Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 241 (2007) 94–111
Journal of Differential Equations
www.elsevier.com/locate/jde

Convergence rate of solutions toward stationary solutions to the compressible Navier–Stokes equation in a half line

Tohru Nakamura [a,*], Shinya Nishibata [b], Takeshi Yuge [b]

[a] *Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan*
[b] *Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan*

Received 28 August 2006

Abstract

We study a large time behavior of a solution to the initial boundary value problem for an isentropic and compressible viscous fluid in a one-dimensional half space. The unique existence and the asymptotic stability of a stationary solution are proved by S. Kawashima, S. Nishibata and P. Zhu for an outflow problem where the fluid blows out through the boundary. The main concern of the present paper is to investigate a convergence rate of a solution toward the stationary solution. For the supersonic flow at spatial infinity, we obtain an algebraic or an exponential decay rate. Precisely, if an initial perturbation decays with the algebraic or the exponential rate in the spatial asymptotic point, the solution converges to the corresponding stationary solution with the same rate in time as time tends to infinity. An algebraic convergence rate is also obtained for the transonic flow. These results are proved by the weighted energy method.

MSC: 35B35; 35B40; 76N15

Keywords: Decay estimate; Isentropic fluid; Eulerian coordinate; Weighted energy method; Stationary wave

1. Introduction

The present paper concerns an asymptotic behavior of a solution to the initial boundary value problem for the compressible Navier–Stokes equation in one-dimensional half space

* Corresponding author.
E-mail address: tohru@math.kyushu-u.ac.jp (T. Nakamura).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.06.016

$\mathbb{R}_+ := (0, \infty)$. We especially study a convergence rate toward a corresponding stationary solution for the problem in which fluid blows out through a boundary. An isentropic or isothermal model of the compressible viscous fluid is formulated in the Eulerian coordinate as

$$\rho_t + (\rho u)_x = 0, \tag{1.1a}$$

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = \mu u_{xx}, \tag{1.1b}$$

where the unknown functions are a mass density ρ and a fluid velocity u. A constant μ is called a viscosity coefficient. A pressure p is given by $p = p(\rho) = K\rho^\gamma$ where $K > 0$ and $\gamma \geqslant 1$ are constants. The initial condition is prescribed by

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{1.2a}$$

$$\lim_{x\to\infty} (\rho_0, u_0)(x) = (\rho_+, u_+), \qquad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0, \quad \rho_+ > 0. \tag{1.2b}$$

The main concern of the present paper is a phenomena in which the gas brows out from the boundary. This is called an outflow problem in [7]. Thus, we adopt a boundary condition

$$u(t, 0) = u_b < 0. \tag{1.3}$$

Note that only one boundary condition (1.3) is necessary and sufficient for the wellposedness of this problem since the characteristic $u(t, x)$ of the hyperbolic equation (1.1a) is negative around the boundary $\{x = 0\}$ due to the condition (1.3).

It is shown in the paper [5] that the solution to the problem (1.1), (1.2) and (1.3) converges to the corresponding stationary solution as time tends to infinity. Here we summarize the results in [5]. The stationary solution $(\tilde\rho, \tilde u)(x)$ is a solution to the system (1.1) independent of a time variable t, satisfying the same conditions (1.2b) and (1.3). Therefore, the stationary solution $(\tilde\rho, \tilde u)$ satisfies the system of equations

$$(\tilde\rho\tilde u)_x = 0, \tag{1.4a}$$

$$\left(\tilde\rho\tilde u^2 + \tilde p\right)_x = \mu\tilde u_{xx} \tag{1.4b}$$

and the boundary and the spatial asymptotic conditions

$$\tilde u(0) = u_b, \qquad \lim_{x\to\infty} (\tilde\rho, \tilde u)(x) = (\rho_+, u_+), \qquad \inf_{x\in\mathbb{R}_+} \tilde\rho(x) > 0, \tag{1.5}$$

where $\tilde p := p(\tilde\rho)$. Integrating (1.4a) over (x, ∞) yields

$$\tilde u = \frac{u_+}{v_+}\tilde v, \tag{1.6}$$

where $v := 1/\rho$ is called a specific volume and $v_+ := 1/\rho_+$. Due to (1.3) and (1.6), we see that

$$u_+ = \frac{v_+}{v(0)} u_b < 0 \tag{1.7}$$

is necessary for the existence of the stationary solution to the problem (1.4) and (1.5). Integrating (1.4b) over (x,∞) and substituting (1.6) in the resultant equality, we get

$$\mu u_+ \tilde{w}_x = F(\tilde{w}), \tag{1.8}$$

$$F(w) := K\rho_+^\gamma\big(w^{-\gamma}-1\big)+\rho_+u_+^2(w-1), \quad \tilde{w}(x) := \frac{\tilde{u}(x)}{u_+} = \frac{\rho_+}{\tilde{\rho}(x)}. \tag{1.9}$$

Let c_+ and M_+ be a sound speed and a Mach number at the spatial asymptotic states, respectively. Then they are given by

$$c_+ := \sqrt{p'(\rho_+)} = \sqrt{\gamma K\rho_+^{\gamma-1}}, \quad M_+ := \frac{|u_+|}{c_+}. \tag{1.10}$$

If $M_+ > 1$, the equation $F(w) = 0$ has the distinct two roots $w = 1$ and $w = w_c$ satisfying $w_c < 1$. If $M_+ = 1$, the equation $F(w) = 0$ admits only one root $w = w_c = 1$. The asymptotic stability of the stationary solution is discussed in the function space $X(0,T)$ defined by

$$\begin{aligned} X(0,T) := \big\{(\varphi,\psi);\, &\varphi \in \mathcal{B}_T^{1+\sigma/2,1+\sigma},\ \psi \in \mathcal{B}_T^{1+\sigma/2,2+\sigma},\ (\varphi,\psi) \in C\big([0,T];H^1(\mathbb{R}_+)\big),\\ &\varphi_x \in L^2\big(0,T;L^2(\mathbb{R}_+)\big),\ \psi_x \in L^2\big(0,T;H^1(\mathbb{R}_+)\big)\big\}. \end{aligned}$$

Proposition 1.1. *(See [5].) Assume that the conditions* (1.3) *and* (1.7) *hold.*

(i) (*Existence*) *The boundary value problem* (1.4) *and* (1.5) *has a unique smooth solution* $(\tilde{\rho},\tilde{u})$ *if and only if* $M_+ \geqslant 1$ *and* $w_c u_+ > u_b$. *If* $M_+ > 1$, *there exist positive constants* λ *and* C *such that the stationary solution satisfies the estimate*

$$\big|\partial_x^k\big(\tilde{\rho}(x)-\rho_+,\tilde{u}(x)-u_+\big)\big| \leqslant C\delta_S e^{-\lambda x} \quad \textit{for } k = 0,1,2,\ldots, \tag{1.11a}$$

where $\delta_S := |u_b - u_+|$. *If* $M_+ = 1$, *the stationary solution satisfies*

$$\big|\partial_x^k\big(\tilde{\rho}(x)-\rho_+,\tilde{u}(x)-u_+\big)\big| \leqslant C\frac{{\delta_S}^{k+1}}{(1+\delta_S x)^{k+1}} \quad \textit{for } k = 0,1,2,\ldots, \tag{1.11b}$$

where C *is a positive constant.*

(ii) (*Stability*) *Suppose that* $M_+ \geqslant 1$ *and* $w_c u_+ > u_b$ *hold. In addition, the initial data* (ρ_0,u_0) *is supposed to satisfy*

