Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field

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A R T I C L E  I N F O
Article history:
Received 12 July 2010
Received in revised form 17 January 2011
Accepted 17 January 2011

Keywords:
Quaternion matrix
Cramer rule
Generalized inverse $A_{T,S}^{(2)}$

A B S T R A C T
In this paper, we establish the determinantal representations of the generalized inverses $A_{T,S}^{(2)}$, $A_{0,2}^{(2)}$, $A_{(T,S),0,1}^{(2)}$ and $A_{(T,S),0,2}^{(2)}$ over the quaternion skew field by the theory of the column and row determinants. In addition, we derive some generalized Cramer rules for the unique solution of some restricted quaternion matrix equations. The findings of this paper extend some known results in the literature.

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1. Introduction

It is well known that after discovering the algebra of quaternions $H = R_1 \oplus R_1 \oplus R_1 \oplus R_1$ on the 16 October 1843, the Irish physicist and mathematician William Rowan Hamilton devoted the remaining years of his life developing the new theory which he believed would have profound applications in physics. As he expected, nowadays quaternions are not only part of contemporary mathematics (algebra and analysis see, e.g. [1–4]), but also widely and heavily used in computer graphics, control theory, quantum physics, signal and color image processing, and so on (see, e.g. [5–7]).

Throughout, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$, the identity matrix with the appropriate size by $I$. For $A \in \mathbb{H}^{m \times n}$, the symbols $A^*$ stands for the conjugate transpose of $A$. The Moore–Penrose inverse of $A$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{H}^{n \times m}$ satisfying

\begin{align*}
& (1) \ AXA = A, \\
& (2) \ XAX = X, \\
& (3) \ (AX)^* = AX, \\
& (4) \ (XA)^* = XA.
\end{align*}

For positive definite matrices $M$ and $N$ of orders $m$ and $n$, respectively, the weighted Moore–Penrose inverse of $A$, which is often denoted by $A_{M,N}^\dagger$, is the unique solution to the equations (1), (2), and the following two equations:

\begin{align*}
& (MAX)^* = MAX, \\
& (NXA)^* = NXA.
\end{align*}
For $A \in \mathbb{R}^{n \times n}$ with $k = \text{Ind}(A)$, the smallest positive number such that $r(A^{k+1}) = r(A^k)$, the Drazin inverse of $A$, denoted by $A^D$, is defined to be the unique matrix $X$ that satisfies (2) and the following equations:

$$A^T X A = A^k, \quad AX = XA.$$  

If $k = 1$, then $X$ is called the group inverse of $A$, and is denoted by $X = A_g$.

In 1970, Steve Robinson [8] gave an elegant proof of Cramer’s rule over the complex number field: rewriting $Ax = b$ as $A \cdot I(i \rightarrow x) = A(i \rightarrow b)$, where $I$ is an identity matrix of order $n$, and taking determinants

$$\det(A) \cdot \det I(i \rightarrow x) = \det(A(i \rightarrow b)).$$

Since $\det I(i \rightarrow x) = x_i, i = 1, \ldots, n$, it follows that

$$x_i = \frac{\det(A(i \rightarrow b))}{\det(A)}, \quad i = 1, \ldots, n,$$

which is called Cramer’s rule. This trick of Robinson has been used to derive a series of Cramer’s rules for the matrix equations (see, e.g. [9–12]).

Note that the definition of the determinant of a square matrix plays a key role in representing the solution of a system of linear equations. Unlike multiplication of real or complex numbers, multiplication of quaternions is not commutative. Many authors had tried to give the definitions of the determinants of a quaternion matrix, (see, e.g. [13–17]). Unfortunately, by their definitions it is impossible for us to give an determinant representation of an inverse matrix. In 1991, Chen [18, 19] offered a new definition and obtained a determinant representation of an inverse matrix over the quaternion skew field. However this determinant also cannot be expanded by cofactors along an arbitrary row or column with the exception of the $n$th row. Therefore, he has not obtained the classical adjoint matrix or its analogue either. Recently, Kyrchei [20] defined the row and column determinants of a square matrix over the quaternion skew field. As applications he not only gave the determinant representation of an inverse matrix, but also derived the generalizations of Cramer’s rule for the left and right systems of linear equations over the quaternion skew field.

Recall that an out inverse of a matrix $A$ over complex field with prescribed range space $T$ and null space $S$ is a solution of the restricted matrix equation

$$XAX = X, \quad R(X) = T, \quad N(X) = S,$$

and is often denoted by $X = A^{(2)}_{T,S}$. Researches on the generalized inverse $A^{(2)}_{T,S}$ or an operator have been actively ongoing for more than 30 years (see, e.g. [21–28]). In 1998, based on the group inverse, Wei [21] gave a representation of the generalized inverse $A^{(2)}_{T,S}$; this was later followed by papers of Djordjević and Wei [22] in 2005, which extended it into Banach space. In 2009, Yu and Wei [25] established a new representation of the generalized inverse $A^{(2)}_{T,S}$ by $(1, S)$-inverse over rings. Clearly, the Moore–Penrose inverse $A^\dagger$, the Weighted Moore–Penrose inverse $A^\dagger_{M,N}$ and the Drazin inverse $A^D$ of $A$ are all generalized inverse $A^{(2)}_{T,S}$ which have some special range and null spaces, respectively. Are there any relationships between them over the quaternion skew field? So it is meaningful to extend the generalized inverse $A^{(2)}_{T,S}$ to the quaternion skew field. For an arbitrary matrix $A \in \mathbb{H}^{m \times n}$, we have the following definitions:

**Definition 1.1.** (1) An out inverse of a matrix $A$ with prescribed right range space $T_1$ and right null space $S_1$ is a solution of the restricted matrix equation

$$XAX = X, \quad R_r(X) = T_1, \quad N_r(X) = S_1,$$

and is denoted by $X = A^{(2)}_{T_1,S_1}$.

(2) An out inverse of a matrix $A$ with prescribed left range space $T_2$ and left null space $S_2$ is a solution of the restricted matrix equation

$$XAX = X, \quad R_l(X) = T_2, \quad N_l(X) = S_2,$$

and is denoted by $X = A^{(2)}_{T_2,S_2}$.

(3) An out inverse of a matrix $A$ with prescribed right range space $T_1$, right null space $S_1$, left range space $T_2$ and left null space $S_2$ is a solution of the restricted matrix equation

$$XAX = X, \quad R_r(X) = T_1, \quad N_r(X) = S_1, \quad R_l(X) = T_2, \quad N_l(X) = S_2,$$

and is denoted by $X = A^{(2)}_{(T_1,T_2),(S_1,S_2)}$.
Motivated by the work mentioned above, we in this paper aim to consider the generalized Cramer rules for the unique solution of the following restricted matrix equations
\[ AXB = D, \quad R_i(X) \subset T_1, \quad N_i(X) \supset S_1, \]  \tag{1.1}
\[ AXB = D, \quad R_i(X) \subset T_2, \quad N_i(X) \supset S_2, \]  \tag{1.2}
\[ AXB = D, \quad R_i(X) \subset T_1, \quad N_i(X) \supset S_2, \quad R_i(X) \subset T_2, \quad N_i(X) \supset S_1, \]  \tag{1.3}
where \( T_i \) and \( S_i \), \( i = 1, 2 \), are some special spaces over \( \mathbb{H} \). The paper is organized as follows. We start with some basic concepts and results about the row and column determinants of a square matrix over the quaternion skew field in Section 2. In Section 3 we give the determinantal representations of the generalized inverse \( A_{(T_1,S_1)}, A_{(T_2,S_2)} \) and \( A_{(T_1,T_2),(S_1,S_2)} \) over the quaternion skew field, respectively. In Section 4, we derive some generalized Cramer rules for the restricted matrix Eqs. (1.1)–(1.3). In Section 5, we show a numerical example to illustrate the main result. To conclude this paper, in Section 6 we propose some further research topics.

