ is defined as follows. For each $c, c' \in C, \tau c = c'$ if, and only if, for any cell $i, c'(i) = \sigma(cN(i))$. These parallel transformations are to be interpreted as describing the allowable one-step changes that can occur in array configurations. It is easy to show that no parallel transformation for a given TA could be defined by two different local transformations.

We now complete the TA definition by defining $I$ to be an arbitrary nonempty subset of $T$, where $T$ is the set of all possible parallel transformations for a given $A, E^d$, and $X$. $I$ can be thought of as specifying which of the possible next-state functions are actually "wired in" for the array of machines and can be executed as "primitive instructions."

For any TA $M = (A, E^d, X, I)$, if there exists a sequence of configurations $c_0, c_1, \ldots, c_m, m \geq 0$, such that $c_i \tau_i = c_{i+1}, 0 \leq i < m$, where the $\tau_i$ are arbitrary elements of $I$, then we say $M$ generates $c_m$ from $c_0$ (in $m$ steps).

### III. The General Completeness Problem

One of the symbols in the state alphabet for any TA considered in this report will be called the quiescent symbol and will be denoted by $0$. A configuration $c$ for a given TA will be called finite if $c(i) = 0$ for all but finitely many cells $i$. We shall be concerned exclusively with finite configurations. The set of all finite configurations for some given TA will be denoted by $\mathcal{C}$. It is easy to see that $\mathcal{C}$ is closed for a given parallel transformation $\tau$, (i.e., for any $c \in \mathcal{C}, c\tau$ is also in $\mathcal{C}$) if, and only if,
where $\sigma(0,0,...,0) = 0$ where $\sigma$ is the local transformation defining $\tau$. The largest subset of $\Gamma$, for some TA $(A, E^2, X, T)$, that contains only such finite-configuration-preserving transformations will be denoted by $\Gamma$. Note that $\#(\Gamma) = \#(\Gamma)/\#(A)$.

We shall often not wish to distinguish configurations that differ only by a "shifting," i.e., $c_1$ and $c_2$ will be called shift-equivalent if, and only if, there exists a $k \in Z^4$ such that for any $i \in E^4$, $c_1(i + k) = c_2(i)$. The equivalence classes on $C$ determined by the relation of shift-equivalence will be called finite patterns, or just patterns. $[c]$ will denote the pattern containing configuration $c$.

A primitive configuration is defined as any configuration $c$ such that $c(0,0,...,0) = a$ for some nonquiescent symbol $a$, and for all $i \neq (0,0,...,0), c(i) = 0$; thus a primitive pattern is one that is quiescent at every cell except one that is nonquiescent.

The general completeness problem for tessellation automata can now be stated: With respect to an arbitrary TA $M = (A, E^2, X, T)$ with the maximum set of finite-configuration-preserving parallel transformations, let $M(c) = \{c' | M$ generates $c'$ from $c\}$, and for $S$ a set of configurations, let $M(S) = \bigcup_{c \in S} M(c)$. The completeness problem for $M$ is then the question whether $M(S_P) = \mathcal{P}_M$ where $S_P$ is the set of primitive configurations and $\mathcal{P}_M$ is the set of all (finite) patterns of $M$. Stated in a metaphorical way, the completeness problem is concerned with whether or not an arbitrary pattern can possibly “evolve” from the most simple nonnull patterns, in a tessellation structure, under full freedom for “environmental” change. This clearly suggests an interpretation of the completeness problem in the evolution theory work mentioned earlier. Another obvious application would be to the work in pattern processing on iterative arrays.

Besides having applications, this problem clearly has an intrinsic mathematical significance which is seen by noting the similarity of pattern generations on a TA and theorem derivations in a formal system.

IV. ONE-DIMENSIONAL BINARY SCOPE-N TESSELATION AUTOMATA

The problem of determining, for an arbitrary TA, whether or not an arbitrary pattern can be generated from a primitive pattern is certain to be one of the major topics in the theory of tessellation automata. This general problem, however, is still unsolved after much effort, and appears very difficult. Here we shall report some specific results concerning this problem for a class of one-dimensional TA with contiguous neighborhood structure. Some remarks concerning the relaxation of these conditions will be made later.

By a one-dimensional, binary, scope-$n$ tessellation automaton, which we often will abbreviate by saying just scope-$n$ TA, we mean a TA of the form:

$$(B, E^1, X, T),$$
where \( B = \{0, 1\} \), and \( X = ((k + 1), (k + 2), \ldots, (k + n)) \), for some integer \( k \) and positive integer \( n \).

Let \( M \) be any scope-\( n \) TA. It can be shown (as an immediate corollary of Proposition IV.1 of Reference [16]) that for any finite \( c \), the set of patterns \( M(c) \) is independent of the scope-\( n \) TA chosen, i.e., if \( M' \) is any other scope-\( n \) TA, then \( M(c) = M'(c) \). We shall denote this set of patterns defined by any one-dimensional binary scope-\( n \) TA from \( c \) by \( A^{(n)}(c) \). It proves convenient in what follows to work with \( X = \{(-1), (0)\} \) for scope-2 TA, \( X = \{(-1), (0), (1)\} \) for scope-3 TA, and \( X = \{(-1), (0), (1), (2)\} \) for scope-4 TA.

Hence, in what follows, when we speak of the one-dimensional binary scope-\( n \) TA we shall mean that \( X \) has been chosen as \( \{(-1), (0), (1), \ldots, (n - 2)\} \).

If we let \( P \) be the set of all finite patterns for \( E^1 \) with respect to \( B \), and \( c_p \) be the (unique) primitive configuration, then the main results of this report can be stated as follows. \( A^{(n)}(c_p) \subseteq P \), \( A^{(n)}(c_p) = P \), for all \( n \geq 4 \).\(^3\) The case for \( A^{(3)} \) is still open. Many related results will also be established along the way.

V. THE INCOMPLETENESS OF SCOPE-2 TA

The eight possible elements of \( T \) for \( A^{(2)} \) are specified by the following eight local transformations:

\[
\begin{array}{cccccccc}
\sigma_0 & \sigma_2 & \sigma_4 & \sigma_6 & \sigma_8 & \sigma_{10} & \sigma_{12} & \sigma_{14} \\
00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
01 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
10 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
11 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

\( \tau_i \) will denote the element of \( T \) specified by \( \sigma_i \), \( i = 0, 2, \ldots, 14 \). Although the arguments for any local transformation for the scope-\( n \) TA are \( n \) tuples in \( B^n \), we shall generally omit commas for simplicity. Note that the subscripts have been obtained from the table columns considered as binary numerals. This convention will be carried over for the scope-3 transformations as well, e.g., \( \tau_{204} \) is the identity mapping on \( C \) for scope-3 TA. Note also that the table for \( \sigma_i \) can easily be constructed given \( i \). For \( \tau_i \in T \), \( i \) will always be even.

We now proceed to establish that there are finite patterns that cannot be generated from \( c_p \) by \( A^{(2)} \).

**Lemma 1.** Let \( c' \) be a configuration such that 01011 appears in consecutive cells of \( E^1 \). Then for any \( i = 0, 2, 4, 8, \) or \( 14 \), there is no configuration \( c \) such that \( c\tau_i = c' \).

\(^3\) We use ‘\( \subset \)’ to indicate proper inclusion.
Proof. We shall treat the case for $\tau_{14}$; the others are handled in a similar fashion. Assume $\sigma_{14} = c'$ where $01011$ is contained in cells $j + 1, j + 2, \ldots, j + 5$ for $c'$. From the definition of $\tau_{14}$, and from the fact that we have chosen $X = [(0, 0)],$ if such a $c$ exists, then $c(j)$ and $c(j + 1)$ must contain $00$. This requires that $c(j + 2) = 1$. Since $\sigma_{14}(10) = \sigma_{14}(11) = 1$, there can be no such configuration $c$.

Lemma 2. For any configurations $c, c'$, if $\sigma_6 = c'$, then for any $i \in \mathbb{E}^1$, $c(i)$ is uniquely determined given $c(i - 1)$ and $c'(i)$; also $c(i)$ is uniquely determined given $c(i + 1)$ and $c'(i + 1)$.

Proof. The first statement follows from the fact that $\sigma_6(10) \neq \sigma_6(11)$ and $\sigma_6(00) \neq \sigma_6(01)$. The second statement follows from the fact that $\sigma_6(01) \neq \sigma_6(11)$ and $\sigma_6(00) \neq \sigma_6(10)$.

It will be convenient to represent (one-dimensional) patterns as follows: If $c \in \mathcal{C}$ is such that $c(k + i) = b_i, 1 \leq i \leq j$, for some integer $k$ and positive integer $j$, and if $c(k + i) = 0$ for all $i < 1$ and all $i > j$, then $[c]$ can be represented by

$$
0 \ b_1 \ b_2 \ \cdots \ b_j \ 0.
$$

Lemma 3. For any configuration $c$, if $[\tau_6] = 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ \cdots \ 0$, $n = 0, 1, 2, \ldots$, then $c$ is not a finite configuration.

Proof. Assume $[c'] = [\tau_6] = 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ \cdots \ 0$, for some $n$. If the leftmost $1$ (which we henceforth denote by $1_L$) in $c'$ is in cell $i$, (i.e., $c'(i) = 1$ and $c'(j) = 0$ for all $j < i$), then by the definition of $\tau_6$, either $c(i - 1) = 1$ and $c(i) = 0$, or else $c(i - 1) = 0$ and $c(i) = 1$.

Case A $[c(i - 1) = 1$ and $c(i) = 0]$. Since $c(i - 1) = 1$ and $c'(i - 1) = 0$, we have by Lemma 2 that $c'(i - 2) = 1$. Assume that for any $j > 1$, $c'(i - j) = 0$ and $c'(i - j) = 0$ and again by Lemma 2, $c'(i - j - 1) = 1$. Therefore, $c'$ contains an infinite repetition of ones to the left and, hence, is not finite.

Case B $[c(i - 1) = 0$ and $c(i) = 1]$. We shall say that $c'$, where $[c'] = 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ \cdots \ 0$, “contains $n$ occurrences of $0 \ 1 \ 1$ to the right of $1 \ 0 \ 1 \ 1$.” We prove now that for any $n$, if $[c'] = [\tau_6] = 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ \cdots \ 0$, then if the rightmost $1$ (abbreviated from now on by $1_R$) of the $k$-th occurrence ($0 \leq k \leq n$) of $0 \ 1 \ 1$ to the right of $1 \ 0 \ 1 \ 1$ is in cell $i_k$, for $c'$, then $c(i_k) = 1$. We say that $1_R$ in $1 \ 0 \ 1 \ 1$ is the $1_R$ of the zero-th occurrence of $0 \ 1 \ 1$ to the right of $1 \ 0 \ 1 \ 1$.

Let the leftmost $1$ of $c'$ be in cell $i$, then $c'(i + 1) = 0$ and $c(i) = 1$ and $c(i + 1)$ is thereby determined (Lemma 2) to be $1$. $c'(i + 2) = 1$ and $c(i + 1) = 1$ requires that $c(i + 2) = 0$. $c'(i + 3) = 1$ and $c(i + 2) = 0$ requires that $c(i + 3) = 1$. This verifies the above claim for $k = 0$. Assume for any $k, 0 < k < n$, if $1_R$ of the $k$-th occurrence of $0 \ 1 \ 1$ is in cell $i_k$, then $c(i_k) = 1$. But then $c'(i_k + 1) = 0$ and $c(i_k) = 1$.
requires that $c(i_k + 1) = 1$. Continuing as above, we find that $c(i_k + 3) = 1$, and our claim is proven.

Now if $c'(j) = 1$, then $c'(j + 1) = 0$ and $c(j) = 1$ requires (Lemma 2) that $c(j + 1) = 1$.
$c'(j + 2) = 0$ and $c(j + 1) = 1$ requires that $c(j + 2) = 1$, etc. Therefore, $c$ contains an infinite repetition of ones to the right and hence is not finite.

**Theorem 4** [The incompleteness of $A^{(2)}(c_p)$]. $A^{(2)}(c_p) \subset \mathcal{P}$, i.e., there are patterns that cannot be generated from $c_p$ on $A^{(2)}$.

**Proof.** Note that $\tau_{10}$ and $\tau_{12}$ preserve patterns. $\tau_{10}$ is the identity on $C$, and $\tau_{12}$ merely “shifts” any configuration one cell to the right. Hence, $\tau_{10}$ and $\tau_{12}$ are useless for the problem at hand. For any $n$, if $\tilde{0}1011(011)^n\tilde{0}$ is in $A^{(2)}(c_p)$, then there exists a sequence $\xi = \tau_{t_1}, \tau_{t_2}, \ldots, \tau_{t_m}$ in $(T)^*$ such that $[c_p\xi] = \tilde{0}1011(011)^n\tilde{0}$. Since finite configurations are preserved by the elements of $T$, there must be a $c \in C$ such that $[c\tau_{t_m}] = \tilde{0}1011(011)^n\tilde{0}$. But this is impossible by Lemmas 1 and 3.