$$(\rho_0-\tilde{\rho},u_0-\tilde{u}) \in H^1(\mathbb{R}_+), \quad (\rho_0,u_0) \in \mathcal{B}^{1+\sigma}(\mathbb{R}_+)\times\mathcal{B}^{2+\sigma}(\mathbb{R}_+) \tag{1.12}$$

for a certain constant $\sigma \in (0,1)$. *Then there exists a positive constant* ε_0 *such that if* $\|(\rho_0-\tilde{\rho},u_0-\tilde{u})\|_1 + \delta_S < \varepsilon_0$ *the initial boundary value problem* (1.1), (1.2) *and* (1.3) *has a unique solution* (ρ,u) *satisfying* $(\rho-\tilde{\rho},u-\tilde{u}) \in X(0,T)$ *for an arbitrary* $T > 0$. *Moreover, the solution* (ρ,u) *converges to the stationary solution* $(\tilde{\rho},\tilde{u})$ *as time tends to infinity*: $\lim_{t\to\infty}\|(\rho,u)(t)-(\tilde{\rho},\tilde{u})\|_\infty = 0$.

The main purpose of the present paper is to investigate the convergence rate of the solution (ρ, u) toward the stationary solution $(\tilde{\rho}, \tilde{u})$ under the assumption that the initial perturbation decays exponentially or algebraically fast in the spatial direction.

Theorem 1.2. *Suppose that the same assumptions as in* (ii) *of Proposition* 1.1 *hold.*

(i) *Suppose that* $M_+ > 1$ *holds. If the initial data satisfies* $(1+x)^{\alpha/2}(\rho_0 - \tilde{\rho}), (1+x)^{\alpha/2}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$ *for a certain positive constant* α*, then the solution* (ρ, u) *to* (1.1), (1.2) *and* (1.3) *satisfies the decay estimate*

$$\left\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\right\|_\infty \leqslant C(1+t)^{-\alpha/2}. \tag{1.13}$$

On the other hand, if the initial data satisfies $e^{(\zeta/2)x}(\rho_0 - \tilde{\rho}), e^{(\zeta/2)x}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$ *for a certain positive constant* ζ*, then there exists a positive constant* α *such that the solution* (ρ, u) *to* (1.1), (1.2) *and* (1.3) *satisfies*

$$\left\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\right\|_\infty \leqslant Ce^{-\alpha t}. \tag{1.14}$$

(ii) *Suppose that* $M_+ = 1$ *holds. There exists a positive constant* ε_0 *such that if the initial data satisfies* $\|(1+x)^{\alpha/2}(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_1 < \varepsilon_0$ *for a certain constant* α *satisfying* $\alpha \in [2, \alpha^*)$*, where* α^* *is a constant defined by*

$$\alpha^*(\alpha^* - 2) = \frac{4}{\gamma + 1} \quad \text{and} \quad \alpha^* > 0, \tag{1.15}$$

then the solution (ρ, u) *to* (1.1), (1.2) *and* (1.3) *satisfies*

$$\left\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\right\|_\infty \leqslant C(1+t)^{-\alpha/4}. \tag{1.16}$$

Remark 1.3. We see that the convergence rate (1.16) for the transonic flow is not as fast as the supersonic flow. Moreover, we assume the condition $\alpha < \alpha^*$, which is necessary for the derivation of the weighted estimate (2.48). Also, this type of assumption is used in [8] for the analysis of the convergence rate toward the traveling wave for a scalar viscous conservation law. It is still open problem whether the assumption $\alpha < \alpha^*$ can be removed or not.

Related results. After the work [1] by Il'in and Oleinik, there are many researches which consider the stability of nonlinear waves. For example, Liu, Matsumura and Nishihara in [6] study the half space problem for the viscous conservation laws. For the one-dimensional half space problem to the compressible Navier–Stokes equation, Matsumura in [7] expects that the asymptotic states of the solutions are classified into more than twenty cases subject to the boundary condition and the spatial asymptotic data. Several problems in this classification have been already studied. For example, Matsumura and Nishihara in [9] consider the case when the asymptotic state becomes one of stationary solutions, rarefaction waves and superposition of them for the inflow problem. The research [5] by Kawashima, Nishibata and Zhu shows the asymptotic stability of the stationary solution for the outflow problem. Following [5], the present paper investigates the convergence rate toward the stationary solution for the outflow problem.

For the multi-dimensional half space problem, Kagei and Kawashima in [2] study the outflow problem and prove the asymptotic stability of a planar stationary wave. Recently, the authors have obtained the convergence rate for this problem. This result also will be published soon.

Outline of the paper. The remainder of the present paper is organized as follows. In Section 2, we begin detailed discussion with a reformulation of the problem (1.1), (1.2) and (1.3) to that for the perturbation from the stationary solution. Then we derive the weighted a priori estimate of the perturbation, which yields the convergence rate toward the stationary solution. In Section 2.1, we consider the supersonic flow. In this case, the algebraic or the exponential rate is obtained, subject to the decay rate of the initial perturbation in the spatial direction in the L^2 norm. The proof is mainly based on the weighted L^2 energy method. In Section 2.2, we obtain the algebraic rate for the transonic flow. Here we need to derive the weighted H^1 a priori estimate.

Notations. For a non-negative integer $l \geqslant 0$, $H^l(\mathbb{R}_+)$ denotes the lth order Sobolev space over $\mathbb{R}_+$ in the L^2 sense with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. The norm $\|\cdot\|_\infty$ means the L^∞-norm over $\mathbb{R}_+$. For $\alpha \in (0,1)$, $\mathcal{B}^{k+\alpha}(\mathbb{R}_+)$ denotes the Hölder space of bounded functions over $\mathbb{R}_+$ which have the kth order derivatives of Hölder continuity with exponent α. Its norm is $|\cdot|_{k+\alpha}$. For a domain $Q_T \subseteq [0,T] \times \mathbb{R}_+$, $\mathcal{B}^{\alpha,\beta}(Q_T)$ denotes the space of the Hölder continuous functions with the Hölder exponents α and β with respect to t and x, respectively. For integers k and l, $\mathcal{B}^{k+\alpha,l+\beta}(Q_T)$ denotes the space of the functions satisfying $\partial_t^i u, \partial_x^j u \in \mathcal{B}^{\alpha,\beta}(Q_T)$ for arbitrary integers $i \in [0,k]$ and $j \in [0,l]$. We abbreviate $\mathcal{B}^{k+\alpha,l+\beta}([0,T] \times \mathbb{R}_+)$ by $\mathcal{B}_T^{k+\alpha,l+\beta}$.

