Dedekind symbols associated with J-forms and their reciprocity law

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Abstract

We define a Dedekind symbol associated with a J-form (J-forms generalize the usual Jacobi forms). Then we prove the reciprocity law for Dedekind symbols. As an example, we give an explicit description for the Dedekind symbol associated with Weierstrass $\wp$-function. The reciprocity law in this case then yields new trigonometric identities. This in turn gives rise to a “generating function” of Apostol reciprocity law for the generalized Dedekind sums.

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1. Introduction and statement of results

A Dedekind symbol is a complex valued function $D$ on $V := \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z} | \gcd(p, q) = 1\}$ satisfying $D(p, q) = D(p, q + p)$. The symbol $D$ is determined by its reciprocity law:

$$D(p, q) - D(q, -p) = R(p, q)$$

uniquely up to an additive constant. The function $R$ is defined on $U := \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | \gcd(p, q) = 1\}$, and is called a reciprocity function associated with the Dedekind
symbol $D$. The function $R$ necessarily satisfies the equation:

$$R(p + q, q) + R(p, p + q) = R(p, q).$$  \hfill (1.2)

In [6], it was shown that if a reciprocity function is a degree $k - 2$ homogeneous polynomial in $p$ and $q$ satisfying (1.2), there is a cusp form $f$ of weight $k$ such that the Dedekind symbol determined by (1.1) is represented by $D_f$ as follows:

$$D_f(p, q) = \int_{i\infty}^{i\infty} \frac{f(\tau)}{q/p} \left( \frac{p\tau}{C_0} - q \right)^{k-2} d\tau.$$  \hfill (1.3)

The converse is also true. In fact, it was shown in [6] that cusp forms of weight $k$ bijectively correspond to Dedekind symbols with degree $k - 2$ homogeneous polynomial reciprocity law.

In this paper, we will generalize the above correspondence, that is, we will define Dedekind symbols associated to Jacobi cusp forms or more generally to J-forms (see Definition 2.1). Let $\phi$ be a Jacobi cusp form of index $m$ and weight $k$. Then the Dedekind symbol $D_{\phi}(p, q; x)$ is defined as follows:

$$D_{\phi}(p, q; x) = \int_{i\infty}^{i\infty} e^{2\pi i m(p\tau - q)x^2} \phi(\tau, (p\tau - q)x) \left( p\tau - q \right)^{k-2} d\tau.$$  \hfill (1.3)

When we try to define Dedekind symbols for a $J$-form (see Definition 2.2 for detail), we might have to modify (1.3) suitably.

In the case of a cusp form $f$, the period polynomial $R_f(p, q)$ defined by

$$R_f(p, q) = \int_{i\infty}^{i\infty} f(\tau) \left( p\tau - q \right)^{k-2} d\tau$$

is the reciprocity function for $D_f(p, q)$ (see [6]). Namely the following holds

$$D_f(p, q) - D_f(q, -p) = R_f(p, q).$$

In the case of a Jacobi cusp form, a “period function” $R_{\phi}(p, q; x)$ can be defined as

$$R_{\phi}(p, q; x) = \int_{i\infty}^{i\infty} e^{2\pi i m(p\tau - q)x^2} \phi(\tau, (p\tau - q)x) \left( p\tau - q \right)^{k-2} d\tau.$$  \hfill (1.4)

Again we have to modify (1.4) to define a “period function” for a $J$-forms (see Definition 3.1 for detail).

Our first result is a reciprocity law of Dedekind symbols for a $J$-form.

**Theorem 1.1.** For a pair of positive integers $(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $\gcd(p, q) = 1$, $x \in \mathbb{R}$ (when $\phi$ is not holomorphic, we assume that $|x|$ is sufficiently small and $0 < x^2 < 4\pi^2/pq$), and a $J$-form $\phi$, a reciprocity law is valid for a Dedekind
symbol:

\[ D_\phi(p, q; x) - D_\phi(q, -p; x) = R_\phi(p, q; x). \]

For example, the Weierstrass \( \wp \)-function is a meromorphic Jacobi form \[8\], and is a J-form in our sense. Hence, we can apply Theorem 1.1 to \( \wp \)-function. We will calculate \( D_\wp(p, q; x) \) and \( R_\wp(p, q; x) \) to obtain the following identity:

**Proposition 1.2.** For \( p, q \in \mathbb{Z}^+ \) such that \( \gcd(p, q) = 1 \), and \( x \in \mathbb{R} \), the following identity holds:

\[
- \frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \left\{ \cot \left( \frac{x}{2i} + \frac{\mu \pi}{p} \right) - \cot \left( \frac{\mu \pi}{p} \right) \right\}
- \frac{1}{4q} \sum_{\mu=1}^{q-1} \cot \left( \frac{\mu p \pi}{q} \right) \left\{ \cot \left( \frac{x}{2i} + \frac{\mu \pi}{q} \right) - \cot \left( \frac{\mu \pi}{q} \right) \right\}
= \frac{1}{4} \cot \frac{px}{2i} \cot \frac{qx}{2i} + \frac{p^2 + q^2 + 1}{12pq} - \frac{1}{4pq} \csc^2 \frac{x}{2i}.
\]

(1.5)

Now let \( x \to \infty \) in (1.5). Then formula (1.5) tends to the following classical Dedekind reciprocity law (refer to [9, pp. 93, 100], [11] or [13]):

\[
\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \cot \left( \frac{\mu p \pi}{p} \right) + \frac{1}{4q} \sum_{\mu=1}^{q-1} \cot \left( \frac{\mu p \pi}{q} \right) \cot \left( \frac{\mu q \pi}{q} \right)
= \frac{p^2 + q^2 + 1 - 3pq}{12pq}.
\]

(1.6)

Subtracting (1.6) from (1.5) in both sides, we can easily obtain the following simpler formula.

