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Multi-point boundary value problems for second-order functional differential equations☆

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ABSTRACT

This paper is concerned with the existence of extreme solutions of multi-point boundary value problem for a class of second-order functional differential equations. We introduce a new concept of lower and upper solutions. By using the method of upper and lower solutions and monotone iterative technique, we obtain the existence of extreme solutions.
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1. Introduction

It is well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations. There exists much literature devoted to the applications of this technique to boundary value problems of ordinary differential equations, see [1–6]. There are also a few papers where the monotone iterative technique is used on nonlinear differential problems with delay, see [7–11]. In [12,13], Nieto and Rodriguez-Lopez introduced a new concept of lower and upper solutions, and considered the periodic boundary value problems for the following first-order functional differential equation

$$\begin{cases} u'(t) = g(t, u(t), u(\theta(t))), & t \in [0, T], \\ u(0) = u(T). \end{cases} \tag{1.1}$$

A similar method has already succeeded to solving nonlinear impulsive integro-differential equations [14] and impulsive functional differential equations [15].

Motivated by [12–15], we consider the multi-point boundary value problems for the functional differential equation

$$\begin{cases} -u''(t) = f(t, u(t), u(\theta(t))), & t \in J = [0, 1], \\ u(0) - au'(0) = cu(\eta), & u(1) + bu'(1) = du(\xi), \end{cases} \tag{1.2}$$

where $f \in C(J \times \mathbf{R}^2, \mathbf{R})$, $0 \le \theta(t) \le t,\ t \in J,\ \theta \in C(J)$, $a \ge 0$, $b \ge 0,\ 0 \le c \le 1,\ 0 \le d \le 1$, $0 < \eta, \xi < 1$.

When $\theta(t) = t$, the boundary value problem (1.2) reduces to multi-point boundary value problems for ordinary differential equations which have been studied in many papers, see [16–19]. To our knowledge, only a few papers paid

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attention to multi-point boundary value problems for functional differential equations. Recently, Jankowski [20] discussed solvability of three-point boundary value problems for a class of second-order differential equations with deviating arguments by using the monotone iterative technique. Our method is different from that of [20].

In this paper, we are concerned with the existence of extreme solutions for the boundary value problem (1.2). The paper is organized as follows. In Section 2, we establish two comparison principles. In Section 3, we consider a linear problem associated to equation (1.2) and then give a proof for the existence theorem. In Section 4, we first introduce a new concept of lower and upper solutions. By using the method of upper and lower solutions with a monotone iterative technique, we obtain the existence of extreme solutions for the boundary value problem (1.2).

2. Comparison principles

In the following, we always assume that the following condition is satisfied.

(H) $a \geq 0,\ b \geq 0,\ 0 \leq c \leq 1, 0 \leq d \leq 1,\ 0 < \eta, \xi < 1,\ a + c > 0,\ b + d > 0.$

For any given function $g \in E = C^2(J, R)$, we denote

$$A_g = \max\left\{\frac{g(0) - ag'(0) - cg(\eta)}{a\pi + c\sin\pi\eta}, \frac{g(1) + bg'(1) - dg(\xi)}{b\pi + d\sin\pi\xi}\right\},$$

$$B_g = \max\{A_g,\ 0\}, \qquad c_g(t) = B_g\sin(\pi t), \qquad r = \pi^2.$$

We now present the main results of this section.

Theorem 2.1. *Assume that $u \in E$ satisfies*

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) \leq 0, & t \in J, \\ u(0) - au'(0) \leq cu(\eta), & u(1) + bu'(1) \leq du(\xi), \end{cases} \tag{2.1}$$

where $a \geq 0, b \geq 0,\ 0 \leq c \leq 1,\ 0 \leq d \leq 1, 0 < \eta, \xi < 1$ and constants $M,\ N$ satisfy

$$M > 0, \qquad N \geq 0, \qquad M + N \leq 2. \tag{2.2}$$

Then $u(t) \leq 0$ for $t \in J$.

Proof. Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$.

If $u(1) = \max_{t\in J} u(t) > 0$, then $u'(1) \geq 0, u(1) \geq u(\xi)$ and

$$du(\xi) \leq u(1) \leq u(1) + bu'(1) \leq du(\xi).$$

So $d = 1$ and $u(\xi)$ is a maximum value.