**VI. Some Further Properties of Scope-2 TA**

For any set $S \subseteq C$ let $A^{(n)}(S) = \bigcup_{c \in S} A^{(n)}(c)$. Note that for any $S \subseteq C$ if all the patterns $\tilde{0}1011(011)^n\tilde{0}$ are in $A^{(2)}(S)$, then these patterns must all be in $S$. Hence, if $S$ is such that $A^{(2)}(S) = \mathcal{P}$, $S$ must be an infinite subset of $C$. This establishes

**Theorem 1.** For any finite set $S$, $S \subseteq C$, $A^{(2)}(S) \subset \mathcal{P}$.

Using the analogy with formal systems, we can interpret this as saying no finite axiomatization of $\mathcal{P}$ is possible for $A^{(2)}$.

In applications such as biological modeling or pattern generation, one would not only be interested in whether or not a particular configuration could be attained by the array, but also whether or not it could be attained by a “monotonic growth.” Organisms generally grow monotonically, and efficient pattern generations would tend to be monotonic. Mathematically, this problem has some application to the question of decidability of the sets of generable patterns.

By the length of $c \in C$, $lg(c)$, we mean the nonnegative integer $j - i + 1$, where $c(i) = 1_L$ and $c(j) = 1_R$. If $c(i) = 0$ for all $i \in E^1$, then $lg(c) = 0$. The length of a pattern will be the length of any representative configuration. We say that pattern $[c]$ is generated from $c_p$ on scope-$n$ TA $A^{(n)} = (B, E^1, \mathit{X}, T)$ by a strictly monotone production if there exists a sequence $c_p = c_0, c_1, \ldots, c_m = c$ such that $lg(c_i) < lg(c_{i+1})$, $c_i r_{j_i} = c_{i+1}$, $r_{j_i} \in T$, $1 \leq i < m$. If $lg(c_i) \leq lg(c_{i+1})$ for $1 \leq i < m$, then the production above is said to be monotone. Let $A^{(n)}_{SM}(c_p)$ and $A^{(n)}_{M}(c_p)$ denote the set of all patterns generated from $c_p$ by strictly-monotone and monotone productions, respectively.
**Lemma 2.** $01001110$ is in $\mathcal{A}^{(2)}(c_p)$ but not in $\mathcal{A}^{(2)}_M(c_p)$.

**Proof.**

\[ [c_p] = 010, \]
\[ c_pT_6 = c_2, \quad [c_2] = 0110, \]
\[ c_2T_{14} = c_3, \quad [c_3] = 01110, \]
\[ c_3T_6 = c_4, \quad [c_4] = 010010, \]
\[ c_4T_6 = c_5, \quad [c_5] = 0110110, \]
\[ c_5T_6 = c_6, \quad [c_6] = 01011010, \]
\[ c_6T_6 = c_7, \quad [c_7] = 01001010, \]
\[ c_7T_6 = c_8, \quad [c_8] = 011011110, \]
\[ c_8T_6 = c_9, \quad [c_9] = 011001110. \]

Note that $\lg(c_8) = 7$ and $\lg(c_9) = 6$. We complete the proof by showing that for any $c \in C$, if $cT = c_9$, then $\tau = T_6$ and $[c] = [c_9]$. Using the same procedure as in the proof of Lemma V.1, we can show that for $i = 2, 4, 14$, there is no $c$ such that $cT_i = c_9$. If there is a configuration $c$ such that $cT_6 = c_9$ and if $c_9(i) = 1_L$, then $c(i - 1)$ and $c(i)$ must contain, respectively, either 01 or 10. 01 leads to $[c] = 011011110$ and 10 leads to $[c] = \cdots 111011111110$, both of which are infinite. If for some $c$, $cT_8 = c_9$, and if $c_9(i) = 1_L$, then $c(i - 1)$ and $c(i)$ must contain 11. By an argument similar to the one used to prove Lemma V.2, we can show that $c$ is uniquely determined and $[c] = 011011110 = [c_9]$.

**Lemma 3.** $01010010$ is in $\mathcal{A}^{(2)}_M(c_p)$ but not in $\mathcal{A}^{(2)}_{SM}(c_p)$.

**Proof.**

\[ [c_p] = 010, \]
\[ c_pT_6 = c_2, \quad [c_2] = 0110, \]
\[ c_2T_{14} = c_3, \quad [c_3] = 01110, \]
\[ c_3T_6 = c_4, \quad [c_4] = 010010, \]
\[ c_4T_6 = c_5, \quad [c_5] = 0110110, \]
\[ c_5T_6 = c_6, \quad [c_6] = 01011010, \]
\[ c_6T_2 = c_7, \quad [c_7] = 01010010. \]

Note that $\lg(c_6) = \lg(c_7)$. It can be shown as in the proof of Lemma V.1, that there is no $c$ such that $cT_i = c_7$, for $i = 8$ or 14. If for some $c$, $cT_6 = c_7$, and if $c_7(i) = 1_L$, then $c(i - 1)$ and $c(i)$ must contain either 01 or 10. The first possibility implies, as before, that $c$ is not finite; the second possibility implies that $\lg(c) = \lg(c_7)$. If there is a $c \in C$ such that $cT_6 = c_7$, then it can be shown that $c(i - 1) = c(i + 4) = 1$, \ldots
hence \( \lg(c) \geq \lg(c_i) \). Finally, if there is a \( c \in C \) such that \( c_{r_2} = c_7 \), and if \( c_i(i) = 1_L \), then \( c(i) = 1 \). If \( c \) is to be shorter than \( c_7 \), then \( c(i + 4) \) and \( c(i + 5) \) must contain 0 0 or 1 0. Since \( \sigma(00) = \sigma(10) = 0 \), this is impossible.

We have now established

**Theorem 4.** \( A_{SM}^{(2)}(c_p) \subset A_{M}^{(2)}(c_p) \subset A_{SM}^{(2)}(c_p) \subset \mathcal{P} \).

**VII. Some Properties of Scope-3 TA**

In this section and in the next we shall present some results relating to the completeness problem for scope-3 TA. For example, we shall establish here that 

\( \sigma_{SM}(c) \subset \mathcal{P} \), and if \( \sigma_{SM}(S) = \mathcal{P} \) then \( S \) must be an infinite set. The question of whether or not \( \sigma_{SM}(c) = \mathcal{P} \) is still open in spite of a great deal of effort.

We begin this section by showing that a pattern exists that cannot be generated from \( c_p \) by the scope-3 TA with a strictly-monotone production.

It is easy to see that the local transformations \( \sigma_{240}, \sigma_{204}, \) and \( \sigma_{170}, \) specified below define parallel transformations ("left shift," "identity," and "right shift") such that for all \( c \in C \), if \( c \) is taken by any one of them to \( c' \), then \( \lceil c \rceil = \lceil c' \rceil \). Hence, for the problem of pattern generation these transformations can be ignored.

\[
\begin{array}{cccc}
000 & 0 & 0 & 0 \\
001 & 0 & 0 & 1 \\
010 & 0 & 1 & 0 \\
011 & 0 & 1 & 1 \\
100 & 1 & 0 & 0 \\
101 & 1 & 0 & 1 \\
110 & 1 & 1 & 0 \\
111 & 1 & 1 & 1 \\
\end{array}
\]

For any \( \tau \in \mathcal{T} \) (for scope-3 TA), we say that \( \tau \) has property \( R \) (property \( L \)) if, and only if, for any \( a, b \in B \), \( \sigma(0ab) \neq \sigma(ab0) \) \( [\sigma(ab0) \neq \sigma(ab1)] \), where \( \sigma \) defines \( \tau \).

Let \( \beta = a_1, a_2, \ldots, a_k, \) \( a_i \in B, \) \( 1 \leq i \leq k \). We will say that \( c \) contains \( \beta \) (at \( j \)) if \( c(j + i) = a_i, \) \( 1 \leq i \leq k \). Also, we will say that \( \beta \) blocks \( \tau \) if for every \( c' \) that contains \( \beta \) there does not exist a \( c \) such that \( c\tau = c' \).

**Lemma 1.** There exists a finite sequence of elements of \( B \), e.g.,

\[
\alpha = 0001011001000110111011110010101001101101,
\]
that blocks any scope-3 parallel transformation in \(T - \{\tau_{304}\}\) that does not have property \(R\) or \(L\).

**Proof.** For any given \(\beta = a_1, a_2, \ldots, a_k, a_i \in B, 1 \leq i \leq k\), and any \(\tau \in T - \{\tau_{304}\}\) (\(\tau_{304}\) is the identity mapping on \(C\) and is an exception), it can be shown that the procedure below will stop with a verification that \(\beta\) blocks \(\tau\) if, and only if, \(\beta\) does block \(\tau\) (see the proof procedure of Lemma V.1).

**Step 1.** List above \(a_1\) all triples \((b_1, b_2, b_3) \in B^3\) such that \(a(b_1, b_2, b_3) = a_1\), where \(a\) defines \(\tau\). If there is no such triple, the process terminates with the conclusion that \(\beta\) blocks \(\tau\), otherwise go to Step 2.

**Step \(i\) \((1 < i \leq k)\).** List above \(a_i\) all triples \((d_1, d_2, d_3) \in B^3\) such that \(a(d_1, d_2, d_3) = a_i\) and for some \(b \in B\), \((b, d_1, d_2)\) has been listed above \(a_{i-1}\) during Step \(i - 1\). If \(i < k\) and at least one triple can be listed above \(a_i\), go to Step \(i + 1\). If no triples can be listed above \(a_i\) then the process terminates with the conclusion that \(\beta\) blocks \(\tau\). If \(i = k\) and at least one triple can be listed above \(a_k\) then the process terminates with no conclusion.

By inspection of the table specifying any \(a\) defining some \(\tau \in T\) it can be readily determined whether or not \(\tau\) has property \(R\) or \(L\). The following are all transformations in \(T - \{\tau_{304}\}\) with property \(R\) or \(L\) or both: \(\tau_{30}, \tau_{60}, \tau_{90}, \tau_{102}, \tau_{106}, \tau_{120}, \tau_{150}, \tau_{154}, \tau_{20}, \tau_{210}, \tau_{240}\) (right shift), \(\tau_{168}, \tau_{170}\) (left shift). Using the above procedure it can be verified that \(\alpha = 00010110001000101011110010001001100\) blocks any \(\tau \in T - \{\tau_{304}\}\) not having property \(R\) or \(L\).

It might be noted that \(\alpha\) has been constructed by combining some short strings each of which blocks only certain transformations. A procedure leading to the shortest string that can be used in place of \(\alpha\) is known but tedious. We suspect the length of \(\alpha\) is close to being the shortest. Although Lemma 1 can be generalized to more general TA, we do not need this here.

**Lemma 2.** Let \(c\) be in \(\hat{C}\) with \(c(i) = 1_L\) and \(c(j) = 1_R\), and let \(\tau \in T\). If \(c\tau = c'\), \(c'(k) = 1_R\) and \(c'(p) = 1_L\), then \(k \leq j + 1\) and \(p \geq i - 1\), hence \(\log(c') \leq \log(c) + 2\).

**Proof.** These are simple consequences of the fact that if \(\sigma\) is a local transformation defining some \(\tau \in T\), then \(\sigma(000) = 0\).

The following descriptive technique will be useful for illustrating some of the discussions to follow. The examples below should suffice to make the positional display conventions clear.

Suppose \([c] = 0100111010001\) and \(c\) is transformed by \(\tau_{44}\) to \(c'\) then we can write

\([c] = 0100111010001\)
\([c'] = 01001011100\),
where a symbol in \([c]\) is written directly above a symbol in \([c']\) if these symbols represent the contents of the same cell for \(c\) and \(c'\).

If we are interested only in the relative positions of \(1_L\) and \(1_R\) for \(c\) and \(c'\), then we would write

\[
[c] = 0110
\]
\[
[c'] = 0110
\]

where the line between the 1's indicates we are unconcerned with the symbols between the 1's.

**Theorem 3.** \(\mathcal{A}^{(3)}_{SM}(c_p) \subset \mathcal{A}^{(2)}_{M}(c_p)\), hence \(\mathcal{A}^{(3)}_{SM}(c_p) \subset \mathcal{P}\).

**Proof.** The pattern

\[
[c'] = 010101110001011001000110111011100101001101101010101010101010
\]

is shown in Appendix A to be in \(\mathcal{A}^{(3)}_{SM}(c_p)\). Since the finite string \(\alpha\) of Lemma 1 is contained in consecutive cells for \(c'\), it follows that if there is some \(c\) such that \(cT = c'\), \(\tau \in T - \{\tau_{204}\}\), then \(\tau\) must have property \(R\) or \(L\). If \(lg(c) < lg(c')\), then by Lemma 2, one of the following cases below must hold.

**Case A.**

\[
[c] = 0110
\]
\[
[c'] = 0110
\]

**Case B**

\[
[c] = 00110
\]
\[
[c'] = 0110
\]

**Case C**

\[
[c] = 00110
\]
\[
[c'] = 0110
\]

The only possible transformations with property \(R\) or \(L\) that could take \(c\) to \(c'\) in the manner of Case A are \(\tau_{60}, \tau_{180}\), and \(\tau_{180}\). This follows from the fact that Case A requires that any transformation taking \(c\) to \(c'\) in this way have a local transformation that takes 001 to 0 and 100 to 1. In a similar way, it follows that only \(\tau_{102}, \tau_{106}\), and \(\tau_{186}\) are possible for Case B, and only \(\tau_{50}, \tau_{56}, \tau_{80}, \tau_{150}, \tau_{154}\), and \(\tau_{216}\) are possible for Case C.