2. A priori estimate

In this section, we derive the a priori estimate of the solution in the H^1 Sobolev space. To this end, we define the perturbation (φ, ψ) from the stationary solution as

$$(\varphi, \psi)(t,x) = (\rho, u)(t,x) - (\tilde{\rho}, \tilde{u})(x). \tag{2.1}$$

Due to (1.1) and (1.4), we have the system of equations for (φ, ψ) as

$$\varphi_t + u\varphi_x + \rho\psi_x = -(\tilde{u}_x\varphi + \tilde{\rho}_x\psi), \tag{2.2a}$$

$$\rho\psi_t + \rho u\psi_x + p'(\rho)\varphi_x - \mu\psi_{xx} = -(\varphi\psi + \tilde{u}\varphi + \tilde{\rho}\psi)\tilde{u}_x - \big(p'(\rho) - p'(\tilde{\rho})\big)\tilde{\rho}_x. \tag{2.2b}$$

The initial and the boundary conditions to (2.2) are derived from (1.2a), (1.3) and (1.5) as

$$(\varphi, \psi)(0,x) = (\varphi_0, \psi_0)(x) := (\rho_0, u_0)(x) - (\tilde{\rho}, \tilde{u})(x), \tag{2.3}$$

$$\psi(t,0) = 0. \tag{2.4}$$

The uniform bound of the solutions in the weighted Sobolev space is derived later. For this purpose, we introduce function spaces $X_\omega(0,T)$ and $X_\omega^1(0,T)$ defined by

$$X_\omega(0,T) := \big\{(\varphi, \psi) \in X(0,T);\ (\sqrt{\omega}\varphi, \sqrt{\omega}\psi) \in C\big([0,T]; L^2(\mathbb{R}_+)\big)\big\},$$

$$X_\omega^1(0,T) := \big\{(\varphi, \psi) \in X(0,T);\ (\sqrt{\omega}\varphi, \sqrt{\omega}\psi) \in C\big([0,T]; H^1(\mathbb{R}_+)\big)\big\}.$$

Here the two types of weight functions are considered:

$$\omega(x) := (1+x)^{\alpha} \quad \text{or} \quad \omega(x) := e^{\alpha x}.$$

Also we use the norms $|\cdot|_{2,\omega}$, $\|\cdot\|_{\mathrm{a},\alpha}$ and $\|\cdot\|_{\mathrm{e},\alpha}$ defined by

$$|f|_{2,\omega} := \left\{ \int_0^{\infty} \omega(x) f(x)^2 \, dx \right\}^{1/2}, \quad \|f\|_{\mathrm{a},\alpha} := |f|_{2,(1+x)^{\alpha}}, \quad \|f\|_{\mathrm{e},\alpha} := |f|_{2,e^{\alpha x}}.$$

The following lemma, concerning the existence of the solution locally in time, is proved by the standard iteration method. Hence we omit the proof.

Lemma 2.1. *If the initial data satisfies* (1.12) *and* $\sqrt{\omega}(\rho_0 - \tilde{\rho}), \sqrt{\omega}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$*, there exists a positive constant* T *depending only on* $|\rho_0|_{1+\sigma}$ *and* $|u_0|_{2+\sigma}$ *such that the initial boundary value problem* (2.2), (2.3) *and* (2.4) *has a unique solution* $(\rho, u) \in X_{\omega}(0, T)$*. Moreover, if the initial data satisfies* (1.12) *and* $\sqrt{\omega}(\rho_0 - \tilde{\rho}), \sqrt{\omega}(u_0 - \tilde{u}) \in H^1(\mathbb{R}_+)$*, there exists a unique solution* (φ, ψ) *in* $X^1_{\omega}(0, T)$*.*

2.1. Supersonic flow

In this subsection, we derive the weighted energy estimate of the solution for the case when $M_+ > 1$ holds. To summarize the a priori estimate, we use the following notations for a weight function $W(t,x) = \chi(t)\omega(x)$ until the end of this subsection:

$$N(t) := \sup_{0 \leqslant \tau \leqslant t} \|(\varphi, \psi)(\tau)\|_1, \tag{2.5}$$

$$M(t)^2 := \int_0^t \chi(\tau)\bigl(\|\varphi_x(\tau)\|^2 + \|\psi_x(\tau)\|_1^2 + \varphi(\tau, 0)^2\bigr)\, d\tau, \tag{2.6}$$

$$\begin{aligned} L(t)^2 := \int_0^t & \chi_t(\tau)\bigl(|(\varphi, \psi)(\tau)|_{2,\omega}^2 + \|(\varphi_x, \psi_x)(\tau)\|^2\bigr) \\ & + \chi(\tau)\bigl(|\psi(\tau)|_{2,\omega_{xx}}^2 + |(\varphi, \psi)(\tau)|_{2,|\tilde{u}_x|\omega}^2\bigr)\, d\tau. \end{aligned} \tag{2.7}$$

Proposition 2.2. *Suppose that the same assumptions as in* (ii) *of Proposition* 1.1 *hold.*

(i) (*Algebraic decay*) *Suppose that* $(\varphi, \psi) \in X_{(1+x)^{\alpha}}(0, T)$ *is a solution to* (2.2), (2.3) *and* (2.4) *for certain positive constants* α *and* T*. Then there exist positive constants* ε_0 *and* C *such that if* $N(T) + \delta_{\mathrm{S}} < \varepsilon_0$*, then the solution* (φ, ψ) *satisfies the estimate*

$$(1+t)^{\alpha+\varepsilon}\|(\varphi, \psi)(t)\|_1^2 + \int_0^t (1+\tau)^{\alpha+\varepsilon}\bigl(\|\varphi_x(\tau)\|^2 + \|\psi_x(\tau)\|_1^2\bigr)\, d\tau$$

$$+\int_0^t (1+\tau)^{\alpha+\varepsilon}\big|(\varphi,\varphi_x)(\tau,0)\big|^2\,d\tau$$
$$\leqslant C\big(\big\|(\varphi_0,\psi_0)\big\|_1^2+\big\|(\varphi_0,\psi_0)\big\|_{\mathrm{a},\alpha}^2\big)(1+t)^{\varepsilon} \tag{2.8}$$

for arbitrary $t\in[0,T]$ and $\varepsilon>0$.

(ii) (*Exponential decay*) *Suppose that $(\varphi,\psi)\in X_{e^{\zeta x}}(0,T)$ is a solution to* (2.2)*,* (2.3) *and* (2.4) *for certain positives ζ and T. Then there exist positive constants ε_0, C, β ($<\zeta$) and α satisfying $\alpha\ll\beta$ such that if $N(T)+\delta_{\mathrm{S}}<\varepsilon_0$, then the solution (φ,ψ) satisfies*

$$e^{\alpha t}\big(\big\|(\varphi,\psi)(t)\big\|_1^2+\big\|(\varphi,\psi)(t)\big\|_{\mathrm{e},\beta}^2\big)+\int_0^t e^{\alpha\tau}\big|(\varphi,\varphi_x)(\tau,0)\big|^2\,d\tau$$
$$+\int_0^t e^{\alpha\tau}\big(\big\|\varphi_x(\tau)\big\|^2+\big\|\psi_x(\tau)\big\|_1^2\big)\,d\tau+\int_0^t e^{\alpha\tau}\big(\big\|(\varphi,\psi)(\tau)\big\|_{\mathrm{e},\beta}^2+\|\psi_x(\tau)\|_{\mathrm{e},\beta}^2\big)\,d\tau$$
$$\leqslant C\big(\big\|(\varphi_0,\psi_0)\big\|_1^2+\big\|(\varphi_0,\psi_0)\big\|_{\mathrm{e},\beta}^2\big). \tag{2.9}$$

To prove Proposition 2.2, we first derive the basic energy estimate. To this end, we define an energy form $\mathcal{E}$, as in [5], by

$$\mathcal{E}:=\frac{1}{2}\psi^2+K\tilde\rho^{\gamma-1}\Phi\left(\frac{\tilde\rho}{\rho}\right),\quad \Phi(s):=s-1-\int_1^s \eta^{-\gamma}\,d\eta. \tag{2.10}$$

Owing to Proposition 1.1, we see that the energy form $\mathcal{E}$ is equivalent to $|(\varphi,\psi)|^2$. Namely, there exist positive constants c and C such that

$$c\big(\varphi^2+\psi^2\big)\leqslant\mathcal{E}\leqslant C\big(\varphi^2+\psi^2\big). \tag{2.11}$$

We also have positive bounds of ρ as

$$0<c\leqslant\rho(t,x)\leqslant C \tag{2.12}$$

for $(t,x)\in[0,T]\times\mathbb{R}_+$.