2. Preliminaries

Let \( \mathbb{H}^{m \times n} \) be the set of all \( m \times n \) matrices over the quaternion algebra. \( \mathbb{H}^r_{m \times n} \) denotes its subset of matrices of rank \( r \). Suppose \( S_n \) is the symmetric group on the set \( \{1, \ldots, n\} \).

**Definition 2.1** ([20]). The \( i \)th row determinant of \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is defined by

\[
\det_A^r = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{\sigma(1)k_1} a_{\sigma(2)k_2} a_{\sigma(3)k_3} \cdots a_{\sigma(n)k_n}
\]

for all \( i = 1, \ldots, n \). The elements of the permutation \( \sigma \) are indices of each monomial. The left-ordered cycle notation of the permutation is written as follows:

\[
\sigma = (i_{k_1}1, i_{k_2}2, i_{k_3}3, \ldots, i_{k_n}n) = (i_{k_1}, i_{k_2}, i_{k_3}, \ldots, i_{k_n}).
\]

The index \( i \) opens the first cycle from the left and other cycles satisfy the following conditions, \( i_{k_2} < i_{k_3} < \cdots \). and \( i_{kr} < i_{kr+1} \) for all \( t = 2, \ldots, r \) and \( s = 1, \ldots, t \).

**Definition 2.2** ([20]). The \( j \)th column determinant of \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is defined by

\[
\det_A^c = \sum_{\tau \in S_n} (-1)^{\tau} a_{j_{k_1}k_1} a_{j_{k_2}k_2} a_{j_{k_3}k_3} \cdots a_{j_{k_n}k_n}
\]

for all \( j = 1, \ldots, n \). The elements of the permutation \( \tau \) are indices of each monomial. The right-ordered cycle notation of the permutation \( \tau \) is written as follows:

\[
\tau = (j_{k_1}1, j_{k_2}2, j_{k_3}3, \ldots, j_{k_n}n) = (j_{k_1}, j_{k_2}, j_{k_3}, \ldots, j_{k_n}).
\]

The index \( j \) opens the first cycle from the right and other cycles satisfy the following conditions, \( j_{k_2} < j_{k_3} < \cdots \). and \( j_{kr} < j_{kr+1} \) for all \( t = 2, \ldots, r \) and \( s = 1, \ldots, t \).

Suppose \( A_j(b) \) denotes the matrix obtained from \( A \) by replacing its \( j \)th column with the column \( b \), and \( A_i(b) \) denotes the matrix obtained form \( A \) by replacing its \( i \)th row with the row \( b \). The following theorem plays a key role in the theory of the column and row determinants.

**Theorem 2.1** ([20]). If a matrix \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is Hermitian, then \( \det_1 A = \cdots = \det_n A = c \det A = c \det A \) for all \( i = 1, \ldots, n \). The following theorem about determinantal representation of an inverse matrix of Hermitian follows immediately:

**Theorem 2.2** ([20]). If a Hermitian matrix \( A = (a_{ij}) \in \mathbb{H}^{m \times n} \) is such that \( \det A \neq 0 \), then there exist a unique right inverse matrix \( (RA)^{-1} \) and a unique left inverse matrix \( (LA)^{-1} \), and \( (RA)^{-1} = (LA)^{-1} = A^{-1} \). They possess the following determinantal representations:

\[
(RA)^{-1} = \frac{1}{\det A} \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{in} & R_{in} & \cdots & R_{nn} \end{bmatrix}, \quad (LA)^{-1} = \frac{1}{\det A} \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{bmatrix}.
\]

Here \( R_{ij}, L_{ij} \) are right and left \( ij \)-th cofactors of \( A \) respectively for all \( i, j = 1, \ldots, n \).
3. Determinantal representations of some generalized inverses

**Definition 3.1.** For an arbitrary matrix \( A \in \mathbb{H}^{m \times n} \), we denote by

- \( \mathcal{R}_r(A) = \{ y \in H^m : y = Ax, x \in H^n \} \), the column right space of \( A \),
- \( \mathcal{N}_r(A) = \{ x \in H^n : Ax = 0 \} \), the right null space of \( A \),
- \( \mathcal{R}_l(A) = \{ y \in H^m : y = xA, x \in H^m \} \), the column left space of \( A \),
- \( \mathcal{N}_l(A) = \{ x \in H^m : xA = 0 \} \), the left null space of \( A \).

Recall that in the right quaternion vector space \( \mathbb{H}^n \), the inner product of vectors was defined as

\[
(x, y)_r = y^*x = \overline{y_1}x_1 + \cdots + \overline{y}_nx_n
\]

and \( \| x \| = \sqrt{(x, x)_r} \) is the norm of a vector \( x \in \mathbb{H}^n \). In the left quaternion vector space the inner product of vectors was defined as

\[
(x, y)_l = xy^* = x_1\overline{y_1} + \cdots + x_n\overline{y}_n
\]

and \( \| x \| = \sqrt{(x, x)_l} \) is the norm of a vector \( x \in \mathbb{H}^{1 \times n} \). Obviously, \( (x, y)_r = ((x^*, y^*)_l)^* \).

**Definition 3.2.** (1) Let \( \delta \) be a subset of the right quaternion vector space \( \mathbb{H}^n \) and the inner product of vectors as defined in (3.1). Denote by \( \delta^\perp \) the collection of the vectors in \( \mathbb{H}^n \) that are right orthogonal to all vectors in \( \delta \), that is

\[
\delta^\perp = \{ x \in \mathbb{H}^n : (x, y)_r = 0 \text{ for all } y \in \delta \}.
\]

(2) Let \( \mathcal{T} \) be a subset of the left quaternion vector space \( \mathbb{H}^{1 \times n} \) and the inner product of vectors as defined in (3.2). Denote by \( \mathcal{T}^\perp \) the collection of the vectors in \( \mathbb{H}^{1 \times n} \) that are left orthogonal to all vectors in \( \mathcal{T} \), that is

\[
\mathcal{T}^\perp = \{ x \in \mathbb{H}^{1 \times n} : (x, y)_l = 0 \text{ for all } y \in \mathcal{T} \}.
\]

**Theorem 3.1.** (1) Suppose \( A \in \mathbb{H}^{m \times n} \), \( T_1 \) is a subspace of \( \mathbb{H}^{r \times 1} \) of dimension \( s \leq r \) and \( S_1 \) is a subspace of \( \mathbb{H}^{m \times 1} \) of dimension \( m - s \). Then \( A \) has a \([2]\)-inverse \( X \) such that \( \mathcal{R}_r(X) = T_1 \), \( \mathcal{N}_r(X) = S_1 \) if and only if

\[
AT_1 \oplus S_1 = \mathbb{H}^{m \times 1}, \quad AT_1 \text{ is denoted as } AT_1 = \{ Ax, x \in T_1 \}.
\]

in which case \( X \) is unique and denoted by \( A^{(2)}_{T_1, S_1} \).

(2) Suppose \( A \in \mathbb{H}^{m \times n} \), \( T_2 \) is a subspace of \( \mathbb{H}^{1 \times m} \) of dimension \( t \leq r \) and \( S_2 \) is a subspace of \( \mathbb{H}^{1 \times n} \) of dimension \( n - t \). Then \( A \) has a \([2]\)-inverse \( X \) such that \( \mathcal{R}_l(X) = T_2 \), \( \mathcal{N}_l(X) = S_2 \) if and only if

\[
T_2A \oplus S_2 = \mathbb{H}^{1 \times n}, \quad T_2A \text{ is denoted as } T_2A = \{ xA, x \in T_2 \},
\]

in which case \( X \) is unique and denoted by \( A^{(2)}_{T_2, S_2} \).