**Proposition 1.3.** For \( p, q \in \mathbb{Z}^+ \) such that \( \gcd(p, q) = 1 \), and \( z \in \mathbb{C} \)

\[
- \frac{1}{p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \cot \left( \frac{z + \mu \pi}{p} \right) - \frac{1}{q} \sum_{\mu=1}^{q-1} \cot \left( \frac{\mu p \pi}{q} \right) \cot \left( \frac{z + \mu \pi}{q} \right)
= - \cot(pz) \cot(qz) + \frac{1}{pq} \csc^2 z - 1.
\]

This proposition can be regarded as reciprocity law of “parameterized cotangent sums” (refer to [3,12, p. 90] for similar sums). The coefficients of Laurent expansions of both sides give us Apostol’s reciprocity law [1, Theorem 1.2], for his generalized
Dedekind sums in each degree. The above theorem encodes Apostol’s reciprocity law in all degrees. Hence it may be regarded as a “generating function” of his reciprocity law.

**Remark 1.1.** After the completion of this paper, it has brought to our attention that Proposition 1.3 may also be derived from the work of Dieter in [4]. However, our proof is completely different from that of Dieter. Indeed, our proof is purported to describe the reciprocity law in terms of the Dedekind symbols of the Jacobi form.

### 2. Dedekind symbols of Jacobi forms

**Definition 2.1.** Let $\mathbb{H}$ denote the upper half complex plane. Let $\phi(t, z)$ be a meromorphic function on $\mathbb{H} \times \mathbb{C}$ which is holomorphic in $t \in \mathbb{H}$. We call $\phi$ a J-form of weight $k \in \mathbb{N}$ and index $m \in \mathbb{N}$ if it satisfies the following three conditions:

1. $\phi\left(\frac{at + b}{ct + d}, \frac{z}{ct + d}\right) = (ct + d)^k e^{2\pi imcz^2/(ct + d)} \phi(t, z)$ for $(a, b, c, d) \in SL(2, \mathbb{Z})$,
2. $\phi$ has a Fourier expansion $\phi(t, z) = \sum_{n=0}^{\infty} \phi_n(z)e^{\pi int}$ at $t = \infty$. Furthermore the first coefficient $\phi_0(z)$ has a Laurent expansion $\phi_0(z) = \sum_{i=0}^{\infty} \phi_{0,i}z^i$ around $z = 0$ for some $i_0 \in \mathbb{Z}$,
3. if $\phi$ is not holomorphic, $\phi(t, z)$ and $\phi_0(z)$ (for a fixed $t \in \mathbb{H}$) have poles only at $\{2\pi int + 2\pi in \mid (m, n) \in \mathbb{Z}^2\}$ and $\{2\pi in \mid n \in \mathbb{Z}\}$, respectively.

Associated with a J-form $\phi$, we introduce two functions $\phi_0^+$ and $\phi_0^-$. Set

$$\phi_0^+(z) = \sum_{i=2-k}^{\infty} \phi_{0,i}z^i \quad \text{and} \quad \phi_0^-(z) = \sum_{i=i_0}^{1-k} \phi_{0,i}z^i.$$ 

Clearly $\phi_0(z) = \phi_0^+(z) + \phi_0^-(z)$.

A Jacobi cusp form [5,10], say $\phi$, is an example of a J-form with $\phi_0(z) = 0$.

In what follows, we use the notations:

$$L_0 := \{it \mid 0 < t < \infty\},$$

$$L_{q/p} := \left\{ \frac{q}{p} + it \mid 0 < t < \infty \right\},$$

$$C_{q/p} := \left\{ \frac{q}{2p} + \frac{q}{2p} e^{\pi it} \mid 0 < t < 1 \right\},$$

$$D_{q/p} := \begin{cases} \{\tau \in \mathbb{H} \mid 0 < \Re(\tau) < q/p \text{ and } q/2p < |\tau - q/2p|\} & \text{if } q > 0, \\ \{\tau \in \mathbb{H} \mid 0 > \Re(\tau) > q/p \text{ and } -q/2p < |\tau - q/2p|\} & \text{if } q < 0. \end{cases}$$

Now we present the first lemma.
Lemma 2.1. Let \( f(t, z) \) be a non-holomorphic J-form. Let \( p \) and \( q \) be integers with \( p > 0 \) and \( q \neq 0 \), and \( x \) be a non-zero real number. We consider the function \( f(t, (pt - q)x) \) as a function of \( t \). Suppose that \( t \in \mathbb{H} \) is a pole of \( f(t, (pt - q)x) \). Then the following assertions hold:

1. \( t \) can be expressed as
   \[
   t = \frac{pqx^2 - 4\pi^2 mn + 2\pi x(mq + np)}{p^2 x^2 + 4\pi^2 m^2}
   \]
   for some integers \( m \) and \( n \),
2. \( t \in L_{q/p} \) if and only if \( m = 0 \),
3. \( t \in C_{q/p} \) if and only if \( n = 0 \),
4. if \( x^2 < 4\pi^2/|pq| \) then \( t \notin L_0 \),
5. if \( x^2 < 4\pi^2/|pq| \) then \( t \notin D_{q/p} \).

Proof. If \( t \) is a pole of \( f(t, (pt - q)x) \), it should satisfy \( (pt - q)x = 2\pi imt + 2\pi in \) for some integers \( m \) and \( n \). This proves assertion (1). Assertions (2) and (3) can be obtained from (1) by easy computation.

We will show (4). Suppose the contrary, so there is at least one \( t \in L_0 \). Then \( R(t) = 0 \). Hence \( |mn| = pq|x|^2/4\pi < 1 \) by (1), and \( mn \neq 0 \) by (2) and (3). This is a contradiction. Thus we have proved (4).

