New treatment of fractional Fornberg–Whitham equation via Laplace transform

Jagdev Singh a, Devendra Kumar b,*, Sunil Kumar c

a Department of Mathematics, Jagan Nath University, Village-Rampura, Tehsil-Chaksu, Jaipur 303 901, Rajasthan, India
b Department of Mathematics, Jagan Nath Gupta Institute of Engineering and Technology, Jaipur 302 022, Rajasthan, India
c Department of Mathematics, National Institute of Technology, Jamshedpur 831 014, Jharkhand, India

Received 26 August 2012; revised 22 October 2012; accepted 23 November 2012
Available online 16 January 2013

Abstract In this paper, a user friendly algorithm based on new homotopy perturbation transform method (HPTM) is proposed to solve nonlinear fractional Fornberg–Whitham equation in wave breaking. The new homotopy perturbation transform method is combined form of Laplace transform, homotopy perturbation method and He’s polynomials. The nonlinear terms can be easily handled by the use of He’s polynomials. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive.

1. Introduction

In recent years, fractional differential equations have gained importance and popularity, mainly due to its demonstrated applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. The fractional differential equations are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering [1–13].

In this paper, the HPTM basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear fractional differential equations by manipulating the homotopy perturbation method. The perturbation methods which are generally used to solve nonlinear problems have some limitations e.g., the approximate solution involves series of small parameters which poses difficulty since majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters some time leads to ideal solution but in most of the cases unsuitable choices lead to serious effects in the solutions. The homotopy perturbation method (HPM) was first introduced by He [14]. The homotopy perturbation method was also studied by many authors to handle linear and nonlinear equations arising in physics and engineering [15–18]. The homotopy perturbation transform method (HPTM) is a combination of Laplace transform method, homotopy perturbation method (HPM) and He’s polynomials. In recent years, many authors have paid
attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform. Among these are Laplace decomposition method [19–23] and homotopy perturbation transform method [24–26].

In this paper, we consider the following nonlinear fractional Fornberg–Whitham equation of the form:

\[ D^\alpha u - D_{xx}u + D_{xx}u = aD_{xx}u - uD_{xx}u + 3D_{xx}u, \quad 0 < \alpha \leq 1, \quad \alpha > 0, \]

with the initial condition

\[ u(x, 0) = \exp \left( \frac{x^2}{2} \right), \]

where \( \alpha \) is parameter describing the order of the fractional Fornberg–Whitham equation. The function \( u(x, t) \) is the fluid velocity, \( t \) is the time and \( x \) is the spatial coordinate. The fractional derivatives are understood in the Caputo sense. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of \( \alpha = 1 \) the fractional Fornberg–Whitham equation reduces to the classical nonlinear Fornberg–Whitham equation. Fornberg and Whitham obtained a peaked solution of the form \( u(x, t) = A \exp \left( -\frac{1}{2}(x - \frac{1}{4}t)^2 \right) \), where \( A \) is an arbitrary constant. The nonlinear fractional Fornberg–Whitham equations have been studied by many authors such as Gupta and Singh [27], Saker et al. [28], and Merdan et al. [29]. For improvement of fractional calculus, the differential transform method has been used to obtain approximate solutions of linear and nonlinear physical problems [30–33].

In the present article, further we apply the homotopy perturbation transform method (HPTM) to solve the nonlinear fractional Fornberg–Whitham equation. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. The fact that the HPTM solves nonlinear problems without using Adomian’s polynomials is a clear advantage of this technique over the decomposition method.

2. Basic definitions

In this section, we mention the following basic definitions of fractional calculus which are used further in the present paper.

**Definition 1.** The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \), of a function \( f(t) \in C_{\mu}, \mu \geq -1 \) is defined as [5]:

\[ J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau, \quad (x > 0), \]

\[ J^\alpha f(t) = f(t). \]

For the Riemann–Liouville fractional integral we have:

\[ J^\alpha f = \frac{1}{\Gamma(\gamma + 1)} \int_0^t (t + \alpha + \Gamma) \Gamma(-\alpha - 1) f(\tau) d\tau. \]

**Definition 2.** The fractional derivative of \( f(t) \) in the Caputo sense is defined as [10]:

\[ D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \]

for \( n - 1 < \alpha \leq n, \quad n \in \mathbb{N}, \quad t > 0. \)

**Definition 3.** The Laplace transform of the Caputo derivative is given by Caputo [10]; see also Kilbas et al. [13] in the form

\[ L[D^n f(t)] = s^n F[s] - \sum_{j=0}^{n-1} s^{n-j-1} f^{(j)}(0^+). \quad (n - 1 < \alpha \leq n). \]

**Definition 4.** The Mittag-Leffler introduced by Mittag-Leffler [34], is defined as

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in C, \quad \text{Re}(z) > 0). \]

3. Basic idea of HPTM

To illustrate the basic idea of this method, we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial conditions of the form:

\[ D^\alpha u(x, t) + R u(x, t) + N u(x, t) = g(x, t), \]

\[ u(x, 0) = h(x), \quad u_t(x, 0) = f(x), \]

where \( D^\alpha u(x, t) \) is the Caputo fractional derivative of the function \( u(x, t) \), \( R \) is the linear differential operator, \( N \) represents the general nonlinear differential operator and \( g(x, t) \) is the source term.