(3) Let \( A, T_1, T_2, S_1 \) and \( S_2 \) be defined as (1) and (2). If \( s = t \), then \( A \) has a \([2]\)-inverse \( X \) such that

\[
\mathcal{R}_r(X) = T_1, \quad \mathcal{N}_r(X) = S_1, \quad \mathcal{R}_l(X) = T_2, \quad \mathcal{N}_l(X) = S_2
\]

if and only if (3.3) and (3.4) are satisfied, in which case \( X \) is unique and denoted by \( A^{(2)}_{T_1, T_2, S_1, S_2} \).

**Proof.** Similar to the proof of Theorem 1.3.8 in [29], we can get (1).

(2) \( \Rightarrow \) Let the row of \( U \in \mathbb{H}^{r \times m} \) be a base for \( T_2 \), the row of \( V^* \in \mathbb{H}^{r \times n} \) be a base for \( S_2^\perp \), that is \( \mathcal{R}_r(U) = T_2 \) and \( \mathcal{N}_r(V) = S_2 \). Then the row of \( UA \) span \( T_2A \). It follows from \( T_2A \oplus S_2 = \mathbb{H}^{1 \times n} \) that \( r(UA) = t \). Suppose \( xUA = 0 \), then \( xUA \in \mathcal{N}_r(V) = S_2 \) and \( xUA \in \mathcal{R}_r(UA) = T_2A \). Recall that \( T_2A \cap S_2 = 0 \), thus \( xUA = 0 \). And note that \( UA \) is of full row rank, then \( x = 0 \). It follows that \( UAV \) is nonsingular. Then we can get

\[
r(V(UAV)^{-1}U) = r(UA) = r(V) = t.
\]

Setting \( X = V(UAV)^{-1}U \), then we have

\[
V(UAV)^{-1}UAV(UAV)^{-1}U = V(UAV)^{-1}U,
\]

saying that \( X \) is an out inverse of \( A \) with

\[
\mathcal{R}_r(V(UAV)^{-1}U) = \mathcal{R}_r(U) = T_2, \quad \mathcal{N}_r(V(UAV)^{-1}U) = \mathcal{N}_r(V) = S_2.
\]

\( \Leftarrow \) Since \( X \in A^{[2]} \), i.e., \( A \in X^{[1]} \), \(XA \) is idempotent. It is easy to verify \( \mathcal{R}_l(XA) \oplus \mathcal{N}_l(XA) = \mathbb{H}^{1 \times n} \). Moreover,

\[
\mathcal{R}_r(XA) = \mathcal{R}_r(X)A = T_2A, \quad \mathcal{N}_r(XA) = \mathcal{N}_r(X) = S_2.
\]

Thus \( T_2A \oplus S_2 = \mathbb{H}^{1 \times n} \).
Uniqueness. Let $X_1$ and $X_2$ be [2] inverse of $A$ having column right space $T_2$ and right null space $S_2$, then $A \in X_1\{1\}, A \in X_2\{1\}$. Note that

$$AX_1 = P_{R}(AX_1), N_i(AX_1) = P_{R}(X_1) = P_{T_2} = N_i(AX_1),$$
$$X_2A = P_{R}(X_2A), N_i(X_2A) = P_{R}(X_2A) = P_{R}(X_2A).$$

and

$$\mathcal{R}(X_2) = T_2, \quad N_i(X_1) = S_2$$

then we can get

$$X_2P_{T_2} = X_2, \quad P_{R}(X_2A), \quad s_2X_1 = X_1.$$ 

It follows that

$$X_2 = X_2P_{T_2}, \quad N_i(X_1) = X_2AX_1 = P_{R}(X_2A), \quad s_2X_1 = X_1.$$ 

(3) In the proof of (2), setting $s = t$ and choosing $U \in \mathbb{H}^{m \times s}$, $V^* \in \mathbb{H}^{n \times s}$ such that the column $U$ is a base of $T_1$, the row of $U^*$ is a base for $S_1^\perp$, the column of $V^*$ is a base for $S_1^\perp$ and the row of $V$ is a base for $T_2$. Recall Definition 3.2, for an arbitrary matrix $A \in \mathbb{H}^{m \times n}$, it is easy for us to verify

$$\mathcal{R}_i(A) = \mathcal{R}(A^*) \quad \text{and} \quad \mathcal{N}_i(A) = \mathcal{N}(A^*).$$

It follows that

$$\mathcal{N}_i(V) = \mathcal{R}_i(V^*)^\perp = S_1,$$
$$\mathcal{N}_i(U) = \mathcal{R}_i(U^*)^\perp = S_2.$$ 

Then we can get

$$X = U(VAU)^{-1}V$$

is an out inverse of $A$ with

$$\mathcal{R}_i(X) = \mathcal{R}_i(U) = T_1, \quad \mathcal{N}_i(X) = \mathcal{N}_i(V) = S_1,$$
$$\mathcal{R}_i(X) = \mathcal{R}_i(V) = T_2, \quad \mathcal{N}_i(X) = \mathcal{N}_i(U) = S_2.$$ 

$\Leftarrow$ Since $X \in A(2), i.e., A \in X(1), AX$ and $XA$ are idempotent. It is easy to verify

$$\mathcal{R}_i(AX) \oplus \mathcal{N}_i(AX) = \mathbb{H}^m, \quad \mathcal{R}_i(XA) \oplus \mathcal{N}_i(XA) = \mathbb{H}^{1 \times n}.$$ 

Moreover,

$$\mathcal{R}_i(AX) = A\mathcal{R}_i(X) = AT_1, \quad \mathcal{N}_i(AX) = \mathcal{N}_i(X) = S_1,$$
$$\mathcal{R}_i(XA) = \mathcal{R}_i(X)A = T_2A, \quad \mathcal{N}_i(XA) = \mathcal{N}_i(X) = S_2.$$ 

Thus (3.3) and (3.4) are satisfied. The uniqueness of $A^{(2)}_{T_1, T_2, S_1, S_2}$ comes directly form the uniqueness of $A^{(2)}_{T_1, S_1}$ or $A^{(2)}_{T_2, S_2}$. 

\textbf{Theorem 3.2.} (1) Suppose $A \in \mathbb{H}^{m \times n}$, $T_1$ is a subspace of $\mathbb{H}^{n \times s}$ of dimension $s \leq r$ and $S_1$ is a subspace of $\mathbb{H}^{m \times 1}$ of dimension $m - s$. In addition, suppose $G \in \mathbb{H}^{n \times m}$ such that $\mathcal{R}_i(G) = T_1, \mathcal{N}_i(G) = S_1$. If $A$ has a [2]-inverse $A^{(2)}_{T_1, S_1}$ then \text{Ind}(AG) = 1. Further, we have

$$A^{(2)}_{T_1, S_1} = G(AG)_{G}.$$  

(2) Suppose $A \in \mathbb{H}^{m \times n}$, $T_2$ is a subspace of $\mathbb{H}^{1 \times m}$ of dimension $t \leq r$ and $S_2$ is a subspace of $\mathbb{H}^{1 \times n}$ of dimension $m - t$. In addition, suppose $G \in \mathbb{H}^{n \times m}$ such that $\mathcal{R}_i(G) = T_2, \mathcal{N}_i(G) = S_2$. If $A$ has a [2]-inverse $A^{(2)}_{T_2, S_2}$ then \text{Ind}(GA) = 1. Further, we have

$$A^{(2)}_{T_2, S_2} = (GA)_{GA}.$$  

(3) Let $A, T_1, T_2, S_1$ and $S_2$ be defined as (1) and (2). If $s = t$, and suppose $G \in \mathbb{H}^{n \times m}$ such that $\mathcal{R}_i(G) = T_1, \mathcal{N}_i(G) = S_1, \mathcal{R}_i(G) = T_2, \mathcal{N}_i(G) = S_2$. If $A$ has a [2]-inverse $A^{(2)}_{T_1, T_2, S_1, S_2}$ then \text{Ind}(AG) = \text{Ind}(GA) = 1. Further, we have

$$A^{(2)}_{T_1, T_2, S_1, S_2} = G(AG)_{GA} = (GA)_{GA}.$$  

\textbf{Proof.} As in the proof of Theorem 2.1 in [21], we can get (1).