Assume \(c\tau_{180} = c'\) in the manner of Case A, and \(c'(i) = 1_L\). Clearly \(c(i - 2), c(i - 1),\) and \(c(i)\) contain 001, respectively. From the definition of \(\tau_{180}\), in order that the triple in \(c(i)\), \(c(i - 1),\) and \(c(i + 1)\) determine the 1 in \(c'(i), c(i + 1)\) must be 1. In order that the triple in \(c(i), c(i + 1),\) and \(c(i + 2)\) determine the 0 in \(c'(i + 1), c(i + 2)\) must be 0. Continuing in this way, we find that there is no possible bit for \(c(i + 4)\) such that \(c(i + 2), c(i + 3),\) and \(c(i + 4)\) determine the 1 in \(c'(i + 3)\).
We can conclude, that there is no $c$ such that $c_{780} = c'$ consistent with the form of Case A.

Using exactly the same technique, starting with $1_L$ of $c'$ when $\tau$ has property $R$ (as above), and starting with $1_R$ of $c'$ when $\tau$ has property $L$ (moving leftward), we can show that no $c$ exists such that $c_{780} = c'$ for Case A, no $c$ exists such that $c_{780} = c'$, for $i = 106$ or $166$ for Case B, and $i = 30, 86, 154,$ or $210$ for Case C. (Note that this technique is to apply the procedure used in the proof of Lemma 1 for only certain triples listed above the starting symbol.)

At this point only $\tau_{70}$ for Case A, $\tau_{702}$ for Case B, and $\tau_{790}$ and $\tau_{150}$ for Case C have yet to be considered. We now show that no finite $c$ exists such that $c_{780} = c'$ for Case A. In a similar way, it can be shown that no finite $c$ exists for $\tau_{702}$ for Case B, or for $\tau_{790}$ or $\tau_{150}$ for Case C.

Assume $c_{780} = c'$ and $c'(i) = 1_L$ and $c'(j) = 1_R$. $c(i-1)$ and $c(i)$, respectively, must contain 01 for Case A. Although both 010 and 011 are taken by $\sigma_{70}$ to 1, $c(i + 1)$ must be 1 in order for $\tau_{70}$ to produce the 0 in cell $i + 1$ for $c'$. Again, although both 110 and 111 are taken by $\sigma_{70}$ to 0, $c(i + 2)$ must be 0 in order for $\tau_{70}$ to produce the 1 in cell $i + 2$ for $c'$. At each step another bit of $c$ is uniquely determined, hence some $c$ does exist such that $c_{780} = c'$. Continuing, we eventually find that $c(j - 1)$, $c(j)$, and $c(j + 1)$ do not contain 100, respectively, as required by Case A. We are forced to conclude that $c$ is not finite.

Without much difficulty, it can be shown that $[c']$ of Theorem 3 can be replaced by any pattern of the form

$$\overline{01010111(000)^n10110010001101111000101001101101010101010101010101010101010101}$$

for $n = 1, 2, ..., $ and the proof of Theorem 3 will carry through unaffected. This implies

**THEOREM 4.** For any finite subset $S \subseteq \overline{C}$, $\mathcal{H}_{SM}^{(3)}(S) \subseteq \mathcal{H}_{SM}^{(3)}(c_p)$, hence $\mathcal{H}_{SM}^{(3)}(S) \subseteq \mathcal{P}$.

Each incompleteness result discussed so far was obtained by constructing a configuration $c'$ such that for any parallel transformation $\tau$, of interest for the case at hand, there was no $c$ such that $c_{780} = c'$, satisfying appropriate constraints. Note the relation of this to the Garden-of-Eden problem, e.g., we have shown that any TA of the form $\langle (0, 1), E^3, [(k), (k+1)], \{\tau\} \rangle$, where $\tau$ is nontrivial, has Garden-of-Eden configurations. We shall see now that we cannot make this statement if we replace neighborhood index $[(k), (k+1)]$ above by $[(k), (k+1), (k+2)]$.

**THEOREM 5.** For any $c' \in \overline{C}$ there exists a $c \in \overline{C}$ and a $\tau_i$, $i = 30, 118, 120,$ or $180$, such that $lg(c) \leq lg(c')$ and $c_{780} = c'$.

This result shows that the proof procedure we have been using cannot be extended to $\mathcal{H}_{SM}^{(3)}$, and hence not to $\mathcal{H}^{(3)}$. The proof of Theorem 5 is rather lengthy and is
presented in Appendix B. It involves constructing a state graph such that if one follows the path specified by the consecutive symbols of any \( c' \in C \) from \( l_L \) to \( l_R \), then from information contained in the consecutive nodes traversed, one can construct a \( c \in C \) and choose a \( r \) satisfying the theorem.

VIII. SOME PRODUCTIONS IN SCOPE-3 TA

In this section we indicate some production schemes possible in \( \mathcal{A}^{(3)}(c_p) \) which should give some insight into the extent of \( \mathcal{A}^{(3)}(c_p) \) and which should illustrate some of the difficulties involved in designing transformation sequences which will generate desired patterns in \( \mathcal{A}^{(3)}(c_p) \).

For the convenience of the reader, the local transformations defining the transformations used in this section are listed below.

<table>
<thead>
<tr>
<th>( \sigma_{140} )</th>
<th>( \sigma_{156} )</th>
<th>( \sigma_{196} )</th>
<th>( \sigma_{198} )</th>
<th>( \sigma_{204} )</th>
<th>( \sigma_{220} )</th>
<th>( \sigma_{224} )</th>
<th>( \sigma_{226} )</th>
<th>( \sigma_{228} )</th>
</tr>
</thead>
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<tr>
<td>000</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
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<td>1</td>
<td>0</td>
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<td>1</td>
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<td>0</td>
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<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>111</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Lemma 1.** For any positive integer \( n \), \( \bar{0} \ 1^n \ \bar{0} \in \mathcal{A}^{(3)}(c_p) \).

**Proof.** \( [c_p \tau_{220}^{-1}] = \bar{0} \ 1^n \ \bar{0} \).

**Proposition 2.** For any positive integers \( n_1, m_1, n_2 \), \( \bar{0} \ 1^n \ 0 \ \bar{0} \ 1^{m_1} \ 0 \ \bar{0} \ \in \mathcal{A}^{(3)}(c_p) \).

**Proof.** If \( n_2 \geq n_1 \), then \( c_p \tau_{220}^{-1} \tau_{198}^{-1} \tau_{198}^{-1} \tau_{220}^{-1} \) produces the required pattern as follows:

\[
\bar{0} \ 1 \ \bar{0}.
\]  

(1)

\( \tau_{220}^{-1} \) applied to (1) yields

\[
\bar{0} \ 1^{n_1+n_2} \ \bar{0}.
\]  

(2)

\( \tau_{198} \) applied to (2) yields

\[
\bar{0} \ 1 \ 0 \ 1^{m_1+n_2-1} \ \bar{0}.
\]  

(3)
\[ \tau_{w_1}^{m_1+n_1-2} \] applied to (3) yields
\[ \bar{0} 1 0^{m_1+n_1-1} 1^{n_1-n_1+1} \bar{0}. \] (4)

\[ \tau_{w_2}^{n_1-1} \] applied to (4) yields the required pattern.
If \( n_1 < n_2 \), then the required pattern is similarly produced by
\[ c_p \tau_{196}^{m_1+n_2-1} \tau_{196}^{m_1+n_1-2} \tau_{226}^{n_1-1} \]

**Lemma 3.** For any positive integers \( m_1, m_2, \ldots, m_{k-1} \),
\[ \bar{0} 1 0^{m_1} 1 0^{m_2} \ldots 1 0^{m_{k-1}} 1 \bar{0} \in \mathcal{A}^{(3)}(c_p). \]

**Proof.**
\[ \bar{0} 1 0^{m_1+m_2+\ldots+m_{k-1}+1} \bar{0} \]
is by Lemma 1 in \( \mathcal{A}^{(3)}(c_p) \). \( \tau_{196} \) applied to (1) yields
\[ \bar{0} 1 0 1^{m_1+m_2+\ldots+m_{k-1}} \bar{0}. \] (2)

[Note that \( \tau_n \) in (1) is shifted one cell to the left in going from (1) to (2).] \( \tau_{226} \) applied to (2) yields
\[ \bar{0} 1 0^{m_1} 1 0^{m_2+m_3+\ldots+m_{k-1}+1} \bar{0}. \] (3)

(Here the two leftmost 1's have been shifted left.) \( \tau_{196}^{m_2-1} \) applied to (3) yields
\[ \bar{0} 1 0^{m_1} 1 0^{m_2+m_3+\ldots+m_{k-1}} \bar{0}. \] (4)

Continuing as above, i.e., by applying
\[ \tau_{226}^{m_3-1} \tau_{196}^{m_2-1} \tau_{226}^{m_4-1} \ldots \tau_{226}^{m_{k-1}-1}, \]
we arrive at the required pattern.

**Proposition 4.** For any positive integers \( n_1, n_2, \ldots, n_k \),
\[ \bar{0} 1^{n_1} 0 1^{n_2} 0 \ldots 1^{n_k} \bar{0} \] is in \( \mathcal{A}^{(3)}(c_p) \).

**Proof.** By Lemma 3 there is a \( \xi \) such that
\[ [c_p \xi] = \bar{0} 1 0^{n_1} 1 0^{n_2} 1 \ldots 0^{n_{k-1}} 1 \bar{0}. \] (1)
Let $n_m$ be the max of $n_i$, $1 \leq i \leq k$. Then $\tau_{220}^{n_m}$ applied to (1) yields

$$0 \, 1^{n_1} \, 0 \, 1^{n_2} \, 0 \, \ldots \, 1^{n_{k-1}} \, 0 \, 1^{n_m} \, \bar{0}.$$  \hfill (2)

If $n_k = n_m$, we are finished. If $n_k \neq n_m$, then $1^{n_m}$ [on the right of (2)] must be shortened to $1^{n_k}$. This can be accomplished by applying $\tau_{224}^{n_m-n_k}$ to (2).

**LEMMA 5.** If each $n_i$, $1 \leq i \leq k-1$, is greater than 1, and each $n_j$, $1 \leq j \leq k$, is odd, then

$$0 \, 1^{n_1} \, 0^{m_1} \, 1^{n_2} \, 0^{m_2} \, \ldots \, 1^{n_{k-1}} \, 0^{m_{k-1}} \, 1^{n_k} \, \bar{0} \quad \text{as in } \mathcal{A}(\mathcal{P})[c_p].$$

**Proof.** Let $h^1 = 1$ and $h^n = (10)^{n-1} \, 1(\bar{h}^3 = 10101, \, h^4 = 1010101, \text{etc.})$, and let $[x]$ be the smallest integer not less than $x$. By Lemma 3,

$$0 \, h^{n_1/2} \, 0^{m_1} \, h^{n_2/2} \, 0^{m_2} \, \ldots \, h^{n_{k-1}/2} \, 0^{m_{k-1}} \, h^{n_k/2} \, \bar{0} \quad \text{is in } \mathcal{A}(\mathcal{P})[c_p].$$

By applying $\tau_{228}$ to this, we obtain the required pattern.

By Lemma 5, and one application of $\tau_{220}$, we have

**PROPOSITION 6.** For any $n_i$, $1 \leq i \leq k$, and any $m_j$, $1 \leq j \leq k-1$,

$$0 \, (1 \, 1)^{n_1} \, 0^{m_1} \, (1 \, 1)^{n_2} \, \ldots \, 0^{m_{k-1}} \, (1 \, 1)^{n_k} \, \bar{0} \quad \text{is in } \mathcal{A}(\mathcal{P})[c_p].$$

Notice that Proposition 6 shows that a simple "encoding" of any pattern is included in $\mathcal{A}(\mathcal{P})[c_p].$

**LEMMA 7.** If each $n_i$, $1 \leq i \leq k$, is either exactly two, or greater than three, and if each $m_j$, $1 \leq j \leq k-1$, is greater than one, then

$$0 \, 1^{n_1} \, 0^{m_1} \, 1^{n_2} \, 0^{m_2} \, \ldots \, 1^{n_{k-1}} \, 0 \, 1^{n_k} \, \bar{0} \quad \text{is in } \mathcal{A}(\mathcal{P})[c_p].$$

**Proof.** Let $[c_1]$ be a pattern satisfying the above form. Let $[c_2]$ obtained from $[c_1]$ by replacing each substring $1^n$ in $[c_1]$, $n_i \geq 5$ and $n_i$ odd, by $1^{n_1} \, 0 \, 1^{n_2}$, where $n_i, n_i \geq 2$, both are even and $n_i + n_i + 1 = n_i$. By Proposition 6, $[c_2]$ as in $\mathcal{A}(\mathcal{P})[c_p]$ and $[c_1] = [c_2] \tau_{226}$.