Lemma 2.3. *Suppose that the same assumptions as in Theorem* 1.2 *hold. Then there exists a positive constant ε_0 such that if $N(T)+\delta_{\mathrm{S}}<\varepsilon_0$, it holds that*

$$\chi(t)\big|(\varphi,\psi)(t)\big|_{2,\omega}^2+\int_0^t \chi(\tau)\big(\big|(\varphi,\psi)(\tau)\big|_{2,\omega_x}^2+\big|\psi_x(\tau)\big|_{2,\omega}^2+\varphi(\tau,0)^2\big)\,d\tau$$
$$\leqslant C\big|(\varphi_0,\psi_0)\big|_{2,\omega}^2+CL(t)^2. \tag{2.13}$$

Proof. Multiplying (2.2b) by ψ, we see that the energy form satisfies the equality

$$(\rho\mathcal{E})_t - G_{1x} + \mu\psi_x^2 = (\mu\psi\psi_x)_x + R_0, \tag{2.14}$$

$$G_1 := -\rho u\mathcal{E} - \big(p(\rho) - p(\tilde\rho)\big)\psi, \tag{2.15}$$

$$R_0 := -\big\{(\rho u - \tilde\rho\tilde u)\psi + p(\rho) - p(\tilde\rho) - p'(\tilde\rho)\varphi\big\}\tilde u_x - \frac{1}{\tilde\rho}\varphi\psi p(\tilde\rho)_x.$$

Here, the positive bound of ρ (2.12) and the Schwarz inequality yield the estimate for R_0 as

$$|R_0| \leqslant C|\tilde u_x|\big(\varphi^2 + \psi^2\big). \tag{2.16}$$

Multiplying (2.14) by a weight function $W(t,x) = \chi(t)\omega(x)$, we have

$$\begin{aligned}&(W\rho\mathcal{E})_t - (WG_1)_x + W_xG_1 + \mu W\psi_x^2\\ &\quad= W_t\rho\mathcal{E} + \left(\mu W\psi\psi_x - \frac{\mu}{2}W_x\psi^2\right)_x + \frac{\mu}{2}W_{xx}\psi^2 + WR_0.\end{aligned} \tag{2.17}$$

Due to the boundary conditions (1.3) and (2.4), the integration of the second term on the left-hand side of (2.17) over $\mathbb{R}_+$ is estimated from below as

$$\int_0^\infty \big\{W\rho u\mathcal{E} + W\big(p - p(\tilde\rho)\big)\psi\big\}_x\,dx = -\chi(t)\rho(t,0)u_{\mathrm b}\mathcal{E}(t,0) \geqslant c\chi(t)\varphi(t,0)^2, \tag{2.18}$$

where we have used the estimates (2.11) and (2.12). The third term on the left-hand side of (2.17) is computed as

$$\begin{aligned}G_1 &= F_1(\varphi,\psi) + R_1,\\ F_1(\varphi,\psi) &:= \frac{K\gamma\rho_+^{\gamma-2}|u_+|}{2}\varphi^2 + \frac{\rho_+|u_+|}{2}\psi^2 - K\gamma\rho_+^{\gamma-1}\varphi\psi,\\ R_1 &:= -\frac{K\gamma\rho_+u_+}{2\rho^2}\big(\tilde\rho^{\gamma-1} - \rho_+^{\gamma-3}\rho^2\big)\varphi^2 - K\rho_+u_+\tilde\rho^{\gamma-1}\left\{\Phi\left(\frac{\tilde\rho}{\rho}\right) - \frac{\gamma}{2}\left(\frac{\tilde\rho}{\rho} - 1\right)^2\right\}\\ &\quad - K\gamma\big(\tilde\rho^{\gamma-1} - \rho_+^{\gamma-1}\big)\varphi\psi - K\tilde\rho^\gamma\left\{\left(\frac{\rho}{\tilde\rho}\right)^\gamma - 1 - \gamma\left(\frac{\rho}{\tilde\rho} - 1\right)\right\}\psi - (\rho u - \rho_+u_+)\mathcal{E}.\end{aligned} \tag{2.19}$$

The condition $M_+ > 1$ yields that the quadratic form $F_1(\varphi,\psi)$ is positive definite since

$$\begin{aligned}F_1(\varphi,\psi) &= \left(\frac{p'(\rho_+)^{3/2}}{2\rho_+}\varphi^2 + \frac{\rho_+\sqrt{p'(\rho_+)}}{2}\psi^2\right)(M_+ - 1) + \frac{\sqrt{p'(\rho_+)}}{2\rho_+}\big(\sqrt{p'(\rho_+)}\varphi - \rho_+\psi\big)^2\\ &\geqslant c\big(\varphi^2 + \psi^2\big).\end{aligned} \tag{2.20}$$

Utilizing (1.11), (2.11) and the inequalities

$$\left|\Phi(s)-\frac{\gamma}{2}(s-1)^2\right| \leqslant C|s-1|^3, \quad \left|s^\gamma - 1 - \gamma(s-1)\right| \leqslant C|s-1|^2 \tag{2.21}$$

for $|s-1| \ll 1$, we have the estimate for R_1 as

$$|R_1| \leqslant C\big(N(t)+\delta_S\big)\big(\varphi^2+\psi^2\big). \tag{2.22}$$

Therefore, integrating (2.17) over $\mathbb{R}_+ \times (0,t)$, substituting (2.16), (2.18), (2.20) and (2.22) in the resultant equality, and then taking $N(T)+\delta_S$ suitably small, we obtain the desired estimate (2.13). □

Next, we obtain the estimate for the first order derivatives of the solution (φ,ψ). As the existence of the higher order derivatives of the solution is not supposed, we need to use the difference quotient for the rigorous derivation of the higher order estimates. Since the argument using the difference quotient is similar to that in the paper [5], we omit the details and proceed with the proof as if it verifies

$$(\varphi,\psi) \in C\big([0,T]; H^2(\mathbb{R}_+)\big), \quad \varphi_x \in L^2\big(0,T; H^1(\mathbb{R}_+)\big), \quad \psi_x \in L^2\big(0,T; H^2(\mathbb{R}_+)\big).$$

Lemma 2.4. *There exists a positive constant ε_0 such that if $N(T)+\delta_S < \varepsilon_0$, then*

$$\begin{aligned}
&\chi(t)\left\|\varphi_x(t)\right\|^2 + \int_0^t \chi(\tau)\big(\left\|\varphi_x(\tau)\right\|^2 + \varphi_x(\tau,0)^2\big)\,d\tau \\
&\quad \leqslant C\big(\|\varphi_{0x}\|^2 + \left|(\varphi_0,\psi_0)\right|_{2,\omega}^2\big) + CL(t)^2 + C\big(N(t)+\delta_S\big)M(t)^2.
\end{aligned} \tag{2.23}$$

Proof. Differentiating (2.2a) in x and multiplying the resultant equality by φ_x yield

$$\begin{aligned}
&\left(\frac{1}{2}\varphi_x^2\right)_t + \left(\frac{1}{2}u\varphi_x^2\right)_x = -\rho\psi_{xx}\varphi_x + R_2, \\
&R_2 := \frac{1}{2}u_x\varphi_x^2 - \varphi_x(2\varphi_x\psi_x + 2\tilde{u}_x\varphi_x + 2\tilde{\rho}_x\psi_x + \tilde{u}_{xx}\varphi + \tilde{\rho}_{xx}\psi).
\end{aligned} \tag{2.24}$$