(2) It is easy to verify that

$$\mathcal{R}_i(GA) = \mathcal{R}_i(G)A = T_2A \quad \text{and} \quad S_2 = \mathcal{N}(G) \subseteq \mathcal{N}(GA).$$
By the assumption of Theorem 3.1, we have
\[ \dim(T_2A) = m - (m-s) = s. \]

Note that
\[ \dim(\mathcal{R}_i(GA)) + \dim(\mathcal{N}_i(GA)) = m, \]
then
\[ \dim(\mathcal{N}_i(GA)) = m - \dim(\mathcal{R}_i(GA)) = m - s = \dim(\mathcal{N}_i(G)). \]

Thus \( \mathcal{N}_i(GA) = \mathcal{N}_i(G) = S_2 \), so that
\[ \mathcal{R}_i(GA) \oplus \mathcal{N}_i(GA) = T_2A \oplus S_2 = \mathbb{H}^{1\times n}. \]

Then \( \text{Ind}(GA) = 1 \). Setting
\[ X = (GA)_gG = GA((GA)^3)^\dagger GAG. \]

By direct verification, we have
\[ XAX = GA((GA)^3)^\dagger GAGA((GA)^3)^\dagger GAG = GA((GA)^3)^\dagger GAG = X, \]
and
\[ \mathcal{R}_i(X) = \mathcal{R}_i((GA)_gG) \subseteq \mathcal{R}_i(G) = T_2, \]
\[ \mathcal{N}_i(X) = \mathcal{N}_i((GA)_gG) \supseteq \mathcal{N}_i((GA)_g) = \mathcal{N}_i(GA) \supseteq \mathcal{N}_i(G) = S_2. \]

Obviously, \( r(X) \leq \dim(T_2) \). On the other hand,
\[ r(X) = r((GA)_gG) \geq r((GA)_gGA) = r(GA) = s = \dim(T_2), \]
thus \( \mathcal{R}_i(X) = T_2 \). Similarly, we have \( \mathcal{N}_i(X) = S_2 \).

(3) Combining the proofs of (1) and (2), we can get (3) easily. \( \square \)

**Theorem 3.3.**

(1) Suppose \( A \in \mathbb{H}^{m \times n} \), then for the Moore–Penrose inverse \( A^\dagger \), the Weighted Moore–Penrose inverse \( A^\dagger_{M,N} \), we have

(a) \[ A^\dagger = \overline{A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)}} = \overline{A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)}} = A^*(AA^*)_g = (A^*A)_gA^*. \]

(b) \[ A^\dagger_{M,N} = \overline{A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)}} = \overline{A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)}} = A^*(AA^*)^\dagger_{MM} = (A^*A)^\dagger_{NN}A^*, \]

where \( A^* = N^{-1}A^*M \).

(2) Suppose \( A \in \mathbb{H}^{m \times n} \), then for the Drazin inverse \( A^D \), we have
\[ A^D = \overline{A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger); A^k(A^k+1)g = (A^k+1)gA^k}, \]

where \( k = \text{Ind}(A) \).

**Proof.** In **Theorem 3.2(1)**, setting \( G = A^* \), and recall \( A^\dagger = A^*(AA^*)^\dagger - A^* \), we can get
\[ A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)} = A^*(AA^*)_g = A^*AA^*((AA^*)^3)^\dagger AA^* = A^*(AA^*)^\dagger = A^\dagger, \]
\[ A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)} = (A^*A)_gA^* = A^*A((A^*A)^3)^\dagger A^*AA^* = (A^*A)^\dagger A^* = A^\dagger, \]

setting \( G = N^{-1}A^*M \), we can get
\[ A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)} = A^*(AA^*)^\dagger_{MM} = (A^*A)^\dagger_{NN}A^* = A^\dagger_{MM}, \]
\[ A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)} = A^*(AA^*)^\dagger_{MM} = (A^*A)^\dagger_{NN}A^* = A^\dagger_{MM}. \]

In **Theorem 3.2(2)**, suppose \( A \in \mathbb{H}^{m \times n} \), then setting \( G = A^k \), where \( k = \text{Ind}(A) \), then we have
\[ A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)} = A^k(A^{k+1})_g = A^D, \]
\[ A^{(2)}_{\mathcal{R}_i(A^\dagger), \mathcal{N}_i(A^\dagger)} = (A^{k+1})gA^k = A^D. \]
Remark 3.1. (1) It follows from Theorem 3.3 that the Moore–Penrose inverse $A^\dagger$, the Weighted Moore–Penrose inverse $A_{M,N}^\dagger$ and the Drazin inverse $A^0$ are special cases of the generalized inverse $A_{(T_1,T_2),(S_1,S_2)}^{(2)}$ over the quaternion skew field $\mathbb{H}$.

(2) In Theorems 3.1 and 3.2, setting $s = r$ or $t = r$ or $s = t = r$, we can get some similar results relate to the generalized inverses $A_{(T_1,T_2),(S_1,S_2)}^{(1,2)}, A_{(T_1,T_2),(S_1,S_2)}^{(1,2)}$ and $A_{(T_1,T_2),(S_1,S_2)}^{(1,2)}$.

Next we show the determinantal representations of the generalized inverses over the quaternion skew field $\mathbb{H}$.

Theorem 3.4. (1) Suppose that $A \in \mathbb{H}_r^{m \times n}, \ T_1 \subset \mathbb{H}^n, \ S_1 \subset \mathbb{H}^m$, 
$$\dim(T_1) = \dim(S_1^+) = t \leq r \quad \text{and} \quad AT_1 \oplus S_1 = \mathbb{H}^m.$$ 
Let $B \in \mathbb{H}_m^{(m-t) \times n}$ and $C^* \in \mathbb{H}_n^{n \times (n-t)}$ be of column rank such that 
$$S_1 = \mathcal{R}_r(B) \quad \text{and} \quad T_1 = \mathcal{N}_t(C).$$

Denote
$$M = (m_{ij}) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$
then the generalized inverse $A_{(T_1,S_1)}^{(2)} = (a_{ij}) \in \mathbb{H}^{n \times m}$ possesses the following determinantal representations:
$$a_{ij} = \frac{\sum_{k=1}^{m+n-t} L_{ik}m^*_{kj}}{\det MM^*}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m;$$
or
$$a_{ij} = \frac{\sum_{k=1}^{m+n-t} m^*_{ik}R_{jk}}{\det MM^*}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m,$$
where $L_{ik}$ are the left $ij$-th cofactors of $M^*M$ and $R_{jk}$ are right $ij$-th cofactors of $MM^*$, respectively for all $i, j = 1, \ldots, m + n - t$.