Applying the Laplace transform (denoted in this paper by \( L \)) on both sides of Eq. (9), we get

\[ L[D^\alpha u(x, t)] + L[R u(x, t)] + L[N u(x, t)] = L[g(x, t)]. \]

Using the property of the Laplace transform, we have

\[ L[u(x, t)] = \frac{h(x)}{s^\alpha} + \frac{f(x)}{s^{\alpha+1}} + \frac{1}{s^{\alpha+2}} L[g(x, t)] - \frac{1}{s^{\alpha+2}} L[R u(x, t)] - \frac{1}{s^{\alpha+2}} L[N u(x, t)]. \]

Operating with the Laplace inverse on both sides of Eq. (12) gives

\[ u(x, t) = G(x, t) - L^{-1} \left[ \frac{1}{s^{\alpha+2}} L[R u(x, t)] + L[N u(x, t)] \right], \]

where \( G(x, t) \) represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

\[ u(x, t) = \sum_{n=0}^{\infty} \rho^n u_n(x, t), \]

and the nonlinear term can be decomposed as

\[ N u(x, t) = \sum_{n=0}^{\infty} \rho^n H_n(u), \]

for some He’s polynomials \( H_n(u) \) [35,36] that are given by

\[ H_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial \rho^n} \left[ N \left( \sum_{n=0}^{\infty} \rho^n u \right) \right] \quad n = 0, 1, 2, 3 \ldots \]

Using (14) and (15) in (13), we get

\[ \sum_{n=0}^{\infty} \rho^n u_n(x, t) = G(x, t) - \rho \left[ L^{-1} \left( \frac{1}{s^{\alpha+2}} \right) \left( \sum_{n=0}^{\infty} \rho^n u_n(x, t) + \sum_{n=0}^{\infty} \rho^n H_n(u) \right) \right]. \]
which is the coupling of the Laplace transform and the homotopy perturbation method using He’s polynomials. Comparing the coefficients of like powers of \( p \), the following approximations are obtained.

\[
p^0 : u_0(x, t) = G(x, t),
\]

\[
p^1 : u_1(x, t) = -L^{-1} \left[ \frac{1}{s^p} L [R u_0(x, t) + H_0(u)] \right],
\]

\[
p^2 : u_2(x, t) = -L^{-1} \left[ \frac{1}{s^p} L [R u_1(x, t) + H_1(u)] \right],
\]

\[
p^3 : u_3(x, t) = -L^{-1} \left[ \frac{1}{s^p} L [R u_2(x, t) + H_2(u)] \right],
\]

For \( H^p(u) \), we find that

\[
\sum_{n=0}^{\infty} p^n H^n(u) = u D_{x,x} u.
\]

The first few, components of He’s polynomials, are given by

\[
H_0(u) = u_0 D_{x,x} u_0, \\
H_1(u) = u_0 D_{x,x} u_1 + u_1 D_{x,x} u_0, \\
H_2(u) = u_0 D_{x,x} u_2 + u_1 D_{x,x} u_1 + u_2 D_{x,x} u_0,
\]

\[
:\ :
\]

Proceeding in this same manner, the rest of the components \( u_{n}(x, t) \) can be completely obtained and the series solution is thus entirely determined.

Finally, we approximate the analytical solution \( u(x, t) \) by truncated series

\[
\lim_{N \rightarrow \infty} \sum_{n=0}^{N} u_n(x, t).
\]

The above series solutions generally converge very rapidly.

\section{4. Solution of the problem}

Consider the following nonlinear time-fractional Fornberg–Whitham equation

\[
D^p_t u - D_{x,x} u + D_t u = u D_{x,x} u - u D_t u + 3 D_{x} u D_{x} u, \quad t > 0, \quad 0 < x \leq 1,
\]

with the initial condition

\[
u(x, 0) = \exp \left( \frac{x}{2} \right).
\]

Applying the Laplace transform on both sides of Eq. (20), subject to initial condition (21), we have

\[
L[u(x, t)] = \frac{1}{s} \exp \left( \frac{x}{2} \right) + \frac{1}{s} L [D_{x,x} u - D_t u + u D_{x,x} u - u D_t u + 3 D_x u D_x u].
\]

The inverse Laplace transform implies that

\[
u(x, t) = \exp \left( \frac{x}{2} \right) + L^{-1} \left[ \frac{1}{s} L [D_{x,x} u - D_t u + u D_{x,x} u - u D_t u + 3 D_x u D_x u] \right].
\]