On the other hand, multiplying (2.2b) by $\rho\varphi_x$ gives

$$\begin{aligned}
&\big(\rho^2\varphi_x\psi\big)_t - \big(\rho^2\varphi_t\psi\big)_x + \rho p'(\rho)\varphi_x^2 = \mu\rho\psi_{xx}\varphi_x + R_3, \\
&R_3 := -\big(2\rho\tilde{\rho}_x\psi - \rho^2\psi_x\big)\varphi_t - \rho^2 u\varphi_x\psi_x - \rho\tilde{u}_x(\varphi\psi + \tilde{u}\varphi + \tilde{\rho}\psi)\varphi_x - \rho\tilde{\rho}_x\big(p'(\rho) - p'(\tilde{\rho})\big)\varphi_x.
\end{aligned} \tag{2.25}$$

Multiply (2.24) by μ and add the resultant equality to (2.25). Then we have

$$\left(\frac{\mu}{2}\varphi_x^2 + \rho^2\varphi_x\psi\right)_t + \left(\frac{\mu}{2}u\varphi_x^2 - \rho^2\varphi_t\psi\right)_x + \rho p'(\rho)\varphi_x^2 = \mu R_2 + R_3. \tag{2.26}$$

Owing to the Schwarz inequality with the aid of (1.11), the right-hand side of (2.26) is estimated as

$$|\mu R_2 + R_3| \leqslant (\varepsilon + C\delta_{\mathrm{S}})\varphi_x^2 + C_\varepsilon\big(\psi_x^2 + |\psi_x|\varphi_x^2\big) + C_\varepsilon|\tilde{u}_x|\big(\varphi^2 + \psi^2\big) \tag{2.27}$$

for an arbitrary positive constant ε, where C_ε is a positive constant depending on ε. Multiplying (2.26) by a weight function $\chi = \chi(t)$, we have

$$\begin{aligned}&\left\{\chi\left(\frac{\mu}{2}\varphi_x^2 + \rho^2\varphi_x\psi\right)\right\}_t + \left\{\chi\left(\frac{\mu}{2}u\varphi_x^2 - \rho^2\varphi_t\psi\right)\right\}_x + \rho p'(\rho)\chi\varphi_x^2\\ &\quad = \chi_t\left(\frac{\mu}{2}\varphi_x^2 + \rho^2\varphi_x\psi\right) + \chi(\mu R_2 + R_3).\end{aligned} \tag{2.28}$$

The boundary condition (1.3) gives the lower estimate of the integration of the second term on the left-hand side of (2.28) as

$$\int_0^\infty \left\{\chi\left(\frac{\mu}{2}u\varphi_x^2 - \rho^2\varphi_t\psi\right)\right\}_x dx \geqslant c\chi(t)\varphi_x(t,0)^2. \tag{2.29}$$

Integrate (2.28) over $\mathbb{R}_+ \times (0,t)$, substitute (2.27), (2.29) and the estimate

$$\int_0^\infty |\psi_x|\varphi_x^2\,dx \leqslant C\|\psi_x\|_1\|\varphi_x\|^2 \leqslant CN(t)\big\|(\varphi_x,\psi_x,\psi_{xx})\big\|^2 \tag{2.30}$$

in the resultant equality, and take ε and δ_{S} suitably small. These computations together with (2.13) give the desired estimate (2.23). □

Lemma 2.5. *There exists a positive constant ε_0 such that if $N(T) + \delta_{\mathrm{S}} < \varepsilon_0$, then*

$$\begin{aligned}&\chi(t)\big\|\psi_x(t)\big\|^2 + \int_0^t \chi(\tau)\big\|\psi_{xx}(\tau)\big\|^2\,d\tau\\ &\quad \leqslant C\Big(\big\|(\varphi_{0x},\psi_{0x})\big\|^2 + \big|(\varphi_0,\psi_0)\big|_{2,\omega}^2\Big) + CL(t)^2 + C\big(N(t)+\delta_{\mathrm{S}}\big)M(t)^2.\end{aligned} \tag{2.31}$$

Proof. Multiplying (2.2b) by $-\psi_{xx}$ gives

$$\left(\frac{1}{2}\rho\psi_x^2\right)_t - (\rho\psi_x\psi_t)_x + \mu\psi_{xx}^2 = R_4, \tag{2.32}$$

$$\begin{aligned}R_4 &:= -\rho_x\psi_x\psi_t + \frac{1}{2}\rho_t\psi_x^2 + \rho u\psi_x\psi_{xx} + p'(\rho)\varphi_x\psi_{xx} + (\rho u - \tilde{\rho}\tilde{u})\tilde{u}_x\psi_{xx}\\ &\quad + \big(p'(\rho) - p'(\tilde{\rho})\big)\tilde{\rho}_x\psi_{xx}.\end{aligned} \tag{2.33}$$

Note that the function R_4 is estimated by using (2.2b) and (1.11) as

$$|R_4| \leqslant \varepsilon\psi_{xx}^2 + C_\varepsilon\big(\varphi_x^2 + \psi_x^2 + \psi_x^4 + \varphi_x^2\psi_x^2\big) + C_\varepsilon|\tilde{u}_x|\big(\varphi^2 + \psi^2\big), \tag{2.34}$$

where ε is an arbitrary positive constant and C_ε is a positive constant depending on ε. Multiplying (2.32) by a weight function $\chi(t)$, we get

$$\left(\frac{1}{2}\chi\rho\psi_x^2\right)_t - (\chi\rho\psi_x\psi_t)_x + \mu\chi\psi_{xx}^2 = \frac{1}{2}\chi_t\rho\psi_x^2 + \chi R_4. \tag{2.35}$$

Integrating (2.35) over $\mathbb{R}_+ \times (0,t)$, substituting (2.13), (2.23) and (2.34) as well as the estimate

$$\int_0^\infty \psi_x^4 + \varphi_x^2\psi_x^2\,dx \leqslant C\|\psi_x\|_1^2\big\|(\varphi_x,\psi_x)\big\|^2 \leqslant CN(t)\big\|(\psi_x,\psi_{xx})\big\|^2 \tag{2.36}$$

in the resultant equality and then letting ε sufficiently small, we obtain the desired estimate (2.31). □

Proof of Proposition 2.2. Summing up the estimates (2.13), (2.23) and (2.31) and taking $N(T)+\delta_{\mathrm{S}}$ suitably small, we have

$$\begin{aligned}
&\chi(t)\big(\big\|(\varphi_x,\psi_x)\big\|^2 + \big|(\varphi,\psi)\big|_{2,\omega}^2\big) + \int_0^t \chi(\tau)\big|(\varphi,\varphi_x)(\tau,0)\big|^2\,d\tau\\
&\quad + \int_0^t \chi(\tau)\big(\big|(\varphi,\psi)(\tau)\big|_{2,\omega_x}^2 + \big\|\varphi_x(\tau)\big\|^2 + \big|\psi_x(\tau)\big|_{2,\omega}^2 + \big\|\psi_{xx}(\tau)\big\|^2\big)\,d\tau\\
&\leqslant C\big(\big\|(\varphi_{0x},\psi_{0x})\big\|^2 + \big|(\varphi_0,\psi_0)\big|_{2,\omega}^2\big) + CL(t)^2.
\end{aligned} \tag{2.37}$$