Next we will show (5) in the case that \( q > 0 \) (the other case is similar, and we omit the proof). Suppose that \( 0 < R(t) < q/p \), namely

\[
0 < R(t) = \frac{pqx^2 - 4\pi^2 mn}{p^2 x^2 + 4\pi^2 m^2} < \frac{q}{p}.
\]

Since \( 0 < R(t) \), we have \( mn < pqx^2/4\pi^2 < 1 \). By (2) and (3), we may assume \( mn \neq 0 \) (since if \( mn = 0 \) then \( t \notin D_{q/p} \)). This imposes that \( mn < 0 \). On the other hand, since \( R(t) < q/p \), we have \( 0 < q/p + n/m \). Combining these, we obtain

\[
-pn(mq + np) < 0.
\]

Now applying (2.1), we will compute the difference \( q^2/4p^2 - |t - q/2p|^2 \). We have

\[
\frac{q^2}{4p^2} - \left| \frac{t - q}{2p} \right|^2 = \frac{q^2 - 4p^2x^2 + 4\pi^2(2pm + qm)^2}{4p^2(x^2 + 4\pi^2 m^2)} - \frac{-\pi^2 pn(mq + np)}{4p^2(x^2 + 4\pi^2 m^2)} > 0
\]

from which we may deduce that \( t \notin D_{q/p} \), proving (5).

This completes the proof. \( \square \)
Next we will define Dedekind symbols associated with a J-form. Hereafter, we always assume that an integral is taken along a geodesic (a vertical line or a semicircle).

**Definition 2.2.** Let \( \phi \) be a J-form of weight \( k \) and index \( m \). For \( p, q \in \mathbb{Z} \) such that \( p > 0 \) and \( x \in \mathbb{R} \) (when \( \phi \) is not holomorphic, we assume that \( |x| \) is sufficiently small and \( 0 < x^2 < 4\pi^2/|pq| \)), we define:

\[
D_\phi(p, q; x) := \int_{\tau_0}^{i\infty} e^{2\pi i mp(\tau - q)x^2} \left\{ \phi(\tau, (p\tau - q)x) - \phi_0((p\tau - q)x)(p\tau - q)^{k-2} \right\} d\tau
\]

\[
+ \int_{q/p}^{\tau_0} e^{2\pi i mp(\tau - q)x^2} \frac{\phi(\tau, (p\tau - q)x)(p\tau - q)^{k-2}}{(p\tau - q)^2} d\tau
\]

\[
- \int_{q/p}^{\tau_0} e^{2\pi i mp(\tau - q)x^2} \phi_0((p\tau - q)x)(p\tau - q)^{k-2} d\tau
\]

\[
- \int_{i\infty}^{\tau_0} e^{2\pi i mp(\tau - q)x^2} \phi_0((p\tau - q)x)(p\tau - q)^{k-2} d\tau
\]

\[
+ \int_{i\infty}^{\tau_0} \frac{\phi_0(x)}{(p\tau - q)^2} d\tau
\]

where \( \tau_0 \) is an arbitrary point in \( L_0 \).

One can easily check the existence of the integrals involved (when \( \phi \) is not holomorphic, by (4) and (5) of Lemma 2.1, we know there is no pole of the integrands on the paths of integration). The definition does not depend on the choice of \( \tau_0 \in L_0 \).

The following lemma can easily be proved once one notices the identity: \( \phi(\tau, z) = \phi(\tau + 1, z) \). We will omit the proof.

**Lemma 2.2.**

\[
D_\phi(p, q + p; x) = D_\phi(p, q; x).
\]

That is, \( D_\phi(p, q; x) \) is a Dedekind symbol.

3. A period function of a J-form

We would like to define a “period function” for a J-form, which is similar to a period polynomial for a cusp form.

**Definition 3.1.** Let \( \phi \) be a J-form of weight \( k \) and index \( m \). For \( p, q \in \mathbb{Z}^+ \) and \( x \in \mathbb{R} \) (when \( \phi \) is not holomorphic, we assume that \( |x| \) is sufficiently small and
0 < x^2 < 4\pi^2 / |pq|), we define:

\[ R_{\phi}(p, q; x) \]

\[ := \int_{t_0}^{i\infty} e^{2\pi i \text{mp}(p \tau - q) x^2} \left\{ \phi(\tau, (p \tau - q)x) - \phi_0((p \tau - q)x)(p \tau - q)^{k-2} \right\} d\tau \]

\[ + \int_{0}^{\tau_0} \left\{ e^{2\pi i \text{mp}(p \tau - q) x^2} \phi(\tau, (p \tau - q)x) - e^{2\pi i \text{mp}(p \tau - q) x^2} \frac{\phi_0((p \tau - q)x)}{\tau^k} \right\} \times (p \tau - q)^{k-2} d\tau \]

\[ - \int_{q/p}^{\tau_0} e^{2\pi i \text{mp}(p \tau - q) x^2} \frac{\phi_0^+(p \tau - q)x)}{\tau^k} (p \tau - q)^{k-2} d\tau \]

\[ + \int_{q/p}^{\tau_0} e^{2\pi i \text{mp}(p \tau - q) x^2} \frac{\phi_0^-(p \tau - q)x)}{\tau^k} (p \tau - q)^{k-2} d\tau \]

\[ \times (p \tau - q)^{k-2} d\tau \]

\[ - \int_{1}^{\tau_0} e^{2\pi i \text{mp}(p \tau - q) x^2} \phi_0((p \tau - q)x)(p \tau - q)^{k-2} d\tau \]

\[ + \int_{0}^{\tau_0} e^{2\pi i \text{mp}(p \tau - q) x^2} \frac{\phi_0((p \tau - q)x)}{\tau^k} (p \tau - q)^{k-2} d\tau \]

where \( \tau_0 \in L_0 \) is chosen arbitrary.

By (4) of Lemma 2.1 and the assumption that \( 0 < x^2 < 4\pi^2 / |pq| \), there is no pole of the integrand on the paths of integration. Hence the integrals exist. The definition does not depend on the choice of \( \tau_0 \). It can be reduced to the following simple form when \( \phi_0(z) = 0 \) (especially when \( \phi \) is a Jacobi cusp form):

\[ R_{\phi}(p, q; x) = \int_{0}^{i\infty} e^{2\pi i \text{mp}(p \tau - q) x^2} f(\tau, (p \tau - q)x)(p \tau - q)^{k-2} d\tau. \]

Note that \( R_{\phi}(p, q; x) \) is generally not a polynomial in \( p \) and \( q \), contrary to the case when \( \phi \) is a cusp form.