If $u(0) = \max_{t\in J} u(t) > 0$, then $u'(0) \leq 0, u(0) \geq u(\eta)$ and

$$cu(\eta) \leq u(0) \leq u(0) - bu'(0) \leq cu(\eta).$$

So $c = 1$ and $u(\eta)$ is a maximum value.

Therefore, there is a $t^* \in (0, 1)$ such that

$$u(t^*) = \max_{t\in J} u(t) > 0, \qquad u'(t^*) = 0, \qquad u''(t^*) \leq 0. \tag{2.3}$$

Suppose that $u(t) \geq 0$ for $t \in J$. From the first inequality of (2.1), we obtain that $u''(t) \geq 0$ for $t \in J$. Hence

$$u(0) = \max_{t\in J} u(t) \quad \text{or} \quad u(1) = \max_{t\in J} u(t).$$

If $u(0) = \max_{t\in J} u(t) > 0$, then $u(t) \equiv K(K$ is a positive constant) for $t \in [0, \eta]$. From the first inequality of (2.1), we have that when $t \in [0, \eta]$,

$$0 < MK \leq Mu(t) + Nu(\theta(t)) \leq u''(t) = 0,$$

which is a contradiction.

If $u(1) = \max_{t\in J} u(t) > 0$, then $u(t) \equiv K(K$ is a positive constant) for $t \in [\xi, 1]$. From the first inequality of (2.1), we have that when $t \in [\xi, 1]$,

$$0 < MK \leq Mu(t) + Nu(\theta(t)) \leq u''(t) = 0,$$

which is a contradiction.

Suppose that there exist $t_1,\ t_2 \in J$ such that $u(t_1) > 0$ and $u(t_2) < 0$. We consider two cases.

Case 1. $u(0) > 0$. Since $u(t_2) < 0$, there is $\kappa > 0,\ \varepsilon > 0$ such that $u(\kappa) = 0, u(t) \geq 0$ for $t \in [0, \kappa)$ and $u(t) < 0$ for all $t \in (\kappa, \kappa + \varepsilon]$. It is easy to obtain that $u''(t) \geq 0$ for $t \in [0, \kappa]$. If $t^* < \kappa$, then $0 < Mu(t^*) \leq u''(t^*) \leq 0$, a contradiction.

Hence $t^* > \kappa + \varepsilon$. Let $t_* \in [0, t^*)$ such that $u(t_*) = \min_{t\in[0,t^*)} u(t)$, then $u(t_*) < 0$. From the first inequality of (2.1), we have

$$u''(t) \geq (M+N)u(t_*), \quad t \in [0, t^*).$$

Integrating the above inequality from $s(t_* \leq s \leq t^*)$ to t^*, we obtain

$$-u'(s) \geq (t^* - s)(M+N)u(t_*), \quad t_* \leq s \leq t^*,$$

and then integrate from t_* to t^* to obtain

$$\begin{aligned}
-u(t_*) &< u(t^*) - u(t_*) \\
&\leq \int_{t_*}^{t^*} (s - t^*)(M+N)u(t_*)\mathrm{d}s \\
&\leq -\frac{M+N}{2}u(t_*)(t^* - t_*)^2 \\
&\leq -\frac{M+N}{2}u(t_*).
\end{aligned}$$

From (2.2), we have that $u(t_*) > 0$. This is a contradiction.

Case 2. $u(0) \leq 0$. Let $t_* \in [0, t^*)$ such that $u(t_*) = \min_{t\in[0,t^*)} u(t) \leq 0$. From the first inequality of (2.1), we have

$$u''(t) \geq (M+N)u(t_*), \quad t \in [0, t^*).$$

The rest of the proof is similar to that of case 1. The proof is complete. □

Theorem 2.2. *Assume that (H) holds and $u \in E$ satisfies*

$$-u''(t) + Mu(t) + Nu(\theta(t)) + [(M+r)c_u(t) + Nc_u(\theta(t))] \leq 0, \quad t \in J,$$

where constants M, N satisfy (2.2), then $u(t) \leq 0$ for $t \in J$.