First, we prove the estimate (2.8). Owing to the Poincaré type inequality

$$\big|\varphi(t,x)\big| \leqslant \big|\varphi(t,0)\big| + \sqrt{x}\big\|\varphi_x(t)\big\| \tag{2.38}$$

which is proved by the similar computation as in [4,10], substituting $\omega(x) = (1+x)^\beta$ and $\chi(t) = (1+t)^\xi$ in (2.37) for $\beta \in [0,\alpha]$ and $\xi \geqslant 0$ gives

$$\begin{aligned}
&(1+t)^\xi\big(\big\|(\varphi,\psi)(t)\big\|_1^2 + \big\|(\varphi,\psi)(t)\big\|_{\mathrm{a},\beta}^2\big) + \int_0^t (1+\tau)^\xi\big|(\varphi,\varphi_x)(\tau,0)\big|^2\,d\tau\\
&\quad + \int_0^t (1+\tau)^\xi\big(\beta\big\|(\varphi,\psi)\big\|_{\mathrm{a},\beta-1}^2 + \|\psi_x\|_{\mathrm{a},\beta}^2 + \big\|(\varphi_x,\psi_{xx})\big\|^2\big)\,d\tau\\
&\leqslant C\big(\big\|(\varphi_0,\psi_0)\big\|_1^2 + \big\|(\varphi_0,\psi_0)\big\|_{\mathrm{a},\beta}^2\big) + C\beta(\beta-1)\int_0^t (1+\tau)^\xi\|\psi\|_{\mathrm{a},\beta-2}^2\,d\tau
\end{aligned}$$

$$+ C\xi \int_0^t (1+\tau)^{\xi-1} \left(\left\| (\varphi, \psi) \right\|_{\mathrm{a},\beta}^2 + \left\| (\varphi_x, \psi_x) \right\|^2 \right) d\tau. \tag{2.39}$$

Therefore, applying an induction to (2.39) gives the desired estimate (2.8). Since this computation is similar those in [3,8,11], we omit the details.

Next, we prove the estimate (2.9). Substitute $\omega(x) = e^{\beta x}$ and $\chi(t) = e^{\alpha t}$ in (2.37) for $\beta < \lambda$ to obtain

$$\begin{aligned}
&e^{\alpha t} \left(\left\| (\varphi, \psi)(t) \right\|_{\mathrm{e},\beta}^2 + \left\| (\varphi_x, \psi_x)(t) \right\|^2 \right) + \int_0^t e^{\alpha\tau} \left| (\varphi, \varphi_x)(\tau, 0) \right|^2 d\tau \\
&\quad + \int_0^t e^{\alpha\tau} \left(\beta \left\| (\varphi, \psi)(\tau) \right\|_{\mathrm{e},\beta}^2 + \left\| (\varphi_x, \psi_{xx})(\tau) \right\|^2 + \left\| \psi_x(\tau) \right\|_{\mathrm{e},\beta}^2 \right) d\tau \\
&\leqslant C \left(\left\| (\varphi_0, \psi_0) \right\|_{\mathrm{e},\beta}^2 + \left\| (\varphi_{0x}, \psi_{0x}) \right\|^2 \right) + C\left(\alpha + \beta^2\right) \int_0^t e^{\alpha\tau} \left\| (\varphi, \psi)(\tau) \right\|_{\mathrm{e},\beta}^2 d\tau \\
&\quad + C\alpha \int_0^t e^{\alpha\tau} \left\| (\varphi_x, \psi_x)(\tau) \right\|^2 d\tau + C\delta_{\mathrm{S}} \int_0^t e^{\alpha\tau} \left(\varphi(\tau, 0)^2 + \left\| (\varphi_x, \psi_x)(\tau) \right\|^2 \right) d\tau.
\end{aligned} \tag{2.40}$$

Here, we have used the Poincaré type inequality (2.38) again. Thus, taking δ_{S}, β and α suitably small, we obtain the desired a priori estimate (2.9). □

2.2. *Transonic flow*

This subsection is devoted to prove the algebraic decay estimate for the transonic case $M_+ = 1$ in Theorem 1.2. To state the a priori estimate of the solution precisely, we use the notations:

$$N_1(t) := \sup_{0 \leqslant \tau \leqslant t} \left\| \left((1+x)^{\alpha/2} \varphi, (1+x)^{\alpha/2} \psi \right)(\tau) \right\|_1,$$

$$M_1(t)^2 := \int_0^t (1+\tau)^{\xi} \left\| (\varphi_x, \psi_x, \psi_{xx})(\tau) \right\|_{\mathrm{a},\beta}^2 d\tau.$$

Proposition 2.6. *Suppose that the same assumptions as in* (ii) *of Proposition* 1.1 *hold. Let* $(\varphi, \psi) \in X^1_{(1+x)^\alpha}(0, T)$ *be a solution to* (2.2), (2.3) *and* (2.4) *for certain positive constants* T *and* $\alpha \in [2, \alpha^*)$, *where* α^* *is defined in* (1.15). *Then there exist positive constants* ε_0 *and* C *such that if* $N_1(T) + \delta_{\mathrm{S}} < \varepsilon_0$, *then the solution* (φ, ψ) *satisfies the estimate*

$$(1+t)^{\alpha/2+\varepsilon} \left\| (\varphi, \psi) \right\|_1^2 + \int_0^t (1+\tau)^{\alpha/2+\varepsilon} \left| (\varphi, \varphi_x)(\tau, 0) \right|^2 d\tau$$

$$+\int_0^t (1+\tau)^{\alpha/2+\varepsilon}\big(\|\varphi_x\|^2+\|\psi_x\|_1^2\big)\,d\tau$$
$$\leqslant C\big\|(\varphi_0,\psi_0,\varphi_{0x},\psi_{0x})\big\|_{\mathrm{a},\alpha}^2(1+t)^{\varepsilon}. \tag{2.41}$$

In order to prove Proposition 2.6, we need to get a lower estimate for $\tilde{u}_x$.

Lemma 2.7. *The stationary solution $\tilde{u}(x)$ satisfies*

$$\tilde{u}_x(x)\geqslant A\left(\frac{u_+}{u_\mathrm{b}}\right)^{\gamma+2}\frac{\delta_\mathrm{S}^2}{(1+Bx)^2},$$
$$A:=\frac{(\gamma+1)\rho_+}{2\mu},\qquad B:=\delta_\mathrm{S}A \tag{2.42}$$

for $x\in(0,\infty)$.