(2) Suppose $A \in \mathbb{H}_r^{m \times n}, \ T_2 \subset \mathbb{H}_t^{1 \times m}, \ S_2 \subset \mathbb{H}_t^{1 \times n}$ and 
$$\dim(T_2) = \dim(S_2^+) = s \leq r \quad \text{and} \quad T_2A \oplus S_2 = \mathbb{H}_t^{1 \times n}.$$ 
Let $B \in \mathbb{H}_m^{(m-s) \times n}$ and $C^* \in \mathbb{H}_n^{(n-s) \times m}$ be of full row rank such that 
$$S_2 = \mathcal{R}_t(B) \quad \text{and} \quad T_2 = \mathcal{N}_t(C).$$

Denote
$$M = (m_{ij}) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$
then the generalized inverse $A_{(T_2,S_2)}^{(2)} = (a_{ij}) \in \mathbb{H}^{n \times m}$ possesses the following determinantal representations:
$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} L_{ik}m^*_{kj}}{\det MM^*}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m;$$
or
$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} m^*_{ik}R_{jk}}{\det MM^*}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m,$$
where $L_{ik}$ are the left $ij$-th cofactors of $M^*M$ and $R_{jk}$ are right $ij$-th cofactors of $MM^*$, respectively for all $i, j = 1, \ldots, m + n - s$.

(3) Let $A, T_1, S_1, T_2, S_2$ be defined as (1) and (2). If $s = t$, and suppose that $B \in \mathbb{H}_m^{m \times (m-s)}$ and $C^* \in \mathbb{H}_n^{n \times (n-s)}$ are of full column rank such that 
$$S_1 = \mathcal{R}_r(B), \quad T_1 = \mathcal{N}_t(C), \quad S_2 = \mathcal{R}_t(C) \quad \text{and} \quad T_2 = \mathcal{N}_t(B).$$

Denote
$$M = (m_{ij}) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$
then the generalized inverse \( A_{T,T}^{(2)}(s_1,s_2) = (a_{ij}) \) possess the following determinantal representations:
\[
a_{ij} = \frac{\sum_{k=1}^{m+n-s} L_{k}m_{kj}^{*}}{\det M^*M}, \quad i = 1, \ldots, n; j = 1, \ldots, m;
\]
or
\[
a_{ij} = \frac{\sum_{k=1}^{m+n-s} m_{ik}^{*}R_{kj}}{\det MM^*}, \quad i = 1, \ldots, n; j = 1, \ldots, m,
\]
where \( L_{ij} \) are the left \( ij \)-th cofactors of \( M^*M \) and \( R_{ij} \) are right \( ij \)-th cofactors of \( MM^* \), respectively for all \( i, j = 1, \ldots, m + n - t \).

**Proof.** For (1), set
\[
X = \begin{bmatrix}
A_{T,T}^{(2)} & (I - A_{T,T}^{(2)}A)^{\dagger} \\
B^{\dagger}(I - AA_{T,T}^{(2)}) & B^{\dagger}(AA_{T,T}^{(2)}A - A)^{\dagger}
\end{bmatrix}
\]
then
\[
MX = \begin{bmatrix}
AA_{T,T}^{(2)} + BB^{\dagger}(I - AA_{T,T}^{(2)}) & A(I - A_{T,T}^{(2)}A)C^{\dagger} - BB^{\dagger}(I - AA_{T,T}^{(2)})AC^{\dagger} \\
CA_{T,T}^{(2)} & C(I - A_{T,T}^{(2)}A)C^{\dagger}
\end{bmatrix}
\]
By the definition of \( A_{T,T}^{(2)} \) we have
\[
AA_{T,T}^{(2)} = P_{AT,S}, \quad I - AA_{T,T}^{(2)} = P_{S,AT}.
\]
Recall (3.5) we have
\[
BB^{\dagger} = P_{R_{k}} = P_{S} \quad \text{and} \quad CA_{T,T}^{(2)} = 0.
\]
Noting that \( R_{k}(C^{\dagger}) = R_{k}(C^{*}) = T^{\perp} \) and \( C \) is of full row rank then
\[
C(I - A_{T,T}^{(2)}A)C^{\dagger} = CC^{\dagger} = I.
\]
Since
\[
AA_{T,T}^{(2)} + BB^{\dagger}(I - AA_{T,T}^{(2)}) = P_{AT,S} + P_{S}P_{S,AT} = P_{AT,S} + P_{S,AT} = I
\]
and
\[
A(I - A_{T,T}^{(2)}A)C^{\dagger} - BB^{\dagger}(I - AA_{T,T}^{(2)})AC^{\dagger} = (I - A_{T,T}^{(2)}A)AC^{\dagger} - (I - AA_{T,T}^{(2)})AC^{\dagger} = 0
\]
then \( MX = I \), hence \( M \) is invertible and \( M^{-1} = X \). Then there exist the inverse \((M^*M)^{-1}\) of the Hermitian matrix \( M^*M \). Multiplying it on the right by \( M^* \), we obtain the left inverse \((LM)^{-1} = (M^*M)^{-1}M^* \). By Theorem 2.2, we have
\[
(LM)^{-1} = (L(M^*M))^{-1}M^*
\]
\[
= \frac{1}{\det M^*M} \begin{bmatrix}
L_{11} & L_{21} & \cdots & L_{n1} \\
L_{12} & L_{22} & \cdots & L_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1n} & L_{2n} & \cdots & L_{nn}
\end{bmatrix} M^*
\]
\[
= \frac{1}{\det M^*M} \begin{bmatrix}
\sum_{k} L_{k1}m_{k1}^{*} & \sum_{k} L_{k1}m_{k2}^{*} & \cdots & \sum_{k} L_{k1}m_{kn}^{*} \\
\sum_{k} L_{k2}m_{k1}^{*} & \sum_{k} L_{k2}m_{k2}^{*} & \cdots & \sum_{k} L_{k2}m_{kn}^{*} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k} L_{km+n-r}m_{k1}^{*} & \sum_{k} L_{km+n-r}m_{k2}^{*} & \cdots & \sum_{k} L_{km+n-r}m_{kn}^{*}
\end{bmatrix}
\]
Then we can get
\[
A_{T,T}^{(2)} = (a_{ij})
\]
possesses the following determinantal representations:

\[ a_{ij} = \frac{\sum_{k=1}^{m+n-r} L_{kj} m_{ij}^k}{\det M^* M}, \quad i = 1, \ldots, n, j = 1, \ldots, m, \]

where \( L_{ij} \) are the left \( ij \)-th cofactors of \( M^* M \). Now we prove formula (3.7). There exist the inverse \((M^*)^{-1}\) of the Hermitian matrix \( MM^* \). Multiplying it on the left by \( M^* \), we obtain the left inverse \((RM)^{-1} = M^*(MM^*)^{-1}\). By Theorem 2.2, we have

\[
(RM)^{-1} = M^* (R (MM^*))^{-1}
\]

\[
= M^* \frac{1}{\det MM^*} \begin{bmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{bmatrix}
\]

\[
= \frac{1}{\det MM^*} \begin{bmatrix} \sum_{k} m_{1k}^* R_{1k} & \sum_{k} m_{2k}^* R_{1k} & \cdots & \sum_{k} m_{nk}^* R_{1k} \\ \sum_{k} m_{1k}^* R_{2k} & \sum_{k} m_{2k}^* R_{2k} & \cdots & \sum_{k} m_{nk}^* R_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k} m_{1k}^* R_{mk}(m+n-r)k & \sum_{k} m_{2k}^* R_{mk}(m+n-r)k & \cdots & \sum_{k} m_{nk}^* R_{mk}(m+n-r)k \end{bmatrix}
\]

Then (3.7) follows immediately.

Similarly, we can get (2) and (3).

\[ \square \]

Remark 3.2. (1) In Theorem 3.3, setting \( s = r \) or \( t = r \) or \( s = t = r \), we can get determinantal representations of the generalized inverses \( A_{11}^{(1,2)} \), \( A_{12}^{(1,2)} \) and \( A_{11}^{(1,2)} \), respectively.