**PROPOSITION 8.** For $1 \leq i \leq k$ and $1 \leq j \leq k-1$, any of the following conditions implies that

$$0 \, 1^{n_1} \, 0^{m_1} \, 1^{n_2} \, 0^{m_2} \, \ldots \, 1^{n_{k-1}} \, 0^{m_{k-1}} \, 1^{n_k} \, \bar{0} \quad \text{as in } \mathcal{A}(\mathcal{P}).$$  \hfill (3)

(A) None of the $n_i$ equals 1, 2, or 4.

(B) None of the $n_i$ equals 1, or 3, and none of the $m_i$ equals 1.

(C) None of the $n_i$ equals 2, and none of the $m_i$ equals 1 or 2.

(D) None of the $n_i$ equals 1 and none of the $m_i$ equals 1, 2, or 3.

(E) None of the $m_i$ equals 1, 2, 3, or 4.
Proof. Case B is exactly that covered by Lemma 7. Any pattern of the form required by Case A can be obtained from a pattern of the form required by Case B by one application of $\tau_{220}$. Finally, the patterns satisfying Cases C, D, and E can be obtained from patterns satisfying Case B by one, two, or three applications of $\tau_{340}$.

Figure 1 is a schematic presentation of the results in Proposition 8.

![Figure 1](image)

**Fig. 1.** Schematic summary of Proposition 8.

**IX. A Pattern Decomposition Lemma**

By a configuration in *standard decomposition* we shall mean one that defines a pattern of the form $0 \alpha \beta 0$ where for some $n, m \geq 0$, $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$, and each $\alpha_i$ is either a finite sequence of zeros of any length or the finite sequence $110$; and $\beta = \beta_1 \beta_2 \cdots \beta_m$, and each $\beta_i$ is either a finite sequence of zeros or the finite sequence $100$. In other words $c \in \mathcal{C}$ is in standard decomposition if, and only if,

$$[c] = 01100*1100* \cdots 1100*1000*1000* \cdots 1000*10,$$

where each occurrence of $0*$ is any finite sequence of zeros, including the sequence of zero length.

Lemma 1 below is established through several lemmas and propositions, many having some intrinsic interest. For readability we have placed these results in Appendix C. It is noted here that a great deal can be said about pattern decomposition. Here we limit ourselves only to that used to contribute to our completeness results. Lemma 1 plays a central role in Sections X and XI below.
A COMPLETENESS PROBLEM FOR TA

LEMMA 1. For any \( c \in C \) there exists a configuration \( c_a \) in standard decomposition such that \( c_a^x = c \) where \( x \in (T)^* \) is composed only from the scope-3 parallel transformations \( \tau_{165} \) and \( \tau_{180} \).

It is noted in passing that any \( c \in C \) can be transformed to a configuration in standard form by a finite number of applications of \( \tau_{165} \) and \( \tau_{180} \), but we do not need this result here.

X. THE COMPLETENESS OF SCOPE-4 TA

In this section we shall establish that \( A^{(4)}(c_a) = \mathcal{P} \).

PROPOSITION 1. For any \( c \in C \) in standard decomposition, there exists a finite sequence \( \xi \) of scope-4 parallel transformations such that \( c_{\xi} = c \).

Proof. Let \([c] = 0 \ldots 0_{n,1} 0 \ldots 0_{n,2} 1 \ldots 0_{n,s} 1 0_{m,1} 0 \ldots 0_{m,s} 1 0 \ldots 0_{m,t} 1 \ldots 0_{m,u} 1 \ldots 0_{m,v} \), \( n_i \geq 1, 1 \leq i \leq r, m_j \geq 2, 1 \leq j \leq s, r, s \geq 0 \).

With respect to the TA \((B, E^4, [(0), (1), (2), (3)], T)\), it will be convenient to specify any element of \( T \) by giving the set of sequences of length four that are transformed by \( a \) to 1, where \( a \) is the local transformation defining the element of \( T \). For example, \( \tau_a \) specified by \( \{0010, 0011, 0100, 0110, 0111, 1100, 1110, 1111\} \), is the element of \( T \) specified by

\[
\sigma_a : \\
0000 0 \\
0001 0 \\
0010 1 \\
0011 1 \\
0100 1 \\
0101 0 \\
0110 1 \\
0111 1 \\
1000 0 \\
1001 0 \\
1010 0 \\
1011 0 \\
1100 1 \\
1101 0 \\
1110 1 \\
1111 1.
\]

Consider

\[
[c_{\xi} \tau_a^{m-1}] = \bar{0} 1 m \bar{0}.
\]

(1)

\( m \) will be specified later.
Let $\tau_b$ be specified by \{0001, 0011, 1001, 1011, 1101, 1111, 1110\}. $\tau_b$ applied to (1) yields
\[0\ 1\ 1\ 0\ 1^{m-2}\ 0. \quad (2)\]

Let $\tau_c$ be specified by \{0110, 1101, 1100, 0100, 1110, 1111\}. $\tau_c$ applied $n_1 - 1$ times to (2) yields
\[0\ 1\ 1\ 0^{n_1} 1^{m-n_1-1}\ 0. \quad (3)\]

$\tau_b$ applied to (3) yields
\[0\ 1\ 1\ 0^{n_1} 1\ 1\ 0^{1^{m-n_1-3}}\ 0. \quad (4)\]

$\tau_c$ applied $n_2 - 1$ times to (4) yields
\[0\ 1\ 1\ 0^{n_1} 1\ 1\ 0^{n_2} 1^{m-n_1-n_2-2}. \quad (5)\]

Continuing in this way, i.e., by applying $\tau_b\tau_c^{n_3-1}\cdots\tau_b\tau_c^{n_r-1}$ to (5), we arrive at
\[0\ 1\ 1\ 0^{n_1} 1\ 1\ 0^{n_2} \cdots 1\ 1\ 0^{n_r} 1^{m'}\ 0, \quad (6)\]
where $m' = m - n_1 - n_2 \cdots - n_r - r$.

Let $\tau_d$ be specified by \{0011, 0110, 1011, 1111, 1110, 0010\}. $\tau_d$ applied to (6) yields
\[0\ 1\ 1\ 0^{m_1} 1\ 1\ 0^{m_2} \cdots 1\ 1\ 0^{m_r} 1\ 0^{1^{m'-2}}\ 0. \quad (7)\]

$\tau_c$ applied $m_1 - 1$ times to (7) yields
\[0\ 1\ 1\ 0^{m_1} 1\ 1\ 0^{m_2} \cdots 1\ 1\ 0^{m_r} 1\ 0^{1^{m'-m_1-1}}\ 0. \quad (8)\]

By applying $\tau_d\tau_c^{m_2-1}\cdots\tau_d\tau_c^{m_r-1}$ to (8) yields
\[0\ 1\ 1\ 0^{m_1} 1\ 1\ 0^{m_2} \cdots 1\ 1\ 0^{m_r} 1\ 0^{m_2} \cdots 1\ 0^{m_s-1} 1^{m'}\ 0, \quad (9)\]
where $m' = m' - m_1 - m_2 \cdots - m_{s-1} - s + 1$.

Let $\tau_e$ be specified by \{0011, 0110, 0010\}. $\tau_e$ applied to (9) yields the desired pattern $[\tau]$ provided $m$ was initially chosen so that $m' \geq 3$. The special cases where $r$ or $s$ is zero can be easily handled.

Corollary IX.3.1, established that for any $\epsilon \in \mathcal{C}$ some $c_s$ in standard decomposition can be transformed to $c$ by some sequence $\xi$ of scope-3 transformations. From this result, Proposition 1, and the fact that all scope-$n$ transformations are included in the set of all scope-$m$ transformations, $m \geq n$, we have

**Theorem 2.** $\mathcal{A}^{(4)}(c_p) = \mathcal{P}$.

**Corollary.** For any $n \geq 4$, $\mathcal{A}^{(n)}(c_p) = \mathcal{P}$. 
XI. A Completeness Result for Two-Dimensional Binary TA

In this section we shall outline a proof of the following result.

**Theorem 1.** There exists a two-dimensional binary TA $(B, E^2, X, T)$ such that for any $c \in C$ there is a $\xi \in (T)^*$ such that $[c, \xi] = [c]$.

More explicitly, our proof outline will be an informal discussion of how for any given $c \in C$ one can construct as a function of $c$, configurations $c_d$ and $c_R$, and how one can specify $\xi_1$, $\xi_2$, $\xi_3 \in (T)^*$ and $r' \in T$ such that $[c, \xi_1] = [c_d]$, $[c, \xi_2] = [c_R]$, and $[c, r' \xi_3] = [c]$, hence the $\xi$ required by the theorem would be $\xi_1 \xi_2 r' \xi_3$.

Let $M = (B, E^2, X, T)$, where

$$X = [(-5, 1), (-4, 1), ..., (3, 1), (4, 1),$$
$$(-5, 0), (-4, 0), ..., (3, 0), (4, 0), (-5, -1), (-4, -1), ..., (3, -1), (4, -1)],$$

i.e., $X$ defines a rectangular neighborhood of height three and width ten.

By row $j$ in $E^2$, $j \in Z$, we shall mean the two-way infinite sequence of cells

$$\ldots, (-2, j), (-1, j), (0, j), (1, j), (2, j), \ldots.$$ 

With respect to $M$, for any $c \in C$ there exists a $c_d \in C$ such that for each $j$, the restriction of $c_d$ to row $j$ is a configuration in standard decomposition, or is such that $c_d(i, j) = 0$ for all $i \in Z$, and there is a $\xi_3 \in (T)^*$ such that $c_d \xi_3 = c$. A rigorous proof of this would be tedious, but from the results in Appendix C it should be clear how this could be done, e.g., the local transformation defining the terms in $\xi_3$ could be independent of all components of $X$ except $(-1, 0), (0, 0), (1, 0)$; and could be defining, in effect, $\tau_{166}$ and $\tau_{180}$. Some sequence of applications of $\tau_{166}$ and $\tau_{180}$ applied to $c$ would lead to the desired $c_d$.

Let $R'$ be the set of cells of $E^2$ enclosed by the smallest rectangle with vertical and horizontal sides where all cells $i$ such that $c_d(i) = 1$ are included in $R'$. Let $(m + 1, n + 1), (m + s, n + 1), (m + s, n + r)$, and $(m + 1, n + r)$ be the cells at the corners of $R'$. We can assume that $r$ and $s$ are both greater than one, since the theorem is easily established for $r$ or $s$ equal to one. Let $R'$ be the rectangle $R'$ extended on the left so as to include only the additional column of cells $(m, n + 1)$, $(m, n + 2), \ldots, (m, n + r)$. Now let $R$ be the rectangle formed by extending $R'$ so as to include four more columns on the left and then one line of cells on the bottom. The four "corner" cells in $R$ would be $(m - 4, n), (m + s, n), (m + s, n + r)$, and $(m - 4, n + r)$. An example is given in Fig. 2(b).
Let \( c_1 \) be the configuration defined using \( R', R', R, \) and \( c_d \), as follows:

(a) If \( i \in R - R' \), then \( c_1(i) = 1 \).
(b) If \( i \in R' - R'' \), then \( c_1(i) = 0 \).
(c) If \( i \in R' \), then \( c_1(i) = c_d(i) \).
(d) If \( i \in R \), then \( c_1(i) = 0 \).

[See Fig. 2(a) and (b) for an example of this.]

Let \( c_R \) be constructed from \( c_1 \) as follows. For any \( (i, j) \in E^2 \), let \( c_R(i, j) = c_1(i, j) \) with the following exceptions. If \( (i + 1, j), (i + 2, j), \ldots, (i + k - 1, j), k \geq 6 \), form a maximal contiguous horizontal line of cells all in \( R \) and all containing 0's in \( c_1 \), i.e., each of the cells \((i, j)\) and \((i + k, j)\) either contain a 1 or are not in \( R \), then the contents of cells \((i + 2, j), (i + 3, j), \ldots, (i + k - 2, j)\) are altered in the following
A Completeness Problem for TA

way to obtain \( c_R \). The unique string of length \( k-3 \) from the set (specified by regular expression) \( 111 \cup (1110)^* \) \( (1 \cup 0 \cup 00 \cup 000) \) is placed in the consecutive cells \( (i + 2, j), \ldots, (i + k - 2, j) \). [See Fig. 2(c).]

Let \( c_s \in \mathcal{C} \) be defined by: \( c_s(i) = 1 \) if \( i \in R \) and \( c_s(i) = 0 \) if \( i \notin R \). Clearly there is a \( \xi_1 \in (T)^* \) such that \( c_s\xi_1 = c_s \). We shall argue now that a \( \xi_2 \in (T)^* \) exists such that \( c_s\xi_2 = c_R \).

Only cell \((m + s, n + r)\) in \( c_s \) would have a neighborhood state\(^8\) of the form

\[
\begin{align*}
0 & 0 0 0 0 0 0 0 0 0 \\
1 & 1 1 1 1 1 0 0 0 0 \\
1 & 1 1 1 1 1 0 0 0 0 .
\end{align*}
\]

We can find a \( \tau \in \overline{T} \) that will change the contents of this cell to that required in \( c_R \), say \( a_1 \), leaving all other cell contents unaltered. Call the resulting configuration \( c_s^{(1)} \).