Proof. Assume that $u(0) - au'(0) \leq cu(\eta)$, $u(1) + bu'(1) \leq du(\xi)$, then $c_u(t) \equiv 0$. By Theorem 2.1, $u(t) \leq 0$.

Assume that $u(0) - au'(0) \leq cu(\eta)$, $u(1) + bu'(1) > du(\xi)$, then

$$c_u(t) = \frac{\sin(\pi t)}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)).$$

Put $y(t) = u(t) + c_u(t)$, $t \in J$, then $y(t) \geq u(t)$ for all $t \in J$, and

$$\begin{aligned}
y'(t) &= u'(t) + \frac{\pi\cos(\pi t)}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)), \quad t \in J, \\
y''(t) &= u''(t) - rc_u(t), \quad t \in J.
\end{aligned}$$

Hence

$$\begin{aligned}
& y(0) = u(0), \qquad y(1) = u(1), \\
& y(\xi) = u(\xi) + \frac{\sin(\pi\xi)}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)), \\
& y'(0) = u'(0) + \frac{\pi}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)), \\
& y'(1) = u'(1) - \frac{\pi}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)). \\
& -y''(t) + My(t) + Ny(\theta(t)) = -u''(t) + Mu(t) + Nu(\theta(t)) + [(M+r)c_u(t) + Nc_u(\theta(t))] \\
& \qquad \leq 0, \\
& y(0) - ay'(0) = u(0) - au'(0) - \frac{a\pi}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)) \\
& \qquad \leq cu(\eta) \leq cy(\eta), \\
& y(1) + by'(1) - dy(\xi) = u(1) + bu'(1) - du(\xi) - \frac{b\pi}{b\pi + d\sin(\pi\xi)}(u(1) + bu'(1) - du(\xi)) \\
& \qquad - \frac{d\sin(\pi\xi)}{b\pi + d\sin\pi\xi}(u(1) + bu'(1) - du(\xi)) \\
& \qquad \leq 0.
\end{aligned}$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$.

Assume that $u(0) - au'(0) > cu(\eta),\ u(1) + bu'(1) \le du(\xi)$, then

$$c_u(t) = \frac{\sin \pi t}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta)).$$

Put $y(t) = u(t) + c_u(t),\ t \in J$, then $y(t) \ge u(t)$ for all $t \in J$, and

$$\begin{aligned} y'(t) &= u'(t) + \frac{\pi\cos(\pi t)}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta)), \quad t \in J,\\ y''(t) &= u''(t) - rc_u(t), \quad t \in J. \end{aligned}$$

Hence

$$\begin{aligned} &y(0) = u(0), \qquad y(1) = u(1),\\ &y(\eta) = u(\eta) + \frac{\sin(\pi\eta)}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta)),\\ &y'(0) = u'(0) + \frac{\pi}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta)),\\ &y'(1) = u'(1) - \frac{\pi}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta)).\\ &-y''(t) + My(t) + Ny(\theta(t)) = -u''(t) + Mu(t) + Nu(\theta(t)) + [(M + r)c_u(t) + Nc_u(\theta(t))]\\ &\qquad \le 0,\\ &y(0) - ay'(0) - cy(\eta) = u(0) - au'(0) - cu(\eta) - \frac{a\pi}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta))\\ &\qquad - \frac{c\sin(\pi\eta)}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta))\\ &\qquad \le 0,\\ &y(1) + by'(1) = u(1) + bu'(1) - \frac{b\pi}{a\pi + c\sin(\pi\eta)}(u(0) - au'(0) - cu(\eta))\\ &\qquad \le du(\xi) \le dy(\xi). \end{aligned}$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$.