Now, applying the homotopy perturbation method, we get

\[
\sum_{n=0}^{\infty} p^n u_n(x, t) = \exp \left( \frac{x}{2} \right) + p \left[ L^{-1} \left[ \frac{1}{s^p} L \left( \sum_{n=0}^{\infty} p^n u_0(x, t) \right) \right] - D_x \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) + \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) \right].
\]

where \( H_n(u), H^p_n(u) \) and \( H^m_n(u) \) are He’s polynomials [36,37] that represents the nonlinear terms. So, the He’s polynomials are given by

\[
\sum_{n=0}^{\infty} p^n H_n(u) = u D_{x,x} u.
\]

The first few, components of He’s polynomials, are given by

\[
H_0(u) = u_0 D_{x,x} u_0, \\
H_1(u) = u_0 D_{x,x} u_1 + u_1 D_{x,x} u_0, \\
H_2(u) = u_0 D_{x,x} u_2 + u_1 D_{x,x} u_1 + u_2 D_{x,x} u_0,
\]

\[
:\ :
\]

and for \( H^p_n(u) \), we find that

\[
\sum_{n=0}^{\infty} p^n H^p_n(u) = u D_{x} u,
\]

\[
H^p_0(u) = u_0 D_{x} u_0, \\
H^p_1(u) = u_0 D_x u_1 + u_1 D_x u_0, \\
H^p_2(u) = u_0 D_x u_2 + u_1 D_x u_1 + u_2 D_x u_0,
\]

\[
:\ :
\]

Comparing the coefficients of like powers of \( p \), we have

\[
p^0 : u_0(x, t) = \exp \left( \frac{x}{2} \right) + \frac{1}{s} L \left[ \frac{1}{s^p} L \left( \sum_{n=0}^{\infty} p^n u_0(x, t) \right) \right] - D_x \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) + \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) - \left( \sum_{n=0}^{\infty} p^n H^p_n(u) \right).
\]

\[
\sum_{n=0}^{\infty} p^n u_n(x, t) = \exp \left( \frac{x}{2} \right) + \frac{1}{s} L \left[ \frac{1}{s^p} L \left( \sum_{n=0}^{\infty} p^n u_0(x, t) \right) \right] - D_x \left( \sum_{n=0}^{\infty} p^n u_n(x, t) \right) + \left( \sum_{n=0}^{\infty} p^n H_n(u) \right) - \left( \sum_{n=0}^{\infty} p^n H^p_n(u) \right).
\]
5. Numerical results and discussion

In this section, we calculate numerical results of the displacement \( u(x,t) \) for different time-fractional Brownian motions \( \alpha = 2/3, 3/4, 1 \) and for various values of \( t \) and \( x \). The numerical results for the approximate solution (30) obtained by using HPTM and the exact solution for various values of \( t, x \) and \( \alpha \) are shown by Fig. 1a–d and those for different values of \( t \) and \( \alpha \) at \( x = 1 \) are depicted in Fig. 2.

It is observed from Fig. 1a–c that \( u(x,t) \) increases with the increase in both \( x \) and \( t \) for \( \alpha = 2/3, 3/4 \) and \( \alpha = 1 \). Fig. 1d clearly show that, when \( \alpha = 1 \), the approximate solution (30) obtained by the present method is very near to the exact solution. It is also seen from Fig. 2 that as the value of \( \alpha \) increases, the displacement \( u(x,t) \) increases but afterward its nature is opposite. Finally, we remark that the approximate solution (30) is in full agreement with the results obtained HPM [27].

6. Conclusions

In this paper, the homotopy perturbation transform method (HPTM) is successfully applied for solving nonlinear fractional Fornberg–Whitham equation. The results obtained by using the HPTM presented here agree well with the results obtained by HPM [27]. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction
Acknowledgments

The authors are extending their heartfelt thanks to the reviewers for their valuable suggestions for the improvement of the article.

References

Jagdev Singh is an Assistant Professor in the Department of Mathematics, Jagan Nath University, Jaipur-303901, Rajasthan, India. His fields of research include special functions, fractional calculus, fluid dynamics, homotopy methods, Adomian decomposition method, Laplace transform method and sumudu transform method.

Devendra Kumar is an Assistant Professor in the Department of Mathematics, Jagan Nath Gupta Institute of Engineering and Technology, Jaipur-302022, Rajasthan, India. His area of interest is special functions, fractional calculus, differential equations, integral equations, homotopy methods, Laplace transform method and sumudu transform method.

Sunil Kumar is an Assistant Professor in the Department of Mathematics, National Institute of Technology, Jamshedpur, 801014, Jharkhand, India. His fields of research include singular integral equation, fractional calculus, homotopy methods, Adomian decomposition method and Laplace transform method.