Only one cell, namely \((m + s - 1, n + r)\) will have a neighborhood state in \( c_s^{(1)} \) of the form

\[
\begin{align*}
0 & 0 0 0 0 0 0 0 0 0 \\
1 & 1 1 1 1 1 0 0 0 0 \\
1 & 1 1 1 1 1 0 0 0 0,
\end{align*}
\]

where \( a_1 \in B \). The contents of this cell can be changed as required by \( c_R \), say to \( a_2 \in B \), by some \( \tau \in \overline{T} \) giving a configuration \( c_s^{(2)} \). Again, only one cell in \( c_s^{(2)} \) has a neighborhood state of the form

\[
\begin{align*}
0 & 0 0 0 0 0 0 0 0 0 0 \\
1 & 1 1 1 1 1 0 0 0 0 \\
1 & 1 1 1 1 1 1 1 1 1 0 .
\end{align*}
\]

The contents of this cell can be changed to that required by \( c_R \) by some \( \tau \in \overline{T} \). In the next steps (along row \( n + r \)) there will in each case be a unique neighborhood state of the form

\[
\begin{align*}
0 & 0 0 0 0 0 0 0 0 0 0 \\
1 & 1 1 1 1 1 1 1 1 1 1 \\
1 & 1 1 1 1 1 1 1 1 1 1 .
\end{align*}
\]

This follows since the definition of \( c_R \) guarantees that not all of \( a_{i+3}, a_{i+2}, a_{i+1}, a_i \) can be ones. When a neighborhood state of the form

\[
\begin{align*}
0 & 0 0 0 0 0 0 0 0 0 0 \\
0 & 1 1 1 1 1 0 0 0 0 0 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 \\
0 & 1 1 1 1 1 1 1 1 1 1 .
\end{align*}
\]

\(^8\) It will be convenient to represent neighborhood states by showing the contents of the neighborhood cells spatially arranged as the cells are visualized as being arranged in the array.
exists, this process of altering row \(n + r\) is terminated. At this point there will be a unique neighborhood state of the form:

\[
\begin{matrix}
a_5 & a_4 & a_3 & a_2 & a_1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{matrix}
\]

where not all of \(a_5, a_4, a_3, a_2, a_1\) are ones. A \(\tau \in \overline{T}\) can be found that will change the 1 below \(a_1\) as required in \(c_R\). We can now move along row \(n + r - 1\) and at each step there will be a unique neighborhood state of the form:

\[
\begin{matrix}
a_{i+8} & a_{i+7} & a_{i+6} & a_{i+5} & a_{i+4} & a_{i+3} & a_{i+2} & a_{i+1} & a_i \\
1 & 1 & 1 & 1 & b_{i+3} & b_{i+2} & b_i \\
1 & 1 & 1 & 1 & d_{i+3} & d_{i+2} & d_i \\
\end{matrix}
\]

where the \(a\)'s, \(b\)'s, and \(d\)'s are in \(B\), not all the \(b\)'s are \(1\)'s and there is a \(k, i + 8 \geq k \geq i + 4\) such that \(a_k = 0\). Continuing this process in turn on each row in \(R\) except the bottom row of \(R\), we arrive at a configuration that can be changed to \(c_R\) by changing the contents of any cell \(i\) from 0 to 1 if cell \(i\) is part of a contiguous sequence of cells \((j + 1, k), (j + 2, k), \ldots, (j + t, k)\), where \(t \geq 3\), each containing a 1. But this can be done in one step by some \(\tau' \in \overline{T}\). Hence, there exists a \(\xi_2\) such that \(c_{i_1} \xi_2 = c_R\), and \(c_{i_2} \xi_2 \tau' \xi_2 = c\).

Extending Theorem 1 for TA of higher dimension is possible by a similar process.

**XII. CONCLUDING REMARKS**

Although this report was concerned mainly with what we called one-dimensional binary scope-\(n\) TA, in Section XI we indicated how a completeness result could be obtained for higher dimensional TA, but again with the binary state alphabet. There is little known at present concerning the completeness problem for TA with nonbinary state alphabets beyond the following which was established in a slightly different framework in [13].

**Theorem 1.** Let \((A, E^1, X, \overline{T})\) be such that for some \(k \in \mathbb{Z}, k \text{ and } k + 1 \text{ are among the components of } X, \text{ and } \#(A) = 2^m \text{ for some } m \geq 2, \text{ then } \{[c] \mid c_\xi = c \text{ for some } \xi \in (\overline{T})^\ast\} = \{[c] \mid c \in C\}.

We moved past the scope-2 TA rather quickly, showing only that there exist patterns not generable in such structures. There are, however, a number of open questions one could formulate, e.g., how can one characterize pattern sets generable by scope-2 TA? Are these decidable sets? We might note that not all symmetric
patterns are in $\mathcal{A}^{(2)}(c_p)$, e.g., $011010110 \notin \mathcal{A}^{(2)}(c_p)$, and not only symmetric patterns are in $\mathcal{A}^{(2)}(c_p)$, e.g., $01001010 \in \mathcal{A}^{(2)}(c_p)$.

We concerned ourselves here exclusively with contiguous neighborhood structure, and although a good deal is known about neighborhood structure in TA [15]-[18], there are many open questions here as well, e.g., would a one-dimensional binary TA with $X = ((0), (1), (3), (4))$ be capable of generating any pattern?

The question of whether certain set inclusions are proper or not has not been studied yet, e.g., for $S \subseteq \mathcal{C}$, $\mathcal{A}^{(2)}_{\mathcal{SM}}(S) \subseteq \mathcal{A}^{(3)}_{\mathcal{SM}}(S) \subseteq \mathcal{A}^{(2)}(S)$, or $\mathcal{A}^{(4)}_{\mathcal{SM}}(c_p) \subseteq \mathcal{A}^{(4)}(c_p) \subseteq \mathcal{A}^{(4)}(c_p)$.

As we mentioned earlier, the pattern decomposition phenomenon appears to have a good deal of intrinsic interest. It might be noted here that had we defined $\pi'$ (see Appendix C) by: for any $c \in \mathcal{C}$, $c\pi' = (c_{\text{std}})_{\tau_{180}}$, the results would have been altered slightly. For example, Theorem C.13 would have appeared as follows.

**Theorem C.13'.** Let $c \in \mathcal{C}$ and let $c_1, c_2, \ldots$ be the sequence of predecessors of $c$ under $\pi'$. Then for some $n$, $c_n$ is of the form

\[
\begin{align*}
010^{n_1}10^{n_2} & \cdots 0^{n_r}10^n110^{m_1}101110^{m_2} \\
& \cdots 0^{n_s}101110^{k_1}10^{k_2} \cdots 0^{k_t}110,
\end{align*}
\]

where $r, s, t \geq 0$, $n_i \geq 2$, $1 \leq i \leq r$; $m_i \geq 1$, $1 \leq i \leq s$; $k_i \geq 1$, $1 \leq i \leq t$, and $p, q \geq 1$. Also, $c_{n+j}$ is of the same form except $p$ and $q$ are replaced by $p + j$ and $q + j$, respectively, $j = 1, 2, \ldots$.

A configuration in standard decomposition would then have defined a pattern of the form

\[
\begin{align*}
010^{n_1}10^{n_2} & \cdots 0^{n_r}10^n110^{m_1}110^{m_2} \cdots 0^{m_s}110.
\end{align*}
\]

If one considers, for any $c \in \mathcal{C}$, the sequence of successors of $c$ under $\pi$ then it can be shown that eventually one will arrive at a pattern of the form

\[
\begin{align*}
010^{n_1}10^{n_2} & \cdots 0^{n_r}10^n11010^{m_1}111010^{m_2} \cdots \\
0^{m_s}111010^n110^{k_1}110^{k_2} \cdots 0^{k_t}110,
\end{align*}
\]

where $n_i \geq 2$, $m_i \geq 1$, and $k_i \geq 1$. $r, s, t, r, s, t, or t$ may be 0.

If one uses $\pi'$ instead when the pattern eventually reached will be of the form

\[
\begin{align*}
0110^{n_1}110^{n_2} & \cdots 0^{n_r}110^{m_1}10110^{m_2}010111 \\
0^{m_s} & \cdots 0^{m_t}010110^n10^{k_1}10^{k_2} \cdots 0^{k_t}110,
\end{align*}
\]

where $m_1 \geq 1$, $m_i \geq 1$, and $k_i \geq 2$. Again, $r, s, t, or t$ may be 0.

In both of these cases an analog of Theorem C.13 may be stated.
It seems interesting to interpret in the following way the results just presented. If one places any finite pattern into a tessellation space with a constant "environment," the pattern will decompose (or dissolve) into atomic "pieces" in a specific way, and in some cases will leave a "residue." The residues are in all cases composed of identical minimal pieces that do not dissolve in the constant environment. However, momentary alterations in the environment will cause the residue to dissolve into the atomic pieces as well.

The reader is referred to [23] (which is an enlarged version of [22]) for some suggestions on certain other directions for research on both structural and behavioral aspects of tessellation structures.

APPENDIX A: The Monotonic Generation of \([c']\) for Theorem VII.3

A monotonic generation of \([c']\) from \(c_p\) is accomplished by the following sequence of transformations:

\[
\begin{align*}
\tau_{166}, & \quad \tau_{154}, \quad \tau_{186}, \quad \tau_{154}, \quad \tau_{150}, \quad \tau_{154}, \quad \tau_{180}, \quad \tau_{90}, \quad \tau_{180}, \quad \tau_{210}, \quad \tau_{90}, \\
\tau_{120}, & \quad \tau_{186}, \quad \tau_{120}, \quad \tau_{90}, \quad \tau_{154}, \quad \tau_{210}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{210}, \quad \tau_{90}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{120}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{120}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{166}, \quad \tau_{180}, \quad \tau_{166}.
\end{align*}
\]

The generation was checked on the Mobidic computer by Mr. Roy Mattson and the printout of the successive configurations appears as Fig. 3.

APPENDIX B: The Proof of Theorem VII.5

With respect to the TA \(\mathcal{A}^{(a)} = (B, E', \{0, 1, -1\}, \tau, T)\), if for any \(c, c' \in \mathcal{C}\) and \(\tau \in T\), \(ct = c'\) and \(lg(c) \leq lg(c')\), then the relative positions of \(l_l\) and \(l_r\) for \(c\) and \(c'\) must be (from Lemma VII.2) as indicated in one of the cases below.

\[
\begin{align*}
[c] &= \bar{0}1100\quad & (A) \\
[c'] &= \bar{0}1100, \\
[c] &= 001100\quad & (B) \\
[c'] &= \bar{0}1100, \\
[c] &= \bar{0}1100\quad & (C) \\
[c'] &= \bar{0}1100, \\
[c] &= \bar{0}1100\quad & (D) \\
[c'] &= \bar{0}1100, \\
[c] &= 01100\quad & (E) \\
[c'] &= \bar{0}0110, \\
[c] &= 01100\quad & (F) \\
[c'] &= \bar{0}1100. 
\end{align*}
\]
A COMPLETENESS PROBLEM FOR TA

With respect to \( c \tau = c' \), then by the substring of \( c \) associated with symbol \( c'(i) \) for \( \tau \) we will mean the triple made up of \( c(i-1), c(i), \) and \( c(i+1) \), in that order. This is of course the argument transformed by \( \sigma \) to \( c'(i) \), where \( \sigma \) defines \( \tau \).

For finite configurations \( c \) and \( c' \), if \( c \tau_{30} = c' \) and \( lg(c) \leq lg(c') \), then the substring of \( c \) associated with \( 0_L \) (i.e., the zero to the immediate left of \( 1_L \)) of \( c' \) is uniquely determined to be 000. This follows from the easily verified fact that only Case (C) could hold for \( \tau_{30} \) in the given situation.

For finite configurations \( c \) and \( c' \), if \( c \tau_{180} = c' \) and \( lg(c) \leq lg(c') \), then after
verifying that in this situation only Case (A) or Case (E) is possible, it follows that only 001 or 011 could be the substring of \( c \) associated with \( 0_L \) of \( c' \).

Similarly, only 001 or 010 could be the substring of \( c \) associated with \( 0_L \) of \( c' \) for \( \tau_{120} \), and only 000 could be the substring of \( c \) associated with \( 0_L \) of \( c' \) for \( \tau_{118} \), both subject to the requirement that \( l_g(c) \leq l_g(c') \).

In summary, for the monotonic case only the following forms are possible.

\[
[c] = 0001 \quad 100, \\
[c'] = [c_{30}] = 0001 \quad 100, \\
[c] = 0001 \quad 100, \\
[c'] = [c_{118}] = 0001 \quad 100, \\
[c] = 0001 \quad 100. \\
\]

We now construct a graph \( G \) as follows. Each node is represented by a circle divided into four quadrants, and each quadrant is associated with a transformation as indicated in Fig. 4(a).

One node of \( G \) will be called the initial node, or node 0. Each quadrant of the circle designating node 0 will contain certain pairs of binary digits determined as follows. For any finite configurations \( c \) and \( c' \) such that \( c_{\tau_i} = c' \), \( i = 30, 118, 120, \) or \( 180 \), and \( l_g(c) \leq l_g(c') \), \( \alpha_1 \alpha_2 \) is written in the quadrant associated with \( \tau_i \) of the circle representing node 0 if, and only if, \( \alpha_1 \alpha_2 \) is the substring of \( c \) associated with \( 0_L \) of \( c' \).