(2) We also can get the determinantal representations of the Moore–Penrose inverse \( A^\dagger \), the Weighted Moore–Penrose inverse \( A_{M,N}^\dagger \) and the Drazin inverse \( A^\partial \), respectively.

4. Cramer rules for the unique solution of the restricted matrix equation

Cramer rules for representations of generalized inverses and solutions of some restricted equations have been studied by many authors, such as Cai and Chen [26] who gave a determinantal representation for the generalized inverse \( A_{11}^{(2)} \) and derived some applications. In Chapter 3 of [29], Wang, Wei and Qiao surveyed the recent results on the Cramer rules. Recently, Kyrchei [30], derived a determinantal representations of the Moore–Penrose inverse over the quaternion skew field and got the corresponding Cramer’s rule. Moreover, in [31] he gave a Cramer rule for the solution of the nonsingular quaternion matrix equation

\[ AXB = C \quad (4.1) \]

as follows:

Lemma 4.1. Suppose that \( A, B, C \in \mathbb{H}^{n \times n} \) are given, and \( X \in \mathbb{H}^{n \times n} \) is unknown. If \( \det(A^* A) \neq 0 \) and \( \det(BB^*) \neq 0 \), then (4.1) has a unique solution, and the solution is

\[ x_{ij} = \frac{r \det(BB^*) \cdot (c_i^A)}{\det(A^* A) \cdot \det(BB^*)}, \]

or

\[ x_{ij} = \frac{c_i^A \cdot (c_i^B)}{\det(A^* A) \cdot \det(BB^*)}, \]

where

\[ c_i^A := [ c_i \det(A^* A), \ldots, c_i \det(A^* A) ] \]

is the row vector and

\[ c_i^B := [ r \det(BB^*) \cdot (d_i), \ldots, r \det(BB^*) \cdot (d_i) ]^T \]

is the column vector and \( d_i, d_j \) are the \( i \)-th row vector and \( j \)-th column vector of \( A^* CB^* \), respectively, for all \( i, j = 1, \ldots, n \).
In this section, we aim to consider the Cramer rules for the unique solution of the restricted matrix Eqs. (1.1)–(1.3), respectively. Denote
\[ R(A, B) = \{ Y_1 = AXB : X \in \mathbb{H}^{n \times p}, \} \quad \text{and} \quad \mathcal{N}_i(A, B) = \{ X_i \in \mathbb{H}^{n \times p} : AXB = 0 \}. \]
\[ R_i(A, B) = \{ Y_1 = AXB : X \in \mathbb{H}^{n \times p}, \} \quad \text{and} \quad \mathcal{N}_i(A, B) = \{ X_i \in \mathbb{H}^{n \times p} : AXB = 0 \}. \]

**Theorem 4.2.** (1) Suppose that \( A \in \mathbb{H}_{n_1}^{m \times s_1}, B \in \mathbb{H}_{p_2}^{q \times v}, \) \( C \in \mathbb{H}^{m \times q}, \) \( T_{11} \subset \mathbb{H}^n, \) \( S_{11} \subset \mathbb{H}^m, \) \( T_{12} \subset \mathbb{H}^q \) and \( S_{12} \subset \mathbb{H}^p \) are known and satisfy
\[
\dim(T_{11}) = \dim(S_{11}^+) = s_1 \leq r_1, \quad \dim(T_{12}) = \dim(S_{11}^+) = t_1 \leq r_2.
\]
\[ AT_{11} \oplus S_{11} = \mathbb{H}^m, \quad BT_{12} \oplus S_{12} = \mathbb{H}^p. \]

Suppose that matrices \( L_1 \in \mathbb{H}_{n-n-s_1}^{m \times (m-s_1)}, \) \( M_1^* \in \mathbb{H}_{n-1}^{n \times (n-s_1)}, \) \( L_2 \in \mathbb{H}_{p-t_1}^{p \times (p-t_1)}, \) \( M_2^* \in \mathbb{H}_{q-t_1}^{q \times (q-t_1)}, \) \( G_1 \in \mathbb{H}^{n \times m} \) and \( G_2 \in \mathbb{H}^{q \times p} \) satisfy
\[ \mathcal{R}_i(G_1) = \mathcal{N}_i(M_1), \quad \mathcal{R}_i(G_2) = \mathcal{R}_i(L_2) = S_{12}. \]
Denote
\[ A_1 = \begin{bmatrix} A & L_1 \\ M_1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & L_2 \\ M_2 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}. \]
If \( C \in \mathcal{R}_i(A_1, G_2B), \) then the restricted matrix equation
\[ AXB = C, \quad \mathcal{R}_i(X) \subset T_{11}, \quad \mathcal{N}_i(X) \supset S_{12} \]
has a unique solution
\[ X = A_{11}^{(2)} C B_{12}^{(2)} \]
and possess the following determinantal representations:
\[ x_{ij} = \frac{r \det(B_i B_1^*)}{\det(A_i^* A_i) \det(B_1^*)}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, q; \]
\[ x_{ij} = \frac{c \det(A_i^* A_i) \det(c_{ij} B_1^*)}{\det(A_i^* A_i) \det(B_1^*)}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, q, \]
where
\[ c_{ij}^A := \begin{bmatrix} c \det(A_i^* A_i) \det(d_{i1}) & \cdots & c \det(A_i^* A_i) \det(d_{in}) \end{bmatrix} \]
is the row vector and
\[ c_{ij}^B := \begin{bmatrix} r \det(B_i B_1^*) \det(d_{i1}) & \cdots & r \det(B_i B_1^*) \det(d_{in}) \end{bmatrix}^T \]
is the column vector and \( d_{ij} \) are the \( i \)th row vector and \( j \)th column vector of \( A_i^* C, B_i^* \), respectively, for all \( i = 1, \ldots, m + n, \quad j = 1, \ldots, p + q. \)

(2) Let \( A, B, C \) be defined as (1). Suppose that \( T_{21} \subset \mathbb{H}_{1 \times n}^1, S_{21} \subset \mathbb{H}_{1 \times n}^1, T_{22} \subset \mathbb{H}_{1 \times p}^1 \) and \( S_{22} \subset \mathbb{H}_{1 \times q}^1 \) such that
\[ \dim(T_{21}) = \dim(S_{21}^+) = s_2 \leq r_1, \quad \dim(T_{22}) = \dim(S_{22}^+) = t_2 \leq r_2, \]
and
\[ T_{21} A \oplus S_{21} = \mathbb{H}_{1 \times n}^1, \quad T_{22} B \oplus S_{22} = \mathbb{H}_{1 \times q}^1. \]
and matrices \( L_1 \in \mathbb{H}_{n-n-s}^{(m-s) \times n}, L_2 \in \mathbb{H}_{p-t}^{(p-t) \times q}, M_1^* \in \mathbb{H}_{m-s}^{m \times m}, M_2^* \in \mathbb{H}_{q-t}^{q \times q}, G_1 \in \mathbb{H}^{n \times m} \) and \( G_2 \in \mathbb{H}^{q \times p} \) satisfy
\[ \mathcal{N}_i(G_1) = \mathcal{R}_i(L_1) = S_{21}, \quad \mathcal{R}_i(G_2) = \mathcal{N}_i(M_2) = T_{22}. \]
Denote
\[ A_1 = \begin{bmatrix} A & M_1 \\ L_1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & M_2 \\ L_2 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}. \]
If \( C \in \mathcal{R}_i(A_1, G_2B), \) then the restricted matrix equation
\[ AXB = C, \quad \mathcal{R}_i(X) \subset T_{22}, \quad \mathcal{N}_i(X) \supset S_{21} \]
(4.6)
has a unique solution
\[ X = A^{(2)}_{r_1, s_2} C B^{(2)}_{T_2, s_2} \tag{4.7} \]
and possess the determinantal representations forms of (4.4)–(4.5).