4. A reciprocity law for Dedekind symbol \( D_{\phi}(p, q; x) \)

We will prove Theorem 1.1 which gives rise to a reciprocity law for \( D_{\phi}(p, q; x) \).

Proof of Theorem 1.1. We only prove the case that \( \phi \) is not holomorphic (the holomorphic case is similar but easier). First we calculate \( D_{\phi}(q, -p; x) \) taking \(-1/\tau_0\).
and \( \sigma \) in place of \( \tau_0 \) and \( \tau \), respectively, in Definition 2.2.

\[
D_{\phi}(q, -p; x) = \int_{i\infty}^{i\infty} \frac{e^{2\pi imq(q\sigma+p)x^2}}{(q\sigma+p)^{k-2}} d\sigma

+ \int_{-\tau_0}^{\tau_0} \left\{ e^{2\pi imq(q\sigma+p)x^2} \phi(\sigma, (q\sigma+p)x) (q\sigma+p)^{k-2} - \frac{\phi_0(x)}{(q\sigma+p)^2} \right\} d\sigma

- \int_{-\tau_0}^{\tau_0} e^{2\pi imq(q\sigma+p)x^2} \phi_0^*(x) (q\sigma+p)^{k-2} d\sigma

- \int_{i\infty}^{-i\infty} e^{2\pi imq(q\sigma+p)x^2} \phi_0((q\sigma+p)x) (q\sigma+p)^{k-2} d\sigma + \int_{i\infty}^{-i\infty} \frac{\phi_0(x)}{(q\sigma+p)^2} d\sigma

= \int_{\tau_0}^{\tau_0} e^{2\pi imq \frac{\tau-q}{p} x^2} \left\{ \phi \left( \frac{-1}{\tau}, \frac{p\tau-q}{\tau} x \right) - \phi_0 \left( \frac{p\tau-q}{\tau} x \right) \right\}

\times \left( \frac{p\tau-q}{\tau} \right)^{k-2} \frac{d\tau}{\tau^2}

+ \int_{q/p}^{\tau_0} e^{2\pi imq \frac{p\tau-q}{\tau} x^2} \phi \left( \frac{-1}{\tau}, \frac{p\tau-q}{\tau} x \right) \left( \frac{p\tau-q}{\tau} \right)^{k-2} \frac{\tau^2 \phi_0(x)}{(p\tau-q)^2} \frac{d\tau}{\tau^2}

- \int_{q/p}^{\tau_0} e^{2\pi imq \frac{p\tau-q}{\tau} x^2} \phi_0 \left( \frac{p\tau-q}{\tau} x \right) \left( \frac{p\tau-q}{\tau} \right)^{k-2} \frac{d\tau}{\tau^2}

- \int_{0}^{\tau_0} \frac{\phi_0(x)}{(p\tau-q)^2} d\tau

= - \int_{0}^{\tau_0} \left\{ e^{2\pi imq(p\tau-q)x^2} \phi(\tau, (p\tau-q)x) - e^{2\pi imq \frac{p\tau-q}{\tau} x^2} \frac{\phi_0(p\tau-q)x}{\tau^k} \right\}

\times (p\tau-q)^{k-2} d\tau

+ \int_{q/p}^{\tau_0} \left\{ e^{2\pi imq(p\tau-q)x^2} \phi(\tau, (p\tau-q)x) (p\tau-q)^{k-2} - \frac{\phi_0(x)}{(p\tau-q)^2} \right\} d\tau

- \int_{q/p}^{\tau_0} e^{2\pi imq \frac{p\tau-q}{\tau} x^2} \phi_0 \left( \frac{p\tau-q}{\tau} x \right) \frac{\tau^k}{(p\tau-q)^{k-2}} d\tau

- \int_{0}^{\tau_0} e^{2\pi imq \frac{p\tau-q}{\tau} x^2} \phi_0 \left( \frac{p\tau-q}{\tau} x \right) (p\tau-q)^{k-2} d\tau + \int_{0}^{\tau_0} \frac{\phi_0(x)}{(p\tau-q)^2} d\tau.

Now subtracting the expression of \( D_{\phi}(q, -p; x) \) from the expression of \( D_{\phi}(p, q; x) \), we obtain the theorem. \( \Box \)
5. Dedekind symbols associated with the Weierstrass \( \wp \)-function

In this section, we will present explicit forms of the Dedekind symbol and period function associated with the Weierstrass \( \wp \)-function. For this, we need a few lemmas. Let \( \phi(\tau, z) \) be a Jacobi form of weight \( k \) and index 0, which has an expansion of the form at \( z = 0 \):

\[
\phi(\tau, z) = \sum_{n=n_0}^\infty f_n(\tau)z^{n-k}.
\]

Then for any \( n \geq n_0, f_n(\tau) \) is a modular form of weight \( n \) (note that \( f_n(\tau) = 0 \) for odd \( n \)). In this case the Dedekind symbol \( D_{\phi}(p, q; x) \) is naturally related to a family of the Dedekind symbols \( \{D_{f_n}(p, q)\}_{n \geq n_0} \) as follows:

**Lemma 5.1.** Let \( \phi(\tau, z) \) be a Jacobi form of weight \( k \) and index 0, which has an expansion at \( z = 0 \) of the form:

\[
\phi(\tau, z) = \sum_{n=n_0}^\infty f_n(\tau)z^{n-k}.
\]

Then the Dedekind symbol \( D_{\phi}(p, q; x) \) is expressed in terms of the Dedekind symbols \( D_{f_n}(p, q) \) as follows:

\[
D_{\phi}(p, q; x) = \sum_{n=n_0}^\infty D_{f_n}(p, q)x^{n-k}.
\]

Furthermore, the period function \( R_{\phi}(p, q; x) \) is expressed as

\[
R_{\phi}(p, q; x) = \sum_{n=n_0}^\infty R_{f_n}(p, q)x^{n-k}.
\]

**Proof.** We will give a proof in the case of the Dedekind symbol \( D_{\phi}(p, q; x) \) for a Jacobi form \( \phi \); the cases of the period function \( R_{\phi}(p, q; x) \) can be proved similarly.