Assume that $u(0) - au'(0) > cu(\eta),\ u(1) + bu'(1) > du(\xi)$, then $c_u(t) = A_u \sin(\pi t)$. Put $y(t) = u(t) + c_u(t),\ t \in J$, then $y(t) \ge u(t)$ for all $t \in J$, and

$$\begin{aligned} y'(t) &= u'(t) + A_u\pi\cos(\pi t), \quad t \in J,\\ y''(t) &= u''(t) - rc_u(t), \quad t \in J. \end{aligned}$$

Hence

$$\begin{aligned} &y(0) = u(0), \qquad y(1) = u(1),\\ &y(\eta) = u(\eta) + A_u\sin(\pi\eta), \qquad y(\xi) = u(\xi) + A_u\sin(\pi\xi),\\ &y'(0) = u'(0) + A_u\pi, \qquad y'(1) = u'(1) - A_u\pi.\\ &-y''(t) + My(t) + Ny(\theta(t)) = -u''(t) + Mu(t) + Nu(\theta(t)) + [(M + r)c_u(t) + Nc_u(\theta(t))]\\ &\qquad \le 0,\\ &y(0) - ay'(0) - cy(\eta) = u(0) - au'(0) - cu(\eta) - aA_u\pi - cA_u\sin(\pi\eta)\\ &\qquad \le 0,\\ &y(1) + by'(1) - dy(\xi) = u(1) + bu'(1) - du(\xi) - bA_u\pi - dA_u\sin(\pi\xi)\\ &\qquad \le 0. \end{aligned}$$

By Theorem 2.1, $y(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for $t \in J$. The proof is complete. □

3. Linear problem

In this section, we consider the boundary value problem

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J,\\ u(0) - au'(0) = cu(\eta), \qquad u(1) + bu'(1) = du(\xi). \end{cases} \tag{3.1}$$

Theorem 3.1. *Assume that* (H) *holds,* $\sigma \in C(J)$ *and constants* M, N *satisfy* (2.2) *with*

$$\mu = \left(\frac{a(1+2b)}{2(a+b+1)} + \frac{1}{8}\left(\frac{1+2b}{a+b+1}\right)^2\right)(M+N) < 1. \tag{3.2}$$

Further suppose that there exist $\alpha, \beta \in E$ *such that*

(h_1) $\alpha \le \beta$ *on* J.
(h_2)

$$-\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) + [(M+r)c_\alpha(t) + Nc_\alpha(\theta(t))] \le \sigma(t), \quad t \in J.$$

(h_3)

$$-\beta''(t) + M\beta(t) + N\beta(\theta(t)) - [(M+r)c_{-\beta}(t) + Nc_{-\beta}(\theta(t))] \ge \sigma(t), \quad t \in J.$$

Then the boundary value problem (3.1) *has a unique solution* $u(t)$ *and* $\alpha \le u \le \beta$ *for* $t \in J$.

Proof. We first show that the solution of (3.1) is unique. Let u_1, u_2 be the solution of (3.1) and set $v = u_1 - u_2$. Thus

$$\begin{cases} -v''(t) + Mv(t) + Nv(\theta(t)) = 0, & t \in J, \\ v(0) - av'(0) = cv(\eta), & v(1) + bv'(1) = dv(\xi). \end{cases}$$

By Theorem 2.1, we have that $v \le 0$ for $t \in J$, that is, $u_1 \le u_2$ on J. Similarly, one can obtain that $u_2 \le u_1$ on J. Hence $u_1 = u_2$.

Next, we prove that if u is a solution of (3.1), then $\alpha \le u \le \beta$. Let $p = \alpha - u$. From boundary conditions, we have that $c_\alpha(t) = c_p(t)$ for all $t \in J$. From (h_2) and (3.1), we have

$$-p''(t) + Mp(t) + Np(\theta(t)) + [(M+r)c_p(t) + Nc_p(\theta(t))] \le 0, \quad t \in J.$$

By Theorem 2.1, we have that $p = \alpha - u \le 0$ on J. Analogously, $u \le \beta$ on J.

Finally, we show that the boundary value problem (3.1) has a solution by five steps as follows.