(3) Let \( A, B, C, T_1, S_1, T_2, S_2, T_{21}, S_{21}, T_{22}, S_{22} \) be defined as (1) and (2). If \( s_1 = s_2, t_1 = t_2 \) suppose that matrices
\[ P_1 \in \mathbb{R}^{(m-n) \times n}, Q_1 \in \mathbb{R}^{(m-n) \times m}, P_2 \in \mathbb{R}^{(q-r) \times q}, Q_2 \in \mathbb{R}^{(p-r) \times p}, G_1 \in \mathbb{R}^{n \times m} \text{ and } G_2 \in \mathbb{R}^{p \times p} \]
\[ \mathcal{R}_1(G_1) = \mathcal{N}_1(P_1) = T_{11}, \quad \mathcal{N}_1(G_2) = \mathcal{R}_1(Q_2) = S_{12}, \]
\[ \mathcal{N}_1(G_1) = \mathcal{R}_1(P_1) = S_{21}, \quad \mathcal{R}_1(G_2) = \mathcal{N}_1(Q_2) = T_{22}. \]
Denote
\[ A_1 = \begin{bmatrix} A & P_1 & Q_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & P_2 & Q_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 & 0 \end{bmatrix}. \]
If \( C \in \mathcal{R}_1(A G_1, G_2 B) \) and \( C \in \mathcal{R}(A G_1, G_2 B) \), then the restricted matrix equation
\[ A X B = C, \quad \mathcal{R}_1(X) \subset T_{11}, \quad \mathcal{N}_1(X) \supset S_{12}, \quad \mathcal{R}_1(X) \subset T_{22}, \quad \mathcal{N}_1(X) \supset S_{21}. \]
has a unique solution
\[ X = A^{(2)}_{r_1, s_2}, S_1, T_{21}, S_{21}, T_{22}, S_{22} \] \[ CB^{(2)}_{T_2} \]
and possess the determinantal representations forms of (4.4)–(4.5).

**Proof.** By the proof of Theorem 3.3.4 in [29], we can get (1). Next we show (2). The proof contains two parts. We first establish that the unique solution of Eq. (4.6) can be expressed as (4.6). From the definition of the left range and null space of a pair of matrices, we have \( C = A G_1 Y G_2 B \) for some matrix \( Y \in \mathbb{R}^{n \times p} \). It follows that
\[ \mathcal{R}_1(C) \subset \mathcal{R}_1(G_2 B) = \mathcal{R}_1(G_2) B = T_{22} B. \]
And note that
\[ \mathcal{N}_1(C) = (\mathcal{R}_1(D^*))^{-1} \supseteq (\mathcal{R}_1(G_1^* A^*))^{-1} = (\mathcal{R}_1(G_1^*) A^*)^{-1} = ((\mathcal{N}_1(G_1))^* A^*)^{-1} = (S_{21} A^*)^{-1}, \]
then we have
\[ A^{(2)}_{r_1, s_1, T_{21}, S_{21}} A^{(2)}_{r_2, s_2, T_{22}, S_{22}} B = C, \quad \text{say that (4.7) is a solution of Eq. (4.6) and also satisfies the restricted conditions since} \]
\[ \mathcal{R}_1(A^{(2)}_{r_1, s_1, T_{21}, S_{21}} \subset \mathcal{R}_1(B^{(2)}_{r_2, s_2, T_{22}}, S_{22}) = T_{22}, \quad \mathcal{N}_1(A^{(2)}_{r_1, s_1, T_{21}, S_{21}} \subset \mathcal{N}_1(B^{(2)}_{r_2, s_2, T_{22}}, S_{22}) = S_{21}. \]
For the uniqueness, if \( X_0 \) is a solution of (4.6) then, \( \mathcal{R}_1(X_0) \subset T_{22}, \mathcal{N}_1(X_0) \supset S_{21}, \) it follows that
\[ A^{(2)}_{r_1, s_1, T_{21}, S_{21}} B^{(2)}_{r_2, s_2, T_{22}} = A^{(2)}_{r_1, s_1} A B B^{(2)}_{s_1, T_{22}, S_{22}} = X_0. \]
Next, we show a Cramer rule for solving Eq. (4.6). Since \( X \) is the solution of (4.6), then we have
\[ \mathcal{R}_1(X) \subset T_{22} = N_1(M_2), \quad \mathcal{N}_1(X) \supset S_{21} = \mathcal{R}_1(L_1), \]
it follows that
\[ XM_2 = 0, \quad L_1 X = 0 \]
\[ \begin{bmatrix} A & M_1 \end{bmatrix} \begin{bmatrix} X & 0 \end{bmatrix} \begin{bmatrix} B & M_2 \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{4.8} \]
From the proof of Theorem 3.4, we see the coefficient matrices of (4.8) are nonsingular and
\[ \begin{bmatrix} X & 0 \end{bmatrix} ^{-1} = \begin{bmatrix} A & M_1 \end{bmatrix} ^{-1} \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} B & M_2 \end{bmatrix} ^{-1} \]
then it follows the Cramer rule of Lemma 4.1 we can get the determinantal representations form of the solution is same to (4.4)–(4.5).

Combining the proof of (1) and (2), it is easy to verify (3). \( \square \)

**Corollary 4.3.** (1) Suppose that \( A \in \mathbb{H}_{m \times n}, B \in \mathbb{H}_{p \times q}, M, N, P \text{ and } Q \) are Hermitian positive definite matrices of orders \( m, n, p \) and \( q \) respectively. The columns of \( U_1 \in \mathbb{H}_{m \times (m-r)}, V_1^* \in \mathbb{H}_{m \times (n-r)}, U_1 \in \mathbb{H}_{m \times (m-r)}, V_1^* \in \mathbb{H}_{m \times (n-r)} \) and \( V_1^* \in \mathbb{H}_{m \times (n-r)} \) form bases for \( \mathcal{N}(A^*), \mathcal{N}(A), \mathcal{N}(B^*) \text{ and } \mathcal{N}(B) \) respectively. Denote
\[ A^* = N^{-1} A^* M, \quad B^* = Q^{-1} B^* P, \]
\[ A_1 = \begin{bmatrix} A & M^{-1} U_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & P^{-1} U_2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 \end{bmatrix}. \]
If \( C \in \mathcal{R}_t(AG_1, G_2B) \) and \( C \in \mathcal{R}_t(AG_1, G_2B) \), then the restricted matrix equation
\[
AXB = C, \quad \mathcal{R}_t(X) \subset \mathcal{N}^{-1}(A^*), \quad \mathcal{N}_t(X) \supset P^{-1}\mathcal{N}_t(B^*), \quad \mathcal{R}_i(X) \subset \mathcal{R}_i(A^*)M, \quad \mathcal{N}_i(X) \supset \mathcal{N}_i(B^*)Q,
\]
has a unique solution
\[
X = A^{j*}_MB^{l*}C^{p*}_{dQ}
\]
and possesses the determinantal representations (4.4)–(4.5).

(2) Suppose that \( A, B \) are same as (1), and the columns of \( U_1 \in \mathbb{H}^{m \times (m-r)}, V_1^* \in \mathbb{H}^{n \times (n-r)}, U_1 \in \mathbb{H}^{m \times (m-t)} \) and \( V_1^* \in \mathbb{H}^{n \times (n-t)} \) form bases for \( \mathcal{N}(A^*), \mathcal{N}(A), \mathcal{N}(B^*) \) and \( \mathcal{N}(B) \) respectively. Denote
\[
A_1 = \begin{bmatrix} A & U_1 \\ V_1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & U_2 \\ V_2 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.
\]
If \( C \in \mathcal{R}_t(AG_1, G_2B) \) and \( C \in \mathcal{R}_t(AG_1, G_2B) \), then the restricted matrix equation
\[
AXB = C, \quad \mathcal{R}_t(X) \subset \mathcal{R}_t(A^*), \quad \mathcal{N}_t(X) \supset \mathcal{N}_t(B^*), \quad \mathcal{R}_i(X) \subset \mathcal{R}_i(A^*), \quad \mathcal{N}_i(X) \supset \mathcal{N}_i(B^*),
\]
has a unique solution
\[
X = A^{j*}CB^{k*}
\]
and possesses the determinantal representations (4.4)–(4.5).