For any \( n, f_n(\tau) \) is a modular form of weight \( n \). Let

\[
f_n(\tau) = \sum_{k=0}^\infty a_n(k)e^{2\pi ik\tau}
\]

be its Fourier expansion. Then we know

\[
\phi_0(z) = \sum_{n=n_0}^\infty a_n(0)z^{n-k}, \quad \phi_0^+(z) = \sum_{n=2}^\infty a_n(0)z^{n-k}
\]

and

\[
\phi_0^-(z) = \sum_{n=n_0}^0 a_n(0)z^{n-k}.
\]
We have

\[
D_{\phi}(p, q; x) = \int_{q/p}^{i \infty} \sum_{n=n_0}^{\infty} \left\{ f_n(\tau) - a_n(0) \right\} (p\tau - q)^{n-2} x^{n-k} d\tau \\
+ \int_{q/p}^{\tau_0} \sum_{n=n_0}^{\infty} a_n(0) (p\tau - q)^{n-2} x^{n-k} d\tau \\
- \int_{q/p}^{\tau_0} \sum_{n=2}^{\infty} a_n(0) (p\tau - q)^{n-2} x^{n-k} d\tau \\
- \int_{i \infty}^{\tau_0} \sum_{n=n_0}^{\infty} a_n(0) (p\tau - q)^{n-2} x^{n-k} d\tau \\
+ \int_{i \infty}^{\tau_0} \sum_{n=n_0}^{\infty} a_n(0) \frac{(p\tau - q)^n}{(p\tau - q)^2} x^{n-k} d\tau \\
= \sum_{n=n_0}^{\infty} \int_{q/p}^{i \infty} \left\{ f_n(\tau) - a_n(0) \right\} (p\tau - q)^{n-2} x^{n-k} d\tau \\
+ \sum_{n=n_0}^{\infty} \left[ \int_{q/p}^{\tau_0} \left\{ f_n(\tau)(p\tau - q)^{n-2} - \frac{a_n(0)}{(p\tau - q)^2} \right\} d\tau \right] x^{n-k} \\
- \sum_{n=2}^{\infty} \left[ a_n(0) \frac{(p\tau_0 - q)\tau^{n-1}}{p(n-1)} \right] x^{n-k} \\
- \sum_{n=n_0}^{\infty} \left[ a_n(0) \frac{(p\tau_0 - q)^{n-1}}{p(n-1)} \right] x^{n-k} \\
- \sum_{n=n_0}^{\infty} \left[ a_n(0) \frac{(p\tau_0 - q)^{-1}}{p} \right] x^{n-k} \\
= \sum_{n=n_0}^{\infty} D_{f_n}(p, q)x^{n-k}.
\]

To justify the interchanges of \(\int\) and \(\sum\) in the equation above, one notes that the series \(\sum_{n=n_0}^{\infty} f_n(\tau)z^{n-k}\) converges uniformly absolutely on the set \(\{\tau \in \mathbb{H} | \Im(\tau) \geq t_0\}\) for any \(t_0 > 0\). \(\Box\)

**Remark 5.1.** Let \(h_k(z) = 1/z^k\). Then one can easily check that \(h_k(z)\) is a J-form of weight \(k\) and index 0, and that \(D_{h_k}(p, q; x) = R_{h_k}(p, q; x) = 0\).

Hereafter, we denote by \(B_n\) the \(n\)th Bernoulli number and by \(\zeta(s)\) the Riemann zeta function. The Weierstrass \(\wp\)-function, introduced by Weierstrass, is a meromorphic function given as follows (we follow the notation in [8]):

\[
\wp(\tau, z) = \frac{1}{z^2} + \sum_{\gamma \in \Delta_\tau(2\mathbb{Z}+\mathbb{Z})} \left( \frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2} \right).
\]
We see that \( \varphi \)-function is a Jacobi form of weight 2 and index 0. It is known that the \( \varphi \)-function has the following expansion at \( z = 0 \) (refer to [8, p. 132]):

\[
\varphi(\tau, z) = \frac{1}{z^2} + 2 \sum_{k=2}^{\infty} G_{2k}(\tau) \frac{z^{2k-2}}{(2k-2)!},
\]

where \( G_{2k}(\tau) \) is the Eisenstein series of weight \( 2k \), namely,

\[
G_{2k}(\tau) = -\frac{B_{2k}}{(4k)} + \sum_{m=1}^{\infty} \frac{1}{m^{2k-1}} e^{2\pi i m \tau}.
\]

**Proposition 5.2.** Let \( x \) be a real number such that \( |x| \) is sufficiently small and \( x \neq 0 \). Then the following identity holds:

\[
D_{\varphi}(p, q; x) = -\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \frac{\mu \pi}{p} \left\{ \cot \left( \frac{x}{2i} + \frac{\mu \pi}{p} \right) - \cot \frac{\mu \pi}{p} \right\} + \sum_{k=2}^{\infty} \frac{p^{2k-2} \zeta(2k-1)}{(2\pi)^{2k-1}} x^{2k-2}. \tag{5.1}
\]

**Proof.** In [7], it was shown that

\[
D_{G_{2k}}(p, q) = \frac{p^{2k-2}(2k-2)!}{(2\pi)^{2k-1}} \sum_{l=1}^{p} \left( 1 - \frac{l}{p} \right) \sum_{m=1}^{\infty} \frac{e^{2\pi ilq/p}}{m^{2k-1}}. \tag{5.2}
\]