Node 0 will, therefore, be represented as seen in Fig. 4(b).

There will be an arc labeled 1 from node 0 to node 1. For \( i = 30, 118, 120, \) or \( 180 \), \( \alpha_1 \alpha_2 \) is written in quadrant associated with \( \tau_i \) for node 1 if, and only if, \( \sigma_i(\alpha_1 \alpha_2 \alpha_3) = 1 \), where \( \sigma_i \) is the local transformation defining \( \tau_i \), and \( \alpha_1 \alpha_2 \) is in the quadrant associated with \( \tau_i \) for node 0. There is no arc labeled 0 leaving node 0.

Now, in general, each node \( j, j > 0 \), will have two arcs leaving it, one labeled 0, the other labeled 1. If there is an arc labeled \( \beta \) leaving some node \( j \), then the node that this arc goes to is represented by a circle where, for \( i = 30, 118, 120, \) or \( 180 \), the quadrant associated with \( \tau_i \) contains \( \alpha_1 \alpha_2 \) if and only if \( \sigma_i(\alpha_1 \alpha_2 \alpha_3) = \beta \) and \( \alpha_1 \alpha_2 \) appears in the quadrant associated with \( \tau_i \) for node \( j \). If no pairs qualify for some quadrant of this circle, then \( \phi \) is written into the quadrant. If for \( \tau_{30}, \tau_{120}, \) or \( \tau_{180} \) (not \( \tau_{118} \)), four pairs can be written in the quadrant, then a \( U \) is written into the
Fig. 4. The Graph $G$. 
quadrant instead of the four pairs. We name this node with some positive integer not already used to name some other node if no \( U \) symbols appear in a quadrant, otherwise we do not label the node with an integer and we shall refer to any such node as a \( U \) node. No arcs will leave \( U \) nodes in \( G \). The entire graph \( G \) is given in Fig. 4(b).

**Lemma 1.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be a sequence of arc labels that determines a path on \( G \) from node 0 to a \( U \) node. Then for any \( m \), and any sequence \( \beta_1, \beta_2, \ldots, \beta_m, \beta_i \in B, \) \( 1 \leq i \leq m \), there exists a \( c \in C \) and a \( \tau_i \), \( i = 30, 120, \) or 180, such that

\[
[\tau_i] = \bar{0} \alpha_1 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_m \bar{0}.
\]

**Proof.** From the construction of \( G \), it is easily seen that if \( \alpha_1 \alpha_2 \cdots \alpha_n \) is a sequence of labels that specifies a path leading from node 0 to a \( U \) node with the \( U \) in the quadrant for \( \tau \), then there must exist a sequence of pairs \( \gamma_1 \gamma_2, \gamma_2 \gamma_3, \gamma_3 \gamma_4, \ldots, \gamma_{n-3} \gamma_n \), \( \gamma_n \gamma_{n+1} \) such that \( \gamma_2 \gamma_3 \) is in the quadrant for \( \tau \) of node 0. Following the arc labeled \( \alpha_1 \) from node 0 would lead to a node with \( \gamma_2 \gamma_3 \) in the quadrant for \( \tau \). Following the arc labeled \( \alpha_2 \) from this node would lead to a node with \( \gamma_3 \gamma_4 \) in the quadrant for \( \tau \), etc. Finally, the arc labeled \( \alpha_n \) would lead to the \( U \) node. It can easily be verified that 00 must always be one of the four substrings giving rise to a \( U \) node. From this fact, and since \( \sigma(000) = 0 \) where \( \sigma \) defines \( \tau \), it follows that there is some \( \gamma_0 \in B \) such that \([c] = \bar{0} \gamma_0 \gamma_1 \gamma_2 \cdots \gamma_{n+1} \bar{0} \) and \([c'] = \bar{0} \alpha_1 \alpha_2 \cdots \alpha_n \bar{0} \) and \( \bar{c} = c' \).

In constructing graph \( G \), had we not written \( U \) in the quadrants and instead continued constructing an extended graph (call it \( G' \)), then any arc leaving what would have been a \( U \) node would go to a node that again contains the four substrings in the same quadrant. This follows again from property \( R \). Hence, any node reached on \( G' \) by following any path from a node with four substrings in some quadrant would lead to another node with four substrings in the corresponding quadrant. The Lemma then follows by an argument similar to the one above now for \( G' \).

**Lemma 2.** The sequence of nodes of \( G \) determined by following any (infinite) path of the form \( \alpha_1 \alpha_2 \cdots \alpha_n \bar{0}, \alpha_i \in B, \) \( 1 \leq i \leq n \), starting with node 0, must eventually lead to a \( U \) node or else eventually become constant in node 6 or constant in node 15.

**Proof.** Since the sequence of path labels eventually becomes constant in 0, one can check on \( G \) that from any node if one follows a path of 0's, one is lead to either a \( U \)-node, to node 6, or to node 15. Since that latter two nodes loop on themselves with 0's, the Lemma is proven.

**Theorem 3 (Theorem VII.5).** For any \( c' \in C \), there exists a \( \tau_i, \) \( i = 30, 118, 120, \) or 180, and there exists a \( c \in \bar{C} \) such that \( \lg(c) \leq \lg(c') \) and \( c \tau_i = c' \).

**Proof.** Since \( c' \) is finite, \([c']\) is of the form \( \bar{0} \alpha_1 \alpha_2 \cdots \alpha_n \bar{0}, \) the \( \alpha_i \) being binary
digits and \( \alpha_1 = 1 \). From Lemma 2 there must be some \( k \) for which the path on \( G \) specified by \( \alpha_1 \alpha_2 \ldots \alpha_n 0^k \) leads from node 0 to either a \( U \) node or else to node 6 or node 15. In the former case, the theorem is proven by Lemma 1. In the latter case, for either node 6 or node 15, there is some quadrant, of the circle representing the node, that contains 00. Let \( \tau \) be the transformation representing that quadrant. From the construction of \( G \) there must be a sequence of pairs of binary digits, \( \gamma_1 \gamma_2, \gamma_2 \gamma_3, \ldots, \gamma_{n+k-1} \gamma_{n+k}, \gamma_{n+k} \gamma_{n+k+1} \) such that \( \gamma_{n+k} = \gamma_{n+k+1} = 0 \) and \( \gamma_1 \gamma_2 \) is in the quadrant of node 0 associated with \( \tau \), \( \gamma_2 \gamma_3 \) is in the quadrant associated with \( \tau \) of the node reached from node 0 by following the arc labeled \( \alpha_1, \gamma_3 \gamma_4 \) is in the quadrant associated with \( \tau \) of the node reached from the last node by following the arc labeled \( \alpha_2 \), etc. Finally, following the arc labeled by the last 0 in the sequence \( \alpha_1 \alpha_2 \ldots \alpha_n 0^k \) leads to node 6 or node 15. Then it is easily seen that there is a \( \gamma_0 \) and a \( c \in C \) such that \( [c] = \bar{0} \gamma_0 \gamma_1 \ldots \gamma_n \bar{0} \) and \( c \tau = c' \) and the theorem is proven.

**APPENDIX C: THE DETAILS OF THE DECOMPOSITION RESULTS**

Our purpose here is to establish Lemma IX.1. This is accomplished after a series of preliminary results, many suggesting some interesting interpretations.

**Proposition 1.** For any \( c \in C \) there is a unique \( c_1 \in C \) such that \( c_1 \tau_{180} = c \) and a unique \( c_2 \in C \) such that \( c_2 \tau_{180} = c \).

**Proof.** For the case of \( \tau_{180} \), it is easily verified that the orientation of \( 1_R \) for \( c_1 \) and any \( c \) such that \( c = c_1 \tau_{180} \) must be as shown below.

\[
\begin{align*}
[c_1] &= 10\bar{0} \\
[c] &= [c_1 \tau_{180}] = 1\bar{0}.
\end{align*}
\]

Since \( \tau_{180} \) has property \( R \), the symbols of \( c_1 \) to the left of \( 1_R \) of \( c_1 \) are uniquely determined by the symbols of \( c \) to the left of \( 1_R \) of \( c \). Hence, the uniqueness of \( c_1 \) is established. The three consecutive symbols of \( c_1 \) mapped by \( \sigma_{180} \) into \( 1_R \) of \( c \) must be one of 0 1 0, 1 0 0, 1 0 1, or 1 1 1. These four possibilities determine the following four forms for \( [c_1] \) and \( [c] \).

\[
\begin{align*}
[c_1] &= 0 0 1 0 \\
[c_1 \tau_{180}] &= [c] = \bar{0} 0 1 \\
[c_1] &= 0 0 1 0 0 \\
[c_1 \tau_{180}] &= [c] = 0 0 0 0 1 \\
[c_1] &= 0 0 1 0 1 \\
[c_1 \tau_{180}] &= [c] = 0 0 0 0 1 \\
[c_1] &= 0 0 1 1 1 \\
[c_1 \tau_{180}] &= [c] = \bar{0} 0 0 1 \\
[c_1] &= 0 0 1 1 0 \\
[c_1 \tau_{180}] &= [c] = 0 0 0 1 \\
[c_1] &= 0 0 1 1 1 \\
[c_1 \tau_{180}] &= [c] = 0 0 0 1 \\
[c_1] &= 0 0 1 1 0 \\
[c_1 \tau_{180}] &= [c] = 0 0 0 1 \\
[c_1] &= 0 0 1 1 1 \\
[c_1 \tau_{180}] &= [c] = 0 0 0 1 
\end{align*}
\]
Note that in each case \( c_1 \) is finite. The case for \( \tau_{166} \) is essentially the same argument above replacing \( R, I_R, \) and \( I_L \) by \( L, I_R, \) and \( I_L \).

The reader can verify that \( \tau_{180} \) and \( \tau_{166} \) are the only nontrivial scope-3 bijections on \( C \) (the transformations yielding identity mappings of \( \mathcal{P} \) being the trivial transformations).

Let \( \pi \) be the transformation on \( C \) defined by: For any \( c \in C \)

\[
\epsilon\pi = (c\tau_{180}) \tau_{166}.
\]

Although it can be shown that \( \pi \) is not a scope-3 transformation, we do not need this result here.

\( \pi \) is, of course, a scope-5 transformation. In general, for any scope-\( n \) transformations \( \rho_1 \) and \( \rho_2, \rho_1\rho_2 \) is a scope-\( (2n - 1) \) transformation.

The specification of the local transformation defining \( \pi \) is

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{array}
\]
A COMPLETENESS PROBLEM FOR TA

1 1 0 1 1 1
1 1 1 0 0 1
1 1 1 0 1 1
1 1 1 1 0 0
1 1 1 1 1 1.

As an immediate consequence of Proposition 1 we have

**Corollary 1.1.** For any \( c' \in \mathcal{C} \) there is a unique \( c \in \mathcal{C} \) such that \( \sigma^n c = c' \).

By the sequence of predecessors of a configuration \( c \) under \( \sigma \) we mean the sequence \( c_1, c_2, c_3, \ldots \) where \( c_1 = c, c_2 = c_1, c_3 = c_2, \ldots \).

When dealing with \( \sigma \) below, we shall use the scope-5 neighborhood index \( X = \{(-2), (-1), (0), (1), (2)\} \), i.e., if \( \sigma^n c = c' \), then \( c'(i) \) is determined from \( c(i-2), c(i-1), c(i), c(i+1), \) and \( c(i+2) \).

**Lemma 2.** If for some \( c \in \mathcal{C} \), \( c(i) = 1_L \), and if \( c_1(i-1) = 1_L \) where \( c_1 = \sigma^{n-1} \), then in the sequence \( c_1, c_2, \ldots \) of predecessors of \( c \) under \( \sigma \), \( c_j(i-j) = 1_L \) for all \( j \geq 1 \). Also, each pattern defined from a \( c_j, j \geq 1 \), is of the form

\[
\overline{011001}10.
\]

**Proof.** Assume \( c \) and \( \sigma^{n-1} c_1 \) have the forms

\[
[c_1] = [\sigma^{n-1}] : \overline{00001} \alpha \beta -----,
\]

\[
[c] = \overline{000001} -----,
\]

where \( \alpha, \beta \in B \). If \( \sigma \) is the local transformation defining \( \sigma \), then since \( \sigma(00010) = 1 \) and \( \sigma(00111) = 1 \), \( \alpha = 1 \) and \( \beta = 0 \). Given any \( [c_j] \) of the form \( \overline{011001} ----- \) with \( c_j(i-j) = 1_L \), assume that \( c_{j+1}(i-j-k) = 1_L \) for some \( k > 0 \) \((c_{j+1} = c_j)\).