Step 1. Let $\bar\alpha(t) = \alpha(t) + c_\alpha(t)$, $\bar\beta(t) = \beta(t) - c_{-\beta}(t)$. We claim that
(1) $-\bar\alpha''(t) + M\bar\alpha(t) + N\bar\alpha(\theta(t)) + [(M+r)c_{\bar\alpha}(t) + Nc_{\bar\alpha}(\theta(t))] \le \sigma(t)$ for $t \in J$.
(2) $-\bar\beta''(t) + M\bar\beta(t) + N\bar\beta(\theta(t)) - [(M+r)c_{-\bar\beta}(t) + Nc_{-\bar\beta}(\theta(t))] \ge \sigma(t)$ for $t \in J$.
(3) $\alpha(t) \le \bar\alpha(t) \le \bar\beta(t) \le \beta(t)$ for $t \in J$.
From (h_2) and (h_3), we have

$$-\bar\alpha''(t) + M\bar\alpha(t) + N\bar\alpha(\theta(t)) \le \sigma(t), \quad t \in J, \tag{3.3}$$

$$-\bar\beta''(t) + M\bar\beta(t) + N\bar\beta(\theta(t)) \ge \sigma(t), \quad t \in J, \tag{3.4}$$

and

$$\bar\alpha(0) - a\bar\alpha'(0) - c\bar\alpha(\eta) = \alpha(0) - a\alpha'(0) - c\alpha(\eta) - (a\pi + c\sin(\pi\eta))B_\alpha \le 0, \tag{3.5}$$

$$\bar\alpha(1) + b\bar\alpha'(1) - d\bar\alpha(\xi) = \alpha(1) + b\alpha'(0) - d\alpha(\xi) - (b\pi + d\sin(\pi\xi))B_\alpha \le 0, \tag{3.6}$$

$$-[\bar\beta(0) - a\bar\beta'(0) - c\bar\beta(\eta)] = -\beta(0) + a\beta'(0) + c\beta(\eta) - (a\pi + c\sin(\pi\eta))B_{-\beta} \le 0, \tag{3.7}$$

$$-[\bar\beta(1) + b\bar\beta'(1) - d\bar\beta(\xi)] = -\beta(1) - b\beta'(0) + d\beta(\xi) - (b\pi + d\sin(\pi\xi))B_{-\beta} \le 0. \tag{3.8}$$

From (3.3)–(3.8), we obtain that $c_{\bar\alpha}(t) = c_{-\bar\beta}(t) \equiv 0$, $t \in J$. Combining (3.3) and (3.4), we obtain that (1) and (2) hold.

It is easy to see that $\alpha \le \bar\alpha$, $\bar\beta \le \beta$ on J. We show that $\bar\alpha \le \bar\beta$ on J. Let $p = \bar\alpha - \bar\beta$, then $p(t) = \alpha(t) - \beta(t) + c_\alpha(t) + c_{-\beta}(t)$. From (3.3)–(3.8), we have

$$-p''(t) + Mp(t) + Np(\theta(t)) \le 0, \quad t \in J,$$

and

$$\begin{aligned} p(0) - ap'(0) - cp(\eta) &= \alpha(0) - a\alpha'(0) - c\alpha(\eta) - (a\pi + c\sin I(\pi\eta))B_\alpha \\ &\quad - \beta(0) + a\beta'(0) + c\beta(\eta) - (a\pi + c\sin(\pi\eta))B_{-\beta} \\ &\le 0, \\ p(1) + bp'(1) - dp(\xi) &= \alpha(1) + b\alpha'(1) - d\alpha(\xi) - (b\pi + d\sin(\pi\xi))B_\alpha \\ &\quad - \beta(1) - b\beta'(1) + d\beta(\eta) - (b\pi + d\sin(\pi\xi))B_{-\beta} \\ &\le 0. \end{aligned}$$

By Theorem 2.1, we have that $p \le 0$ on J, that is, $\bar\alpha \le \bar\beta$ on J. So (3) holds.

Step 2. Consider the boundary value problem

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), & t \in J, \\ u(0) - au'(0) = \lambda, & u(1) + bu'(1) = \delta, \end{cases} \tag{3.9}$$

where $\lambda \in \mathbf{R}$, $\delta \in \mathbf{R}$. We show that the boundary value problem (3.9) has a unique solution $u(t, \lambda, \delta)$.