We can now reformulate the above expression using Hurwitz zeta function \( \zeta(s, z) \) as follows (the proof will be given in Lemma 5.3):

\[
D_{G_{2k}}(p, q) = \frac{(2k-2)!}{p(2\pi i)^{2k-1}} \times \left\{ -i \sum_{\mu=1}^{p-1} \cot \frac{\mu \pi}{p} \zeta \left( 2k - 1, \frac{\mu}{p} \right) + \frac{1}{2} p^{2k-1} \zeta(2k-1) \right\}. \tag{5.3}
\]
Hence, by Lemma 5.1, we have

\[ D_{\psi}(p, q; x) = 2 \sum_{k=2}^{\infty} D_{G_2}(p, q) \frac{x^{2k-2}}{(2k-2)!} \]

\[ = 2 \sum_{k=2}^{\infty} \frac{(2k-2)!}{p(2\pi i)^{2k-1}} \left\{ -i \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \zeta \left( 2k - 1, \frac{\mu}{p} \right) \right\} x^{2k-2} \]

\[ + \frac{1}{2} p^{2k-2} (2k-1) \zeta(2k-1) \]

\[ = -\sum_{k=2}^{\infty} \frac{i}{p(2\pi i)^{2k-1}} \left\{ \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \zeta \left( 2k - 1, \frac{\mu}{p} \right) \right\} x^{2k-2} \]

\[ + \sum_{k=2}^{\infty} \frac{p^{2k-2} \zeta(2k-1)}{(2\pi i)^{2k-1}} x^{2k-2}. \] (5.4)

Next we will expand the first term of the right-hand side of (5.1) applying Lemma 5.4 which we will prove later:

\[ -\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \cot \left( \frac{x}{2i} + \frac{\mu \pi}{p} \right) \]

\[ = -\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \pi \cot \left( \pi \left( \frac{x}{2\pi i} + \frac{\mu}{p} \right) \right) \]

\[ = -\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \pi \cot \left( \frac{\mu \pi}{p} \right) \]

\[ - \frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \sum_{k=1}^{\infty} \left\{ (-1)^k \zeta \left( k + 1, \frac{\mu}{p} \right) \right\} \left( \frac{x}{2\pi i} \right)^k \]

(by Lemma 5.4)

\[ = -\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu q \pi}{p} \right) \cot \left( \frac{\mu \pi}{p} \right) \]

\[ - \frac{1}{4p} \sum_{k=2}^{\infty} \left\{ \sum_{\mu=1}^{p-1} 2 \cot \left( \frac{\mu q \pi}{p} \right) \zeta \left( k + 1, \frac{\mu}{p} \right) \right\} \left( \frac{x}{2\pi i} \right)^k \]

(because \( \cot((p-\mu)q \pi/p) = -\cot(\mu q \pi/p) \))
\[
\frac{1}{4p} \sum_{\mu=1}^{\frac{p-1}{2}} \cot \left( \frac{\mu \pi}{p} \right) \cot \left( \frac{\mu \pi}{p} \right) - \sum_{k=2}^{\infty} \frac{i}{p(2\pi i)^{2k-1}} \left( \sum_{\mu=1}^{\frac{p-1}{2}} \cot \left( \frac{\mu \pi}{p} \right) \zeta \left( 2k - 1, \frac{\mu}{p} \right) \right) x^{2k-2}.
\]

Hence, we have the following expression for the right-hand side of (5.1) which is valid for sufficiently small \( x \in \mathbb{R} \):

\[
- \frac{1}{4p} \sum_{\mu=1}^{\frac{p-1}{2}} \cot \left( \frac{\mu \pi}{p} \right) \left\{ \cot \left( \frac{x}{2} + \frac{\mu \pi}{p} \right) - \cot \frac{\mu \pi}{p} \right\} + \sum_{k=2}^{\infty} \frac{p^{2k-2} \zeta(2k-1)}{(2\pi i)^{2k-1}} x^{2k-2}
\]

\[
= - \sum_{k=2}^{\infty} \frac{i}{p(2\pi i)^{2k-1}} \left( \sum_{\mu=1}^{\frac{p-1}{2}} \cot \left( \frac{\mu \pi}{p} \right) \zeta \left( 2k - 1, \frac{\mu}{p} \right) \right) x^{2k-2} + \sum_{k=2}^{\infty} \frac{p^{2k-2} \zeta(2k-1)}{(2\pi i)^{2k-1}} x^{2k-2}. \tag{5.5}
\]

From (5.4) and (5.5), we obtain identity (5.1), thereby completing the proof. \( \square \)

Now we will give proofs to identity (5.3), and Lemma 5.4 used in the proof of Proposition 5.2.

**Lemma 5.3.**

\[
D_{2k}(p, q) = \frac{(2k - 2)!}{p(2\pi i)^{2k-1}} \left\{ -\frac{i}{2} \sum_{\mu=1}^{\frac{p-1}{2}} \cot \left( \frac{\mu \pi}{p} \right) \zeta \left( 2k - 1, \frac{\mu}{p} \right) + \frac{1}{2} p^{2k-1} \zeta(2k-1) \right\}.
\]

**Proof.** First we have

\[
\sum_{m=1}^{\infty} \frac{e^{2\pi imq/p}}{m^{2k-1}} = \sum_{\mu=1}^{\frac{p}{2}} \sum_{n=0}^{\infty} \frac{e^{2\pi in\mu/p}}{(pn + \mu)^{2k-1}} = \frac{1}{p^{2k-1}} \sum_{\mu=1}^{\frac{p}{2}} e^{2\pi \mu q/p} \zeta \left( 2k - 1, \frac{\mu}{p} \right).
\]