**Corollary 4.4.** (1) Suppose that \( A \in \mathbb{H}^{m \times n}, \text{Ind}(A) = k_1, r(A^k) = r_1 < n, B \in \mathbb{H}^{p \times p}, \text{Ind}(B) = k_2, r(B^k) = r_2 < p, \) and \( U_1 \in \mathbb{H}^{m \times (m-r)}, V_1^* \in \mathbb{H}^{n \times (n-r)}, U_1 \in \mathbb{H}^{m \times (m-t)} \) and \( V_1^* \in \mathbb{H}^{n \times (n-t)} \) form bases for \( \mathcal{N}(A^k), \mathcal{N}(A^k), \mathcal{N}(B^k) \) and \( \mathcal{N}(B^k) \), respectively. Denote
\[
A_1 = \begin{bmatrix} A & U_1 \\ V_1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & U_2 \\ V_2 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.
\]
If \( C \in \mathcal{R}_t(AG_1, G_2B) \) and \( C \in \mathcal{R}_t(AG_1, G_2B) \), then the restricted matrix equation
\[
AXB = C, \quad \mathcal{R}_t(X) \subset \mathcal{R}_t(A^k), \quad \mathcal{N}_t(X) \supset \mathcal{N}_t(B^k), \quad \mathcal{R}_i(X) \subset \mathcal{R}_i(A^k), \quad \mathcal{N}_i(X) \supset \mathcal{N}_i(B^k),
\]
has a unique solution
\[
X = A^{j*}CB^{k*}
\]
and possesses the determinantal representations (4.4)–(4.5).

(2) Suppose that \( A \in \mathbb{H}^{m \times n}, \text{Ind}(A) = 1, r(A) = r_1 < n, B \in \mathbb{H}^{p \times p}, \text{Ind}(B) = 1, r(B) = r_2 < p, \) and \( U_1 \in \mathbb{H}^{m \times (m-r)}, V_1^* \in \mathbb{H}^{n \times (n-r)}, U_1 \in \mathbb{H}^{m \times (m-t)} \) and \( V_1^* \in \mathbb{H}^{n \times (n-t)} \) form bases for \( \mathcal{N}(A), \mathcal{N}(A^*), \mathcal{N}(B) \) and \( \mathcal{N}(B^*) \), respectively. Denote
\[
A_1 = \begin{bmatrix} A & U_1 \\ V_1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B & U_2 \\ V_2 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.
\]
If \( C \in \mathcal{R}_t(AG_1, G_2B) \) and \( C \in \mathcal{R}_t(AG_1, G_2B) \), then the restricted matrix equation
\[
AXB = C, \quad \mathcal{R}_t(X) \subset \mathcal{R}_t(A), \quad \mathcal{N}_t(X) \supset \mathcal{N}_t(B), \quad \mathcal{R}_i(X) \subset \mathcal{R}_i(A), \quad \mathcal{N}_i(X) \supset \mathcal{N}_i(B),
\]
has a unique solution
\[
X = A^{j*}_gCB^{k*}_g
\]
and possesses the determinantal representations (4.4)–(4.5).

5. An example

In this section, we give an example to illustrate our results. Let us consider the restricted matrix equation
\[
AXB = C, \quad \mathcal{R}_t(X) \subset \mathcal{R}_t(A^*), \quad \mathcal{N}_t(X) \supset \mathcal{N}_t(B^*), \quad \mathcal{R}_i(X) \subset \mathcal{R}_i(A^*), \quad \mathcal{N}_i(X) \supset \mathcal{N}_i(B^*),
\]
where
\[
A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & j \\ j & -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 - j & 1 + j \\ i - k & i + k \end{bmatrix}.
\]
By setting
\[
U_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} i & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} j \\ 1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} j & 1 \end{bmatrix},
\]
then we have the columns of $U_1$, $V_1^*$, $U_2$ and $V_2^*$ form the bases for $\mathcal{R}_r(A^*)$, $\mathcal{R}_l(A^*)$, $\mathcal{N}_l(B^*)$ and $\mathcal{N}_l(B^*)$, respectively. It follows Corollary 4.3 (2) that

$$A_1 = \begin{bmatrix} 1 & i & j \\ i & -1 & 1 \\ j & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & j & 1 \\ j & -1 & 1 \\ j & 1 & i-k \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1-j & 1+j & 0 \\ 1-i & i-k & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

and

$$A_1^*A_1 = \begin{bmatrix} 3 & i & 0 \\ -j & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B_1^*B_1^* = \begin{bmatrix} 3 & -j & 0 \\ j & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_1^*C_1B_1^* = \begin{bmatrix} 4-4j & 0 & 0 \\ 0 & 4i+4k & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

It is easy to get

$$\mathcal{R}_r(C) \in \mathcal{R}_r(AA^*, B^*B) \quad \text{and} \quad \mathcal{R}_l(C) \in \mathcal{R}_l(AA^*, B^*B).$$

Then Eq. (5.1) has a solution. We shall find it by (4.4). First we obtain the row-vectors $c_i^{A_1}$, for all $i = 1, 2, 3$. Note that

$$c_i \det(A_i^*A_1) = c_i \begin{bmatrix} 4-4j & i & 0 \\ -4i+4k & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 8(3 - j - i(-i + k)) = 16(1-j),$$

similarly, we can get

$$c_i^{A_1} = \begin{bmatrix} 16(1-j) & 16(-1-j) & 0 \\ 16(-i+k) & 16(i+j) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Then by (4.4) we have

$$x_{11} = \frac{r \det(B_1B_1^*)_1(c_1^{A_1})}{\det(A_1^*A_1) \det(B_1^*B_1^*)} \begin{bmatrix} 16(1-j) & 16(-1-j) & 0 \\ j & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{16 \times 16}{32(3 - 3j + j - 1)} = \frac{1 - j}{4},$$

and so forth. Continuing in this way, we have

$$x_{12} = \frac{-1 - j}{4}, \quad x_{21} = \frac{-i + k}{4}, \quad x_{22} = \frac{i + k}{4}.$$

Then

$$X = \frac{1}{4} \begin{bmatrix} -1 - j & -1 - j \\ -i + k & -i + k \end{bmatrix}$$

is the unique solution to Eq. (5.1).

6. Conclusion

In this paper, we have derived the determinantal representation of the generalized inverse $A_{(11-1),1}^{(2)}$, $A_{(11-2),2}^{(2)}$ and $A_{(11-2),2}^{(2)}$ over the quaternion skew field by the theory of the column and row determinants, respectively, and given the Cramer rules for the unique solution of the restricted matrix Eqs. (1.1)–(1.3). Some corresponding results on special cases such as the Moore–Penrose inverse, the Weighted Moore–Penrose inverse and the Drazin inverse have been presented.

Motivated by the work in this paper, it would be of interest to investigate the Minors of the generalized inverse $A_{(11-1),1}^{(2)}$, $A_{(11-2),2}^{(2)}$ and $A_{(11-2),2}^{(2)}$.

We will show the results in another paper.
Acknowledgements

The authors are very much indebted to the anonymous referees for their constructive and valuable comments and suggestions which greatly improve the original manuscript of this paper.

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