It is easy to check that the boundary value problem (3.9) is equivalent to the integral equation

$$u(t) = \frac{\delta(t+a) + (1-t+b)\lambda}{a+b+1} + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{a+b+1} \begin{cases} (a+t)(1+b-s), & 0 \le t \le s \le 1, \\ (a+s)(1+b-t), & 0 \le s \le t \le 1. \end{cases}$$

It is easy to see that $C(J)$ with norm $\|u\|_0 = \max_{t\in J} |u(t)|$ is a Banach space. Define a mapping $\Phi : C(J) \to C(J)$ by

$$(\Phi u)(t) = \frac{\delta(t+a) + (1-t+b)\lambda}{a+b+1} + \int_0^1 G(t,s)[\sigma(s) - Mu(s) - Nu(\theta(s))]\mathrm{d}s.$$

For any $x,\ y \in C(J)$, we have

$$(\Phi x)(t) - (\Phi y)(t) = \int_0^1 G(t,s)[M(y(s) - x(s)) + N(y(\theta(s)) - x(\theta(s)))]\mathrm{d}s.$$

Since

$$\max_{t\in J} \int_0^1 G(t,s)\mathrm{d}s = \frac{a(1+2b)}{2(a+b+1)} + \frac{1}{8}\left(\frac{1+2b}{a+b+1}\right)^2,$$

the inequality (3.2) implies that $\Phi : C(J) \to C(J)$ is a contraction mapping. Thus there exists a unique $u \in C(J)$ such that $\Phi u = u$. The boundary value problem (3.9) has a unique solution.

Step 3. We show that for any $t \in J$, the unique solution $u(t, \lambda, \delta)$ of the boundary value problem (3.9) is continuous in λ and δ. Let $u(t, \lambda_i, \delta_i),\ i = 1, 2$ be the solution of

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), & t \in J, \\ u(0) - au'(0) = \lambda_i, & u(1) + bu'(1) = \delta_i, \quad i = 1, 2. \end{cases} \tag{3.10}$$

Then

$$u(t, \lambda_i, \delta_i) = \frac{\delta_i(t+a) + (1-t+b)\lambda_i}{a+b+1} + \int_0^1 G(t,s)[\sigma(s) - Mu(s, \lambda_i, \delta_i) - Nu(\theta(s), \lambda_i, \delta_i)]\mathrm{d}s, \quad i = 1, 2. \tag{3.11}$$

From (3.11), we have that

$$\begin{aligned} \|u(t, \lambda_1, \delta_1) - u(t, \lambda_2, \delta_2)\|_0 &\le |\lambda_1 - \lambda_2| + |\delta_1 - \delta_2| + (M+N)\|u(t, \lambda_1, \delta_1) - u(t, \lambda_2, \delta_2)\|_0 \max_{t\in J} \int_0^1 G(t,s)\mathrm{d}s \\ &\le |\lambda_1 - \lambda_2| + |\delta_1 - \delta_2| + \mu\|u(t, \lambda_1, \delta_1) - u(t, \lambda_2, \delta_2)\|_0. \end{aligned}$$

Hence

$$\|u(t, \lambda_1, \delta_1) - u(t, \lambda_2, \delta_2)\|_0 \le \frac{1}{1-\mu}(|\lambda_1 - \lambda_2| + |\delta_1 - \delta_2|).$$

Step 4. We show that

$$\bar{\alpha}(t) \le u(t, \lambda, \delta) \le \bar{\beta}(t) \tag{3.12}$$

for any $t \in J$, $\lambda \in [c\bar{\alpha}(\eta),\ c\bar{\beta}(\eta)]$ and $\delta \in [d\bar{\alpha}(\xi),\ d\bar{\beta}(\xi)]$, where $u(t, \lambda, \delta)$ is the unique solution of the boundary value problem (3.9).

Let $m(t) = \bar{\alpha}(t) - u(t, \lambda, \delta)$. From (3.3), (3.5), (3.6) and (3.9), we have that $m(0) - am'(0) \le cm(\eta)$, $\ m(1) + bm'(1) \le dm(\xi)$ and

$$\begin{aligned} -m''(t) + Mm(t) + Nm(\theta(t)) &= -\bar{\alpha}''(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) + u''(t, \lambda) - Mu(t, \lambda, \delta) - Nu(\theta(t), \lambda, \delta) \\ &\le \sigma(t) - \sigma(t) \le 0. \end{aligned}$$

By Theorem 2.1, we obtain that $m \le 0$ on J, that is, $\bar{\alpha}(t) \le u(t, \lambda, \delta)$ on J. Similarly, $u(t, \lambda, \delta) \le \bar{\beta}(t)$ on J.