Furthermore, for \( \mu \not\equiv 0 \pmod{p} \), we have

\[
\sum_{l=1}^{\frac{p}{2}} \frac{\left( \frac{l}{p} - \frac{1}{2} \right) e^{2\pi i \mu q/p}}{2(1 - e^{2\pi i \mu q/p})} = \frac{1}{p} \sum_{l=1}^{\frac{p}{2}} \frac{e^{2\pi i \mu q/p}}{1 - e^{2\pi i \mu q/p}} = \frac{1}{p} \left( \frac{-p}{1 - e^{2\pi i \mu q/p} + p} \right) + p
\]

\[
= -\frac{1 + e^{2\pi i \mu q/p}}{2(1 - e^{2\pi i \mu q/p})} + \frac{1}{2} = -\frac{i}{2} \cot \left( \frac{\mu \pi}{p} \right) + \frac{1}{2},
\]
and, for \( \mu \equiv 0 \pmod{p} \), we have
\[
\sum_{l=1}^{p} \left( \frac{l}{p} - \frac{1}{2} \right) e^{2\pi i mlq/p} = \sum_{l=1}^{p} \left( \frac{l}{p} - \frac{1}{2} \right) = \frac{1}{p} \left( p(p+1) \right) - \frac{p}{2} = \frac{1}{2}
\]
Combining these and (5.2), we obtain

\[
D_{G_{2k}}(p, q)
\]
\[
= \frac{p^{2k-2}(2k-2)!}{(2\pi i)^{2k-1}} \sum_{l=1}^{p} \left( \frac{l}{p} - \frac{1}{2} \right) \sum_{m=1}^{\infty} \frac{e^{2\pi i mlq/p}}{m^{2k-1}}
\]
\[
= \frac{p^{2k-2}(2k-2)!}{(2\pi i)^{2k-1}} \sum_{l=1}^{p} \left( \frac{l}{p} - \frac{1}{2} \right) \left\{ \frac{1}{p^{2k-1}} \sum_{\mu=1}^{p} e^{2\pi i mlq/p} \zeta \left( 2k - 1, \frac{\mu}{p} \right) \right\}
\]
\[
= \frac{p^{2k-2}(2k-2)!}{p(2\pi i)^{2k-1}} \sum_{l=1}^{p} \left( \frac{l}{p} - \frac{1}{2} \right) \left\{ \zeta \left( 2k - 1, \frac{\mu}{p} \right) \sum_{l=1}^{p} \left( \frac{l}{p} - \frac{1}{2} \right) e^{2\pi i mlq/p} \right\}
\]
\[
= \frac{(2k-2)!}{p(2\pi i)^{2k-1}} \left\{ \sum_{\mu=1}^{p} \zeta \left( 2k - 1, \frac{\mu}{p} \right) \left( -i \frac{1}{2} \cot \left( \frac{\mu \pi}{p} \right) + \frac{1}{2} \right) + \frac{1}{2} \zeta \left( 2k - 1, \frac{p}{p} \right) \right\}
\]
\[
= \frac{(2k-2)!}{p(2\pi i)^{2k-1}} \left\{ \frac{i}{2} \sum_{\mu=1}^{p} \cot \left( \frac{\mu \pi}{p} \right) \zeta \left( 2k - 1, \frac{\mu}{p} \right) + \frac{1}{2} p^{2k-1} \zeta \left( 2k - 1 \right) \right\}
\]
completing the proof. \( \square \)

**Lemma 5.4.** Let \( \alpha \) be a real number such that \( 0 < \alpha < 1 \). Then, for sufficiently small \( y \in \mathbb{R} \), we have the following expansion at \( y = 0 \):

\[
\pi \cot(\pi(y + \alpha)) = \pi \cot(\pi \alpha) + \sum_{k=1}^{\infty} (-1)^k \left( \zeta(k + 1, \alpha) - \zeta(k + 1, 1 - \alpha) \right) y^k.
\]

**Proof.** Applying a well-known formula \( \cot(x) = \lim_{N \to \infty} \sum_{n=-N}^{N} 1/(x + n\pi) \) for \( |x| < \pi \), we have

\[
\pi \cot(\pi(y + \alpha))
\]
\[
= \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{y + n + \alpha} = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{n + \alpha} \sum_{k=0}^{\infty} \left( \frac{-y}{n + \alpha} \right)^k
\]
\[
= \sum_{k=0}^{\infty} \left\{ \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{(n + \alpha)^{k+1}} \right\} (-y)^k
\]
\[
\pi \cot(\pi x) + \sum_{k=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{1}{(n+z)^{k+1}} + (-1)^{k+1} \sum_{n=0}^{\infty} \frac{1}{(n+1-z)^{k+1}} \right\} (-y)^k
\]

\[
= \pi \cot(\pi x) + \sum_{k=1}^{\infty} \left\{ (-1)^k \zeta(k+1,z) - \zeta(k+1,1-z) \right\} y^k.
\]

Now we consider the period function \(R_{\phi}(p,q;x)\) associated to the Weierstrass \(\phi\)-function. We will give an explicit form to the period function.

**Proposition 5.5.** Let \(x\) be a real number such that \(|x|\) is sufficiently small and \(x \neq 0\). Then the following identity holds:

\[
R_{\phi}(p,q;x) = \frac{1}{4} \cot \left( \frac{p^x}{2i} \right) \cot \left( \frac{q^x}{2i} \right) + \frac{p^2 + q^2 + 1}{12pq} - \frac{1}{4pq} \csc^2 \left( \frac{x}{2i} \right) - \sum_{k=2}^{\infty} \frac{1}{(2\pi i)^{2k-1}} \frac{\zeta(2k-1)}{2(2k-1)!} (q^{2k-2} - p^{2k-2}) x^{2k-2}.
\]

(5.6)

**Proof.** In [7], we showed that

\[
R_{G2}(p,q) = - \sum_{n=-1}^{2k-1} \frac{(2k-2)!B_{n+1}B_{2k-n-1}}{2(n+1)!(2k-n-1)!} p^n q^{2k-2-n}
\]

\[
- \frac{(2k-2)!\zeta(2k-1)}{2(2\pi i)^{2k-1}} (q^{2k-2} - p^{2k-2}) - \frac{B_{2k}}{4k} p^{-1} q^{-1}.
\]

(5.7)