Consider the substring 00001 of \( c_{j+1} \) containing \( 1_L \). Since \( c_j(i-j-k-2) = c_j(i-j-k-1) = c_j(i-j-k) = 0 \), and since \( \sigma(00010) = \sigma(00111) = 1 \), if follows that \([c_{j+1}]\) must be of the form \( \overline{011001} ----- \). Since \( \sigma(01101) = \sigma(01100) = 1 \), \( c_j(i-j-k+1) = 1 \), and since \( c_j(i-j) = 1_L \), we can conclude: If \( k > 0 \), then \( k = 1 \). Now assume \( k \leq 0 \). Since \( \sigma(00001) = 0, k > -2 \). There are two cases left to be considered: \( k = 0 \) and \( k = -1 \). If \( k = -1 \), then \([c_j]\) and \([c_{j+1}]\) are of the forms

\[
[c_{j+1}] = \overline{000001} \alpha -----,
\]

\[
[c_j] = \overline{000110} -----.
\]

Since \( \sigma(00011) = 0, \alpha = 0 \). Since \( \sigma(00100) = \sigma(00101) = 0, k = -1 \) is impossible. If \( k = 0 \), then \([c_j]\) and \([c_{j+1}]\) are of the forms

\[
[c_{j+1}] = \overline{000110} \alpha \beta \gamma -----,
\]

\[
[c_j] = \overline{000110} -----.
\]
Since $\sigma(00010) = 1$, $\alpha = 1$. Since $\sigma(00110) = 0$, $\beta = 1$. Since $\sigma(01111) = 0$, $\gamma = 0$. Finally, since $\sigma(11101) = \sigma(11100) = 1$, $k = 0$ is impossible.

**Corollary 2.1.** If for some $c \in \mathcal{C}$, $c(i) = 1_L$ and the pattern defined from $c$ is of the form $0110\ldots$, then in the sequence $c_1, c_2, \ldots$ of predecessors of $c$ under $\pi$, $c_j(i - j) = 1_L$ for all $j \geq 1$, and each pattern defined from a $c_j$, $j \geq 1$, is of the form $0110\ldots010$.

**Lemma 3.** If for some $c \in \mathcal{C}$, $c(i) = 1_R$, and if $q(i + 1) = 1_R$ where $q = c \pi - 1$, then in the sequence $c_1, c_2, \ldots$ of predecessors of $c$ under $\pi$, $c_j(i + j) = 1_R$ for all $j \geq 1$. Also, each pattern defined from a $c_j$, $j \geq 1$, is of the form $01\ldots0010$.

The proof of this lemma is very much like the proof of the last one and is omitted.

**Corollary 3.1.** If for some $c \in \mathcal{C}$, $c(i) = 1_R$ and the pattern defined from $c$ is of the form $0010\ldots$, then in the sequence $c_1, c_2, \ldots$ of predecessors of $c$ under $\pi$, $c_j(i + j) = 1_R$ for all $j \geq 1$, and each pattern defined from a $c_j$, $j \geq 1$, is of the form $01\ldots0010$.

**Lemma 4.** Let $c \in \mathcal{C}$ with $c(i) = 1_L$. If $c_1 = c \pi^{-1}$ and $c_1(j) = 1_L$, then $j \geq i - 1$. Also, if $c(i) = 1_R$ and $c_1(j) = 1_R$, then $j \leq i + 1$.

**Proof.** Let $c_1$ and $c$ be such that

$$[c \pi^{-1}] = [c_1] = 000001\alpha_1 \alpha_2 \alpha_3$$

$$[c] = 0000000.$$

Since $\sigma(00010) = 1$, $\alpha_1 = 1$. Since $\sigma(00111) = 1$, $\alpha_2 = 0$. Since $\sigma(01100) = \sigma(01101) = 1$, there is no $\alpha_3$ for this situation. The proof of the second part of the Lemma is similar.

**Lemma 5.** Let $c_1 = c \pi^{-1}$ and suppose $c(i) = c(i + 1) = 0$ for some $i$. If the substring of $c_1$ associated with $c(i)$ is in the set \{00110, 00011, 00001, 00000\}, then the substring of $c_1$ associated with $c(i + 1)$ is also in this set.

**Proof.** This can be easily verified from the definition of $\pi$.

**Lemma 6.** Let $c \in \mathcal{C}$ and $c_1 = c \pi^{-1}$. The substring of $c_1$ associated with $0_L$ of $c$ is in the set \{00110, 00011, 00001, 00000\}.

**Proof.** The substring of $c_1$ associated with any 0 three or more cells to the left
of $0_L$ of $c$ must be 00000. This follows easily from Lemma 4, Corollary IX.1.1, and the fact that $\sigma(00000) = 0$. Let cell $i$ contain such a 0 for $c$, then the substrings that can possibly be associated with cells $i, i + 1, i + 2, i + 3,$ and $i + 4$, no matter what string $c$ is, are shown below.

\[
\begin{array}{cccccc}
00110 & 00110 & 00110 \\
00011 & 00011 & 00011 \\
00001 & 00001 & 00001 & 00001 \\
00000 & 00000 & 00000 & 00000 \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

cell $i$  cell $i + 1$  cell $i + 2$  cell $i + 3$  cell $i + 4$.

The proof is completed by a simple induction argument using Lemma 5.

**Corollary 6.1.** Let $c \in \bar{C}$ and $c_1 = c\pi^{-1}$. The substring of $c_1$ associated with $1_L$ of $c$ is in the set \{00010, 00111, 01100, 01101\}.

**Lemma 7.** Let $[c]$ be of the form $0110\alpha_1 \alpha_2 \cdots \alpha_k \bar{0}$, $k \geq 0$, $\alpha_i \in B$, $1 \leq i \leq k$, and let $c_1 = c\pi^{-1}$. Then $[c_1] = 0110\beta_1 \beta_2 \cdots \beta_m \bar{0}$, for some $m \geq 0$, $\beta_i \in B$, $1 \leq i \leq m$, and the relative positions of $[c]$ and $[c_1]$ are

\[
[c\pi^{-1}] = [c_1] = 0110\beta_1 \beta_2 \\
[c] = \bar{0}0110\alpha_1
\]

If $[c'] = \bar{0}\alpha_1 \alpha_2 \cdots \alpha_k \bar{0}$ and $c_1' = c'\pi^{-1}$, where $[c_1'] = \bar{0}\beta_1 \beta_2 \cdots \beta_m \bar{0}$, then $m' = m$ and $\beta_i' = \beta_i$, $1 \leq i \leq m$, and the relative positions of $[c']$ and $[c_1']$ are

\[
[c_1'] = \bar{0}\beta_1 \beta_2 \\
[c'] = \bar{0}0\alpha_1
\]

**Proof.** Suppose $\alpha_1 = 0$. Then the possible substrings of $c_1$ that can be associated with the symbols $0_L$ to $\alpha_1$ of $c$ are

\[
\begin{array}{cccc}
10000 & 00000 \\
11000 & 10001 & 00001 \\
01101 & 11001 & 10011 & 00011 \\
00110 & 01100 & 11011 & 10110 & 00110 \\
0_L & 1_L & 1 & 0 & \alpha_1 = 0
\end{array}
\]

From Lemma 6 the possible substrings associated with $\alpha_1$ for $c'$ are also limited to these same substrings. The substring that actually is to be associated (from Corollary IX.1.1 it is unique) with $\alpha_1$ is determined by $\alpha_2 \cdots \alpha_k \bar{0}$, and hence must be the same for $c_1$ and $c_1'$. It follows that $m' = m$ and the $\beta_i' = \beta_i$, $1 \leq i \leq m$. 
Suppose $\alpha_1 = 1$. Then the possible substrings of $c_i$ that can possibly be associated with the symbols $0_L$ to $\alpha_4$ of $c$ are

$\begin{array}{cccc}
10000 & 00010 \\
11000 & 10001 \\
01101 & 11011 \\
00110 & 01100 \\
\hline
0_L & 1_L & 1 & 0 \alpha_1 = 1.
\end{array}$

From Corollary 6.1, the possible substrings associated with $\alpha_1$ for $c'$ are also limited to these same substrings. The rest of the proof follows in the same way as above.

**Corollary 7.1.** Let $[c]$ be of the form $0110^m 110^r 110 \alpha_1 \ldots \alpha_k \bar{0}$, $r \geq 0$, $n_i \geq 1$, $1 \leq i \leq r$, $k \geq 0$, $\alpha_i \in B$, $1 \leq i \leq k$. Let $c_i = c_{n_i}$, then $[c_i] = 0110^m 110^r \alpha_1 \ldots \beta_m \bar{0}$, for some $m \geq 0$, and some $\beta_i \in B$, $1 \leq i \leq m$, and the relative positions of $[c]$ and $[c_i]$ are

$\begin{array}{cccc}
[c_{n_i}] = [c_i] & 0110 & 0110 \beta_1 \beta_2 \\
[c] = 0110 & 0110 & \alpha_1 \bar{0}.
\end{array}$

If $[c'] = \bar{0} \alpha_1 \alpha_2 \ldots \alpha_k \bar{0}$ and $c_1' = c'_{n_i}$, where $[c_1'] = \bar{0} \beta_1' \ldots \beta_m' \bar{0}$, then $m' = m$ and $\beta_i' = \beta_i$, $1 \leq i \leq m$, and the relative positions of $[c']$ and $[c_1']$ are

$\begin{array}{c}
[c_1'] = \bar{0} \beta_1 \beta_2 \\
[c'] = \bar{0} \alpha_1 \bar{0}.
\end{array}$

**Lemma 8.** Let $c_1 = c_{n_i}$ and suppose $c(i) = c(i - 1) = 0$ for some $i$. If the substring of $c_1$ associated with $c(i)$ is in the set $\{00100, 01000, 10000, 00000\}$, then the substring of $c_1$ associated with $c(i - 1)$ is also in this set.

**Proof.** This can be easily verified from the definition of $\pi$.

**Lemma 9.** Let $c \in \mathcal{C}$ and $c_i = c_{n_i}$. The substring of $c_1$ associated with $0_R$ of $c$ is in the set $\{00100, 01000, 10000, 00000\}$.

**Proof.** The substring of $c_1$ associated with any 0 three or more cells to the right of $0_R$ must be 000000 (from Lemma 8 and Corollary IX.1.1). Let cell $i$ contain $0_R$, then the substrings associated with cells $i$, $i + 1$, $i + 2$, and $i + 3$ are

$\begin{array}{cccc}
00000 & 00000 \\
01000 & 00000 \\
00100 & 10000 \\
00000 & 00000 \\
\hline
0_L & 0 & 0 & 0 \alpha_1 = 1.
\end{array}$
Corollary 9.1. Let $c \in \mathcal{C}$ and $c_1 = c_{n^{-1}}$. The substring of $c_1$ associated with $1_R$ of $c$ is in the set \{11000, 10100, 00010, 10010\}.

Lemma 10. Let $[c]$ be of the form $\bar{0} \alpha_k \cdots \alpha_2 \alpha_1 0011 \bar{0}$, $k \geq 0$, $\alpha_1 \in B$, $1 \leq i \leq k$, and let $c_1 = c_{n^{-1}}$. Then $[c_1] = \bar{0} \beta_m \cdots \beta_2 \beta_1 0011 \bar{0}$, for some $m \geq 0$, $\beta_i \in B$, $1 \leq i \leq m$, and the relative positions of $[c]$ and $[c_1]$ are

$$[c_{n^{-1}}] = [c_1] = \beta_2 \beta_1 0011 \bar{0}$$

$$[c] = \alpha_1 0011 \bar{0}.$$ If $[c'] = \bar{0} \alpha_k \cdots \alpha_2 \alpha_1 0 \bar{0}$ and $c'_1 = c'_{n^{-1}}$, where $[c'_1] = \bar{0} \beta'_m \cdots \beta'_2 \beta'_1 \bar{0}$, then $m' = m$ and $\beta'_i = \beta_i$, $1 \leq i \leq m$, and the relative positions of $[c']$ and $[c'_1]$ are

$$[c'_1] = \beta_2 \beta_1 \bar{0}$$

$$[c'] = \alpha_1 0 \bar{0}.$$ Proof. Suppose $\alpha_1 = 0$. Then the possible substring of $c_1$ that can be associated with symbols $0_R$ to $\alpha_1$ of $c$ are

$$\begin{array}{cccccc}
0000 & 0000 & 0001 & 0010 & 0100 & 0101 \\
0100 & 0100 & 0101 & 0000 & 0010 & 1000 \\
1000 & 1000 & 1001 & 1001 & 1010 & 0010 \\
\hline
\alpha_1 = 1 & 0 & 0 & 1_R & 0_R
\end{array}$$

From Lemma 9 the possible substrings associated with $\alpha_1$ for $c'$ are also limited to these same substrings. The substring that actually is to be associated with $\alpha_1$ (from Corollary IX.1.1 it is unique) is determined by $\bar{0} \alpha_k \cdots \alpha_2 \alpha_1$, and hence must be the same for $c_1$ and $c'_1$. It follows that $m' = m$ and the $\beta_i = \beta'_i$, $1 \leq i \leq m$.

Suppose $\alpha_1 = 1$. Then the possible substrings of $c_1$ that can possibly be associated with the symbols $0_R$ to $\alpha_1$ of $c$ are

$$\begin{array}{cccccc}
0001 & 0000 & 0010 & 1000 & 0100 & 0101 \\
1000 & 0100 & 0101 & 0001 & 0010 & 0010 \\
\hline
\alpha_1 = 1 & 0 & 0 & 1_R & 0_R
\end{array}$$

From Corollary 9.1, the possible substrings associated with $\alpha_1$ for $c'$ are also limited to these same substrings. The rest of the proof follows in the same way as above.