Proof. Since we have $F(1)=F'(1)=0$ and $1<\tilde{w}<u_\mathrm{b}/u_+$, the function $F(\tilde{w})$ defined in (1.9) satisfies

$$\frac{\gamma+1}{2}c_+^2\rho_+\left(\frac{u_+}{u_\mathrm{b}}\right)^{\gamma+2}(\tilde{w}-1)^2\leqslant F(\tilde{w})\leqslant\frac{\gamma+1}{2}c_+^2\rho_+(\tilde{w}-1)^2. \tag{2.43}$$

Substituting the equality $F(\tilde{w})=\mu u_+(\tilde{w}-1)_x$ in (2.43) and solving the resultant differential inequality with respect to $\tilde{w}-1$ yield

$$\frac{\delta_\mathrm{S}}{|u_+|}\frac{1}{1+Bx}\leqslant\tilde{w}(x)-1\leqslant\frac{\delta_\mathrm{S}}{|u_+|}\frac{1}{1+(u_+/u_\mathrm{b})^{\gamma+2}Bx}. \tag{2.44}$$

Then, substituting (2.44) in (2.43) with the aid of (1.8) gives the desired estimate (2.42). □

We also need the estimate for the Mach number $\tilde{M}$ on the stationary solution $(\tilde{\rho},\tilde{u})$ defined by

$$\tilde{M}(x):=\frac{|\tilde{u}(x)|}{\sqrt{p'(\tilde{\rho}(x))}}. \tag{2.45}$$

Lemma 2.8. *There exists a positive constant C such that*

$$\frac{\gamma+1}{2|u_+|}\frac{\delta_\mathrm{S}}{1+Bx}-C\frac{\delta_\mathrm{S}^2}{(1+Bx)^2}\leqslant\tilde{M}(x)-1\leqslant C\frac{\delta_\mathrm{S}}{1+Bx}. \tag{2.46}$$

Proof. Owing to $M_+=1$ and (1.6), we see that the equality

$$\tilde{M}-1=\tilde{w}^{(\gamma+1)/2}-1=\frac{\gamma+1}{2}(\tilde{w}-1)+\frac{(\gamma+1)(\gamma-1)}{8}\eta^{(\gamma-3)/2}(\tilde{w}-1)^2 \tag{2.47}$$

holds for a certain $\eta \in (1, \tilde{w})$. Substituting (2.44) in (2.47) immediately yields the desired estimate (2.46). □

By using Lemmas 2.7 and 2.8, we obtain the weighted L^2 estimate of (φ, ψ).

Lemma 2.9. *There exists a positive constant ε_0 such that if $N_1(T) + \delta_S < \varepsilon_0$, then*

$$(1+t)^\xi \left\|(\varphi,\psi)\right\|_{a,\beta}^2 + \int_0^t (1+\tau)^\xi \left(\varphi(\tau,0)^2 + \beta\delta_S^2 \left\|(\varphi,\psi)\right\|_{a,\beta-2}^2 + \|\psi_x\|_{a,\beta}^2\right) d\tau$$
$$\leqslant C\left\|(\varphi_0,\psi_0)\right\|_{a,\beta}^2 + C\xi \int_0^t (1+\tau)^{\xi-1} \left\|(\varphi,\psi)\right\|_{a,\beta}^2 d\tau + C\delta_S \int_0^t (1+\tau)^\xi \|\varphi_x\|^2 d\tau \tag{2.48}$$

for $\beta \in [0,\alpha]$ and $\xi \geqslant 0$.

Proof. The equality (2.17) is written to

$$(W\rho\mathcal{E})_t + \left(-WG_1 - \mu W\psi\psi_x + \frac{\mu}{2}W_x\psi^2\right)_x + \mu W\psi_x^2 + W_x G_1 + G_2$$
$$= W_t\rho\mathcal{E} - \frac{\mu}{\tilde{\rho}} W\tilde{u}_{xx}\varphi\psi,$$
$$G_2 := W\tilde{u}_x\left(\rho\psi^2 + \left(p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})\varphi\right)\right) - \frac{\mu}{2}W_{xx}\psi^2, \tag{2.49}$$

where G_1 is defined in (2.15) and $W := (1+Bx)^\beta(1+t)^\xi$. By the same computation as in deriving (2.19), we rewrite the terms G_1 and G_2 to

$$G_1 = F_2 + R_5, \quad G_2 = F_3 + R_6,$$
$$F_2 := \left(\frac{p'(\rho_+)^{3/2}}{2\rho_+}\varphi^2 + \frac{\rho_+\sqrt{p'(\rho_+)}}{2}\psi^2\right)(\tilde{M}-1) + \frac{p'(\tilde{\rho})}{2\tilde{\rho}}\left(\sqrt{p'(\tilde{\rho})}\varphi - \tilde{\rho}\psi\right)^2,$$
$$R_5 := -\frac{\tilde{\rho}p'(\tilde{\rho})\tilde{u}}{2}\left(\frac{1}{\rho^2} - \frac{1}{\tilde{\rho}^2}\right)\varphi^2 - p(\tilde{\rho})\tilde{u}\left(\Phi\left(\frac{\tilde{\rho}}{\rho}\right) - \frac{\gamma}{2}\left(\frac{\tilde{\rho}}{\rho} - 1\right)^2\right)$$
$$- (\rho u - \tilde{\rho}\tilde{u})\mathcal{E} - \left(p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})\varphi\right)\psi$$
$$+ \left\{\left(\frac{p'(\tilde{\rho})^{3/2}}{2\tilde{\rho}} - \frac{p'(\rho_+)^{3/2}}{2\rho_+}\right)\varphi^2 + \left(\frac{\tilde{\rho}\sqrt{p'(\tilde{\rho})}}{2} - \frac{\rho_+\sqrt{p'(\rho_+)}}{2}\right)\psi^2\right\}(\tilde{M}-1),$$
$$F_3 := W\tilde{u}_x\left(\rho_+\psi^2 + \frac{1}{2}p''(\rho_+)\varphi^2\right) - \frac{\mu}{2}W_{xx}\psi^2,$$
$$R_6 := W\tilde{u}_x\left\{(\rho - \rho_+)\psi^2 + \frac{1}{2}\left(p''(\tilde{\rho}) - p''(\rho_+)\right)\varphi^2\right\}$$
$$+ W\tilde{u}_x\left(p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})\varphi - \frac{1}{2}p''(\tilde{\rho})\varphi^2\right). \tag{2.50}$$

By utilizing Lemmas 2.7 and 2.8 with the aid of the fact that $\beta < \alpha^*$, we obtain the lower estimate of $W_x F_2 + F_3$ as

$$\begin{aligned} W_x F_2 + F_3 \geqslant{}& \frac{K\gamma\rho_+^{\gamma-2}A}{4}\left\{(\gamma+1)\beta + 2\left(\frac{u_+}{u_\mathrm{b}}\right)^{\gamma+2}(\gamma-1)\right\}\delta_\mathrm{S}^2(1+t)^\xi(1+Bx)^{\beta-2}\varphi^2 \\ &+ \frac{\rho_+ A}{4}\left\{4\left(\frac{u_+}{u_\mathrm{b}}\right)^{\gamma+2} - (\gamma+1)\beta(\beta-2)\right\}\delta_\mathrm{S}^2(1+t)^\xi(1+Bx)^{\beta-2}\psi^2 \\ &- C\beta\delta_\mathrm{S}^3(1+t)^\xi(1+Bx)^{\beta-3}\left(\varphi^2+\psi^2\right) \\ \geqslant{}& c\delta_\mathrm{S}^2(1-C\delta_\mathrm{S})(1+t)^\xi(1+Bx)^{\beta-2}\left(\varphi^2+\psi^2\right) \end{aligned} \tag{2.51}$$

for $\beta \in (0, \alpha]$. On the other hand, the estimates (1.11), (2.21) and (2.46) yield

$$|W_x R_5 + R_6| \leqslant C\left(N_1(t) + \delta_\mathrm{S}^2\right)\delta_\mathrm{S}(1+t)^\xi(1+Bx)^{\beta-2}\left(\varphi^2+\psi^2\right). \tag{2.52}$$

Finally, integrate (2.49) over $\mathbb{R}_+ \times (0, t)$, substitute (2.51) and (2.52) in the resultant equality, and take $N_1(t)$ and δ_S suitably small to satisfy $N_1(t) \ll \delta_\mathrm{S}^2$ and $\delta_\mathrm{S} \ll 1$. This procedure yields the desired estimate (2.48) for $\beta \in (0, \alpha]$.