Hence, by Lemma 5.1, we obtain

\[
R_{\phi}(p,q;x) = 2 \sum_{k=2}^{\infty} R_{G2}(p,q) \frac{x^{2k-2}}{(2k-2)!}
\]

\[
= 2 \sum_{k=2}^{\infty} \left\{ - \sum_{n=-1}^{2k-1} \frac{(2k-2)!B_{n+1}B_{2k-n-1}}{2(n+1)!(2k-n-1)!} p^n q^{2k-2-n} \right\}
\]

\[
- \frac{(2k-2)!\zeta(2k-1)}{2(2\pi i)^{2k-1}} (q^{2k-2} - p^{2k-2})
\]

\[
- \frac{B_{2k}}{4k} p^{-1} q^{-1} \right\} \frac{x^{2k-2}}{(2k-2)!}
\]

\[
= - \frac{1}{pq} \sum_{k=2}^{\infty} \sum_{n=0}^{k} \frac{B_{2n}B_{2k-2n}p^{2n}q^{2k-2n}}{(2n)!(2k-2n)!} x^{2k-2}
\]
\[ - \sum_{k=2}^{\infty} \frac{\zeta(2k - 1)}{(2\pi i)^{2k-1}}(q^{2k-2} - p^{2k-2})x^{2k-2} \]
\[ - \frac{1}{pq} \sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k - 2)!} x^{2k-2}. \]

(5.8)

Next we will expand the right-hand side of (5.6) using following formulae (by our convention, 0! = 1):

\[
\cot(x) = \frac{1}{x} \left( \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} \frac{1}{x^{2n}} \right),
\]
\[
\csc^2(x) = - \frac{1}{x^2} \left( \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1) 2^{2n} B_{2n}}{(2n)!} \frac{1}{x^{2n}} \right).
\]

Then the right-hand side of (5.6) becomes

\[
\frac{1}{4} \cot\left(\frac{px}{2i}\right)\cot\left(\frac{qx}{2i}\right) + \frac{p^2 + q^2 + 1}{12pq} - \frac{1}{4pq} \csc^2\left(\frac{x}{2i}\right)
\]
\[ - \sum_{k=2}^{\infty} \frac{\zeta(2k - 1)}{(2\pi i)^{2k-1}}(q^{2k-2} - p^{2k-2})x^{2k-2} \]
\[ = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} \left( \frac{px}{2i} \right)^{2n} - \frac{2i}{q} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} \left( \frac{qx}{2i} \right)^{2n} \]
\[ + \frac{p^2 + q^2 + 1}{12pq} - \frac{1}{4pq} (-1) \left( \frac{2i}{x} \right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1) 2^{2n} B_{2n}}{(2n)!} \left( \frac{x}{2i} \right)^{2n} \]
\[ - \sum_{k=2}^{\infty} \frac{\zeta(2k - 1)}{(2\pi i)^{2k-1}}(q^{2k-2} - p^{2k-2})x^{2k-2} \]
\[ = - \frac{1}{pq} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{B_{2n} B_{2k-2n} p^{2n} q^{2k-2n}}{(2n)!(2k - 2n)!} x^{2k-2} + \frac{p^2 + q^2 + 1}{12pq} \]
\[ - \frac{1}{pq} \sum_{n=0}^{\infty} \frac{(2n - 1) B_{2n}}{(2n)!} x^{2n-2} \]
\[ - \sum_{k=2}^{\infty} \frac{\zeta(2k - 1)}{(2\pi i)^{2k-1}}(q^{2k-2} - p^{2k-2})x^{2k-2} \]
\[ = - \frac{1}{pq} \sum_{k=2}^{\infty} \sum_{n=0}^{\infty} \frac{B_{2n} B_{2k-2n} p^{2n} q^{2k-2n}}{(2n)!(2k - 2n)!} x^{2k-2} - \frac{1}{pq} \sum_{n=0}^{\infty} \frac{B_{2n}}{2n(2n - 2)!} x^{2n-2} \]
\[ - \sum_{k=2}^{\infty} \frac{\zeta(2k - 1)}{(2\pi i)^{2k-1}}(q^{2k-2} - p^{2k-2})x^{2k-2} \]

(noting that \( B_0 = 1 \) and \( B_2 = 1/6 \)).

(5.9)
Observe that the final expression in (5.9) and the final expression in (5.8) do coincide. Thus, we have the claimed identity (5.6), thereby completing our proof for the period function \( R_y(p, q; x) \).

6. Proving the propositions

Finally, we can give proofs to Propositions 1.2 and 1.3.

**Proof of Propositions 1.2 and 1.3.** Applying Theorem 1.1, Propositions 5.2 and 5.5, we obtain

\[
-\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu \pi}{p} \right) \left\{ \cot \left( \frac{x}{2i} + \frac{\mu \pi}{p} \right) - \cot \left( \frac{\mu \pi}{p} \right) \right\}
- \frac{1}{4q} \sum_{\mu=1}^{q-1} \cot \left( \frac{\mu \pi}{q} \right) \left\{ \cot \left( \frac{x}{2i} + \frac{\mu \pi}{q} \right) - \cot \left( \frac{\mu \pi}{q} \right) \right\}
= \frac{1}{4} \cot \left( \frac{px}{2i} \right) \cot \left( \frac{qx}{2i} \right) + \frac{p^2 + q^2 + 1}{12pq} - \frac{1}{4pq} \csc^2 \frac{x}{2i}
\]

for \( x \in \mathbb{R} \) such that \( |x| \) is sufficiently small and \( x \neq 0 \). Identity (6.1) can be extended to any \( x \in \mathbb{R} \), considering meromorphic continuation.

Taking \( \lim_{x \to \infty} \) of both sides of (6.1), we obtain

\[
\frac{1}{4p} \sum_{\mu=1}^{p-1} \cot \left( \frac{\mu \pi}{p} \right) \cot \left( \frac{\mu \pi}{p} \right) + \frac{1}{4q} \sum_{\mu=1}^{q-1} \cot \left( \frac{\mu \pi}{q} \right) \cot \left( \frac{\mu \pi}{q} \right)
= \frac{p^2 + q^2 + 1 - 3pq}{12pq}.
\]

This is well-known as the classical Dedekind reciprocity law (refer to [9, pp. 93,100]). This is the assertion in Proposition 1.2.

To establish the identity in Proposition 1.3, first replace \( x/2i \) with \( z \) considering meromorphic continuation. Then we can derive from (6.1) and (6.2) the identity we are after:

\[
-\cot(pz) \cot(qz) + \frac{1}{pq} \csc^2 z - 1.
\]

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References