Step 5. Let $D = [c\bar{\alpha}(\eta),\ c\bar{\beta}(\eta)] \times [d\bar{\alpha}(\xi),\ d\bar{\beta}(\xi)]$. Define a mapping $F : D \to \mathbf{R}^2$ by

$$F(\lambda, \delta) = (u(\eta, \lambda, \delta), u(\xi, \lambda, \delta))$$

where $u(t, \lambda, \delta)$ is the unique solution of the boundary value problem (3.9). From Step 4, we have

$$F(D) \subset D.$$

Since D is a compact convex set and F is continuous, it follows by Schauder's fixed point theorem that F has a fixed point $(\lambda_0, \delta_0) \in D$ such that

$$u(\eta, \lambda_0, \delta_0,) = \lambda_0, \qquad u(\xi, \lambda_0, \delta_0) = \delta_0.$$

Obviously, $u(t, \lambda_0, \delta_0)$ is unique solution of the boundary value problem (3.1). This completes the proof. □

4. Main results

Let $M \in \mathbf{R}$, $N \in \mathbf{R}$. We first give the following definition.

Definition 4.1. A function $\alpha \in E$ is called a lower solution of the boundary value problem (1.2) if

$$-\alpha''(t) + (M + r)c_\alpha(t) + Nc_\alpha(\theta(t)) \le f(t, \alpha(t), \alpha(\theta(t))), \quad t \in J.$$

Definition 4.2. A function $\beta \in E$ is called an upper solution of the boundary value problem (1.2) if

$$-\beta''(t) - (M + r)c_{-\beta}(t) - Nc_{-\beta}(\theta(t)) \ge f(t, \beta(t), \beta(\theta(t))) \quad t \in J.$$

Our main result is the following theorem.

Theorem 4.1. *Assume that* (H) *holds. If the following conditions are satisfied*

(H_1) *$\alpha,\ \beta$ are lower and upper solutions for boundary value problem (1.2) respectively, and $\alpha(t) \le \beta(t)$ on J.*

(H_2) *The constants $M,\ N$ in definition of upper and lower solutions satisfy (2.2) and (3.2) and*

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) \ge -M(x - \bar{x}) - N(y - \bar{y}),$$

for $\alpha(t) \le \bar{x} \le x \le \beta(t),\ \alpha(\theta(t)) \le \bar{y} \le y \le \beta(\theta(t)),\ t \in J$.

Then, there exist monotone sequences $\{\alpha_n\},\ \{\beta_n\}$ with $\alpha_0 = \alpha,\ \beta_0 = \beta$ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t),\ \lim_{n\to\infty} \beta_n(t) = \varrho(t)$ uniformly on J, and $\rho,\ \varrho$ are the minimal and the maximal solutions of (1.2) respectively, such that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \alpha_n \le \rho \le x \le \varrho \le \beta_n \le \cdots \le \beta_2 \le \beta_1 \le \beta_0$$

on J, where x is any solution of (1.2) such that $\alpha(t) \le x(t) \le \beta(t)$ on J.

Proof. Let $[\alpha, \beta] = \{u \in E : \alpha(t) \le u(t) \le \beta(t),\ t \in J\}$. For any $\gamma \in [\alpha, \beta]$, we consider the boundary value problem

$$\begin{cases} -u''(t) + Mu(t) + Nu(\theta(t)) = f(t, \gamma(t), \gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)), & t \in J, \\ u(0) - au'(0) = cu(\eta), \qquad u(1) + bu'(1) = du(\xi). \end{cases} \tag{4.1}$$

Since α is a lower solution of (1.2), from (H_2), we have that

$$\begin{aligned} -\alpha''(t) + M\alpha(t) + N\alpha(\theta(t)) &\le f(t, \alpha(t), \alpha(\theta(t))) + M\alpha(t) + N\alpha(\theta(t)) - (M + r)c_\alpha(t) - Nc_\alpha(\theta(t)) \\ &\le f(t, \gamma(t), \gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)) - (M + r)c_\alpha(t) - Nc_\alpha(\theta(t)). \end{aligned}$$

Similarly, we have that

$$-\beta''(t) + M\beta(t) + N\beta(\theta(t)) \ge f(t, \gamma(t), \gamma(\theta(t))) + M\gamma(t) + N\gamma(\theta(t)) + (M + r)c_{-\beta}(t) + Nc_{-\beta}(\theta(t)).$$

By Theorem 3.1, the boundary value problem (4.1) has a unique solution $u \in [\alpha, \beta]$. We define an operator Ψ by $u = \Psi\gamma$, then Ψ is an operator from $[\alpha, \beta]$ to $[\alpha, \beta]$.