Corollary 10.1. Let $[c]$ be of the form $\bar{0} \alpha_k \cdots \alpha_2 \alpha_1 0^n 1 0^{n-1} 1 \cdots 0^1 1 \bar{0}$, $r \leq 0$, $n_1 \leq 2$, $1 \leq i \leq r$, $k \leq 0$, $\alpha_i \in B$, $1 \leq i \leq k$. Let $c_1 = c_{n^{-1}}$. Then $[c_1] = \ldots$
\[ 0 \beta_n \cdots \beta_2 \beta_1 0^{n_r} 1 0^{m_s-1} 1 \cdots 0^{m_s} 1 0^{m_s+1} 0, \] for some \( m \leq 0 \), and some \( \beta_i \in B, 1 \leq i \leq m \), and the relative positions of \([c]\) and \([c_1]\) are

\[
[c_1] = \beta_1 \beta_2 0 1 0 0 0 1 0
\]

\[
[c] = \alpha_1 0 1 0 0 0 1 0 0.
\]

If \([c'] = 0 \alpha_k \cdots \alpha_2 \alpha_1 0 \) and \( c_1' = c' \pi^{-1} \), where \([c'] = 0 \beta_{m'} \cdots \beta_{z} \beta_{1}' 0 \), then \( m' = m \) and \( \beta_i' = \beta_i, 1 \leq i \leq m \).

We can summarize what has been proven up to this point as follows. Let \([c] = 0 1 1 0^r 1 0 1^s \cdots 1 1 0^r \delta_1 \cdots \delta_k 0^{m_1} 1 0^{m_2} \cdots 0^{m_s} 1 0\), where each \( n_i \) is greater than zero, each \( m_i \) is greater than one, and the \( \delta \)'s are arbitrary elements of \( \{0, 1\} \).

If \( c_1 = c \pi^{-1} \), then \([c_1]\) and its relative position to \([c]\) is

\[
[c_1] = 0 1 1 0^n 0 1 1 0^{n_s-1} \gamma_1 \cdots \gamma_{k+2} 0^{m_s-1} 0 1 0^{m_s} 0 1 0
\]

\[
[c] = 0 0 1 1 0^n 0 1 1 0^{n_s} 0 1 1 0^s 1 0^{m_s-1} 1 0 0^{m_s-1} 1 0 0
\]

and the substring \( \gamma_1 \cdots \gamma_{k+2} \) of \( c_1 \) is determined solely from the substring \( \delta_1 \cdots \delta_k \) of \( c \) as if \([c]\) were \( 0 \delta_k \cdots \delta_1 0 \).

In Lemma 3 we showed that when \( 1_R \) for any \( c \in C \) is in a cell \( i \) and when \( 1_L \) for \( c_1 = c \pi^{-1} \) is in cell \( i + 1 \), then \([c_1]\) is of the form \( \alpha 0^n 0 1 0 \), \( n \geq 1 \), where \( \alpha \) is in a cell \( j, j < i \). We have seen that the substring \( 0 0 1 \) containing \( 1_R \) will continue to move right in the sequence of predecessors for \( \pi \) and will not influence the symbols to its left. Also, from Lemma 2, when \( 1_L \) for any \( c \in C \) is in a cell \( i \) and when \( 1_L \) for \( c_1 = c \pi^{-1} \) is in a cell \( i - 1 \), then \([c_1]\) is of the form \( \alpha 1 1 0^n \alpha \cdots \alpha 1 0 \), \( n \geq 1 \), where \( \alpha \) is in a cell \( j, j > i \). We have seen that the substring \( 1 1 0 \) containing \( 1_L \) will continue to move left in the sequence of predecessors for \( \pi \) and will not influence the symbols to its right.

By the \( \pi \)-decomposition remainder of \( c, c \in \bar{C} \), we mean the substring \( \alpha_1 \cdots \alpha_k \), \( k \geq 0 \), of \( c \) of shortest length where \( \alpha_1 = \alpha_k = 1 \) and if \( k \geq 3 \), then \( \alpha_1 \alpha_2 \alpha_3 \neq 1 1 0 \) and \( \alpha_{k-2} \alpha_{k-1} \alpha_k \neq 0 0 1 \), when \([c]\) is considered to be of the form

\[
\bar{0} 1 1 0^n 0 \cdots 1 1 0^r \alpha_1 \cdots \alpha_k 0^{m_1} 1 0^{m_2} 1 0^{m_s} 1 0\]

\( r \geq 0, n_i \geq 1, 1 \leq i \leq r, s \geq 0, m_j \geq 2, 1 \leq j \leq s \). We have proven

**Proposition 11.** Let \( c \in \bar{C} \), and let \( c_1, c_2, \ldots \) be the sequence of predecessors of \( c \) under \( \pi \). Then if \( \alpha_1 \alpha_2 \cdots \alpha_k \) is the \( \pi \)-decomposition remainder for \( c_i, i = 1, 2, \ldots \), then if for \( c_i \) we have that \( \alpha_i \) is not in a cell to the left of that which will contain \( \alpha_i \), and \( \alpha_k \) is not in a cell to the right of that which contains \( \alpha_k \).

This proposition tells us that in the sequence of predecessors of \( c \) for \( \pi \) the length of the \( \pi \)-decomposition remainder will never increase from one step to the next,
hence is monotone decreasing. Therefore, the sequence of remainder lengths will eventually either go to zero, or will become constant.

**Theorem 12.** For any $c \in C$, if in the sequence of predecessors under $\pi$ the sequence of $\pi$-decomposition remainder lengths becomes constant, then these remainders will become constant in the form $111010^n111010^n_t \ldots 0^n_t11101$, $n_i \geq 1$, $1 \leq i \leq t$, $t \geq 0$. Also, if the remainder becomes constant and if cell $i$ contains the leftmost symbol of the remainder, then cell $i$ will continue to hold this leftmost symbol in the sequence of predecessors from then on (in other words, the remainder is then held fixed in the same cells).

**Proof.** From Lemma 2, Corollary 6.1, and the definition of $\pi$, if the $\pi$-decomposition length is constant going from some $c \in C$ to $c_1 = c_1^{-1}$, then $[c_1]$ must be of the form

$[c_1] = \text{001110}$

If $c_2 = c_1^{-1}$, then $[c_2]$ is of the form

$[c_2] = \text{001110}$

If $c_3 = c_2^{-1}$, then $[c_3]$ is of the form

$[c_3] = \text{00111010}$

If $c_4 = c_3^{-1}$, then $[c_4]$ is of one of the following forms:

$[c_4'] = \text{0011101000}$

$[c_4'] = \text{0011101001}$

$[c_4] = \text{0011101011}$

$[c_3] = \text{00111010}$

It is easy to show that $[c_4]$ would lead to a $[c_4^{-1}(\pi^{-1})^g]$ of the form

$[c_4'] = \text{0011101011}$

$[c_4]$ would lead to a $[c_4'(\pi^{-1})^g]$ of the form

$[c_4'] = \text{00111010011101}$

and $[c_4']$ would lead to a $[c_4'\pi^{-1}]$ of the same form as $[c_4']$. By an induction argument we can conclude that if in the sequence of predecessors, the remainder is to be of constant length, then it must be of the form

$111010^n111010^n_t \ldots 0^n_t11101$, $n_i \geq 1$, $1 \leq i \leq r$. 

571/4/2-6
We shall refer to remainders of this form as reduced remainders, and to \( \pi \)-decompositions with reduced remainders as reduced \( \pi \)-decompositions. A \( \pi \)-decomposition with a null remainder will be referred to as a standard decomposition.

The following theorem summarizes what has been proven so far.

**Theorem 13.** Let \( c \in C \), and let \( c_1, c_2, \ldots \) be the sequence of predecessors of \( c \) under \( \pi \). Then there exists a \( k, k \geq 1 \) such that for any \( n, n \geq k \), \( c_n \) is a standard decomposition, i.e., \( [c_n] \) is of the form

\[
\begin{align*}
0110^n110^n & \cdots 0^{n_{\pi}} 110^p 111010^m 111010^n \\
& \cdots 0^{n_{\pi}} 111010^p 10^{k_1} 10^{k_2} 1 \cdots 0^{k_1} 1 \overline{0},
\end{align*}
\]

where \( r, s, t \geq 0; n_i \geq 1, 1 \leq i \leq r; m_i \geq 1, 1 \leq i \leq s; k_i \geq 2, 1 \leq i \leq t; \) and \( p, q > 1 \). Also \( [c_{n+1}] \) is of the same form as \([c_n]\) except \( p \) and \( q \) are each replaced by \( p + j \) and \( q + j \), respectively, \( j = 1, 2, \ldots \).

We now proceed to show that for any \( c \in C \) there exists a standard decomposition \( c' \) and a finite sequence \( \xi \) of scope-3 transformations such that \( c' \xi = c \).

**Lemma 14.** Let \( c \) be a reduced \( \pi \) decomposition with a nonnull remainder and let \( c' = c(\tau_{\theta})^2 \). Then \( c' \) is a \( \pi \) decomposition with a nonreduced remainder of length equal to that of the remainder of \( c \).

**Proof.** We shall show now that for any reduced \( \pi \) decomposition

\[
[\begin{array}{c}
\begin{array}{c}
0110^n110^n \\
\vdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0^{n_{\pi}} 110^p 111010^m 111010^n \\
\vdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0^{n_{\pi}} 111010^p 10^{k_1} 10^{k_2} 1 \cdots 0^{k_1} 1 \overline{0}
\end{array}
\end{array}
\end{array}
\]

\((r, s, t \geq 0; n_i \geq 1, 1 \leq i \leq r; m_i \geq 1, 1 \leq i \leq s; k_i \geq 2, 1 \leq i \leq t; p, q > 1)\)

when \( c \) is transformed by \( (\tau_{\theta})^2 \) the resulting pattern will have a form,

\[
\begin{align*}
0110^n 110^n & \cdots 0^{n_{\pi}} 110^p \alpha_1 \alpha_2 \cdots \alpha_u 0^a 10^{k_1} 10^{k_2} 1 \cdots 0^{k_1} 1 \overline{0},
\end{align*}
\]

where \( \alpha_i \in B, 1 \leq i \leq u \), and where \( u \) is equal to the length of the remainder of \( c \). Also \( \alpha_1 \alpha_2 \cdots \alpha_u \) is determined solely by the remainder of \( c \). We now argue that \( \alpha_1 = \alpha_u = 1 \) and that \( u \) is the length of the remainder of \( c \).

It can be verified that the forms and relative positions of \([c]\) and \([c(\tau_{\theta})^2]\) are

\[
\begin{align*}
[c(\tau_{\theta})^2] &= 0010^n 10^{n_{\pi}} \quad 0^{n_{\pi}} 010^p 010^n 010^p 010^n 010^p 010^n 010^n \\
[c] &= 0110^n 110^n \quad 0^{n_{\pi}} 110^p 111010^n.
\end{align*}
\]
Since the substring of $c_{166}^{-1}$ associated with the leftmost 1 of the remainder of $c$ is 001, we can proceed to verify that $[c]$ and $[c_{166}^{-1}]$ are related as follows:

$$[c_{166}^{-1}] = 0010^n10^n2 \quad 0^n10^p0101110^m1^{-1},$$
$$[c] = 0110^n110^n2 \quad 0^n110^p1110100^m1^{-1},$$
$$- 0101110^v10110^v_{k_1-2}0110^v_{k_2-2} \quad 0110,$$
$$- 1110100^v100^v_{k_1-2}1000^v_{k_2-2} \quad 1000.$$

In a similar way we can verify that $[c_{166}^{-1}]$ and $[c_{(166)}^{-2}]$ are related as follows:

$$[c_{(166)}^{-2}] = 000110^n10^n10^n21011 \quad 0110^p1,$$
$$[c_{166}] = 001000^n1100^n100 \quad 1000^p1,$$
$$0111110^m1-10111110^m1 \quad 0111110^v1,$$
$$1011100^m1-1101100^m1 \quad 1011100^v1,$$
$$010^v_{k_1}010^v_{k_2} \quad 0^v_{k_1}010,$$
$$110^v_{k_1}110^v_{k_2} \quad 0^v_{k_1}110.$$

**Theorem 15 (Lemma IX.1).** For any $c \in C$, there exists a $c' \in C$ defining a pattern $c'$ in standard decomposition, $0110^n110^n2 \cdots 0^n110^n10^m10^n \cdots 0^n110$, such that $c' \xi = c$ for some finite sequence $\xi$ of scope-3 transformations.

*Proof.* If $c_1 = c(\pi^{-1})^n$ is a $\pi$ decomposition of $c$ with a reduced remainder, then obtain $c_2 = c_1(\tau_{168})^2$. If $c_3 = c_2(\tau_{168})^n$ is a $\pi$ decomposition with a reduced remainder, the length of its remainder must be less than that of $c_1$. By continuing this process a finite number of times we must arrive at a pattern $[c']$ that can be transformed to $c$ by a sequence of $\tau_{180}$ and $\tau_{166}$ transformations. ::

**Acknowledgments**

We wish to acknowledge the contributions made by Giorgio Ingargiola who, during an early stage of this work, suggested the approach that led to Theorems V.4 and VII.3; by Jerry Cooper who helped in finding state graph $G$ of Appendix B; and by Jim Thatcher who pointed out a number of errors and a number of ways of improving our presentation.

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