We shall show that

(a) $\alpha \le \Psi\alpha,\ \Psi\beta \le \beta$.

(b) Ψ is nondecreasing in $[\alpha, \beta]$.

From $\Psi\alpha \in [\alpha, \beta]$ and $\Psi\beta \in [\alpha, \beta]$, we have that (a) holds. To prove (b), we show that $\Psi\nu_1 \le \Psi\nu_2$ if $\alpha \le \nu_1 \le \nu_2 \le \beta$. Let $\nu_1^* = \Psi\nu_1,\ \nu_2^* = \Psi\nu_2$ and $p = \nu_1^* - \nu_2^*$, then by (H_2) and boundary conditions, we have that

$$\begin{aligned} -p''(t) + Mp(t) + Np(\theta(t)) \ &= \ f(t, \nu_1(t), \nu_1(\theta(t))) + M\nu_1(t) + N\nu_1(\theta(t)) \\ &\quad - f(t, \nu_2(t), \nu_2(\theta(t))) - M\nu_2(t) - N\nu_2(\theta(t)) \\ &\le \ 0, \end{aligned}$$

$$p(0) - ap'(0) = cp(\eta), \qquad p(1) + pu'(1) = dp(\xi).$$

By Theorem 2.1, $p(t) \le 0$ on J, which implies that $\Psi\nu_1 \le \Psi\nu_2$.

Define the sequences $\{\alpha_n\}, \{\beta_n\}$ with $\alpha_0 = \alpha, \beta_0 = \beta$ such that $\alpha_{n+1} = \Psi\alpha_n, \beta_{n+1} = \Psi\beta_n$ for $n = 0, 1, 2, \ldots$. From (a) and (b), we have

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_2 \le \beta_1 \le \beta_0$$

on $t \in J$, and each $\alpha_n, \beta_n \in E$ satisfies

$$\begin{cases} -\alpha_n''(t) + M\alpha_n(t) + N\alpha_n(\theta(t)) = f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t))) + M\alpha_{n-1}(t) + N\alpha_{n-1}(\theta(t)), & t \in J, \\ \alpha_n(0) - a\alpha_n'(0) = c\alpha_n(\eta), \qquad \alpha_n(1) + b\alpha_n'(1) = d\alpha_n(\xi). \end{cases}$$

$$\begin{cases} -\beta_n''(t) + M\beta_n(t) + N\beta_n(\theta(t)) = f(t, \beta_{n-1}(t), \beta_{n-1}(\theta(t))) + M\beta_{n-1}(t) + N\beta_{n-1}(\theta(t)), & t \in J, \\ \beta_n(0) - a\beta_n'(0) = c\beta_n(\eta), \qquad \beta_n(1) + b\beta_n'(1) = d\beta_n(\xi). \end{cases}$$

Therefore there exist ρ, ϱ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = \varrho(t)$ uniformly on J. Clearly, ρ, ϱ are solutions of (1.2).

Finally, we prove that if $x \in [\alpha_0, \beta_0]$ is any solution of (1.2), then $\rho(t) \le x(t) \le \varrho(t)$ on J. To this end, we assume, without loss of generality, that $\alpha_n(t) \le x(t) \le \beta_n(t)$ for some n. Since $\alpha_0(t) \le x(t) \le \beta_0(t)$, from property (b), we can obtain

$$\alpha_{n+1}(t) \le x(t) \le \beta_{n+1}(t), \quad t \in J.$$

Hence we can conclude that

$$\alpha_n(t) \le x(t) \le \beta_n(t), \quad \text{for all } n,$$

Passing to the limit as $n \to \infty$, we obtain

$$\rho(t) \le x(t) \le \varrho(t), \quad t \in J.$$

This completes the proof. □

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