Next, we prove (2.48) for $\beta = 0$. Substituting $W = (1+t)^\xi$ in (2.49) and integrating the resultant equality over $\mathbb{R}_+ \times (0, t)$, we get

$$\begin{aligned} &(1+t)^\xi\left\|(\varphi,\psi)(t)\right\|^2 + \int_0^t (1+\tau)^\xi\left(\varphi(\tau,0)^2 + \|\psi_x\|^2\right)d\tau \\ &\quad\leqslant C\left\|(\varphi_0,\psi_0)\right\|^2 + C\xi\int_0^t (1+\tau)^{\xi-1}\left\|(\varphi,\psi)\right\|^2 d\tau \\ &\qquad + C\int_0^t (1+\tau)^\xi \int_0^\infty |\tilde{u}_{xx}\varphi\psi|\,dx\,d\tau. \end{aligned} \tag{2.53}$$

Here, we have used the fact that $G_2 \geqslant 0$ holds. Applying the Poincaré type inequality (2.38) to the third term on the right-hand side of (2.53) with the aid of (1.11b), we obtain the estimate (2.48) for the case of $\beta = 0$. □

In order to complete the proof of Proposition 2.6, we need to obtain the weighted estimate of (φ_x, ψ_x).

Lemma 2.10. *There exists a positive constant ε_0 such that if $N_1(T) + \delta_\mathrm{S} < \varepsilon_0$, then*

$$(1+t)^\xi\left\|(\varphi_x,\psi_x)\right\|_{\mathrm{a},\beta}^2 + \int_0^t (1+\tau)^\xi\left(\varphi_x(\tau,0)^2 + \left\|(\varphi_x,\psi_{xx})(\tau)\right\|_{\mathrm{a},\beta}^2\right)d\tau$$

$$\leqslant C\big\|(\varphi_0,\psi_0,\varphi_{0x},\psi_{0x})\big\|_{\mathrm{a},\beta}^2 + C\xi\int_0^t (1+\tau)^{\xi-1}\big\|(\varphi,\psi,\varphi_x,\psi_x)(\tau)\big\|_{\mathrm{a},\beta}^2\,d\tau \tag{2.54}$$

for $\beta\in[0,\alpha]$ and $\xi\geqslant 0$.

Proof. Since the derivation of the estimate (2.54) is similar to that of (2.23) and (2.31), we only give the outline of the proof. Multiplying (2.26) by $W=(1+Bx)^\beta(1+t)^\xi$, we have

$$\begin{aligned}
&\left\{W\left(\frac{\mu}{2}\varphi_x^2+\rho^2\varphi_x\psi\right)\right\}_t + \left\{W\left(\frac{\mu}{2}u\varphi_x^2-\rho^2\varphi_t\psi\right)\right\}_x + \rho p'(\rho)W\varphi_x^2\\
&\quad = W_t\left(\frac{\mu}{2}\varphi_x^2+\rho^2\varphi_x\psi\right) + W_x\left(\frac{\mu}{2}u\varphi_x^2-\rho^2\varphi_t\psi\right) + W(\mu R_2+R_3).
\end{aligned} \tag{2.55}$$

Integrating (2.55) over $\mathbb{R}_+\times(0,t)$ and substituting (2.48) gives the estimate for φ_x as

$$\begin{aligned}
&(1+t)^\xi\|\varphi_x\|_{\mathrm{a},\beta}^2 + \int_0^t (1+\tau)^\xi\big(\varphi_x(\tau,0)^2+\|\varphi_x\|_{\mathrm{a},\beta}^2\big)\,d\tau\\
&\quad\leqslant C\big\|(\varphi_0,\psi_0,\varphi_{0x})\big\|_{\mathrm{a},\beta}^2 + C\xi\int_0^t (1+\tau)^{\xi-1}\big\|(\varphi,\psi,\varphi_x)\big\|_{\mathrm{a},\beta}^2\,d\tau\\
&\qquad + C\big(N_1(t)+\delta_{\mathrm{S}}\big)M_1(t)^2.
\end{aligned} \tag{2.56}$$

Here, we have used the inequalities

$$\begin{aligned}
\big\{(1+Bx)^\beta\big\}_x\left|\frac{\mu}{2}u\varphi_x^2-\rho^2\varphi_t\psi\right| &\leqslant C\delta_{\mathrm{S}}(1+Bx)^\beta\varphi_x^2 + C(1+Bx)^\beta\psi_x^2\\
&\quad + C\beta(1+Bx)^{\beta-2}\big(\varphi^2+\psi^2\big),
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
(1+Bx)^\beta|\mu R_2+R_3| &\leqslant (\varepsilon+C\delta_{\mathrm{S}})(1+Bx)^\beta\varphi_x^2 + C_\varepsilon(1+Bx)^\beta\big(\psi_x^2+|\psi_x|\varphi_x^2\big)\\
&\quad + C_\varepsilon\delta_{\mathrm{S}}(1+Bx)^{\beta-4}\big(\varphi^2+\psi^2\big),
\end{aligned} \tag{2.58}$$

where ε an arbitrary positive constant. We note that the third term on the right-hand side of (2.58) is estimated by applying Poincaré type inequality (2.38) for the case of $\beta=0$. In deriving (2.56), we have also used the estimate for $|\psi_x|\varphi_x^2$ as

$$\int_0^\infty (1+Bx)^\beta|\psi_x|\varphi_x^2\,dx \leqslant C\|\psi_x\|_1\|\varphi_x\|_{\mathrm{a},\beta}^2 \leqslant CN_1(t)\big\|(\varphi_x,\psi_x,\psi_{xx})\big\|_{\mathrm{a},\beta}^2.$$

Next, we prove the estimate for ψ_x. Multiply (2.32) by $W=(1+t)^\xi(1+Bx)^\beta$ to get

$$\left(\frac{1}{2}W\rho\psi_x^2\right)_t - (W\rho\psi_x\psi_t)_x + \mu W\psi_{xx}^2 = \frac{1}{2}W_t\rho\psi_x^2 - W_x\rho\psi_x\psi_t + WR_4. \tag{2.59}$$

Integrate (2.59) in $\mathbb{R}_+ \times (0,t)$ and substitute (2.48) and (2.56) in the resultant equality with using the inequalities

$$\begin{aligned}&\big\{(1+Bx)^{\beta}\big\}_x|\rho\psi_x\psi_t| + (1+Bx)^{\beta}|R_4|\\&\quad\leqslant \varepsilon(1+Bx)^{\beta}\psi_{xx}^2 + C_\varepsilon(1+Bx)^{\beta}\big(\varphi_x^2+\psi_x^2+\psi_x^4+\varphi_x^2\psi_x^2\big) + C_\varepsilon\delta_{\mathrm{S}}(1+Bx)^{\beta-4}\big(\varphi^2+\psi^2\big)\end{aligned}$$

and

$$\int_0^\infty (1+Bx)^{\beta}\big(\psi_x^4+\varphi_x^2\psi_x^2\big)\,dx \leqslant C\|\psi_x\|_1^2\big\|(\varphi_x,\psi_x)\big\|_{\mathrm{a},\beta}^2 \leqslant CN_1(t)\big\|(\psi_x,\psi_{xx})\big\|_{\mathrm{a},\beta}^2.$$

This procedure yields

$$\begin{aligned}&(1+t)^{\xi}\|\psi_x\|_{\mathrm{a},\beta}^2 + \int_0^t (1+\tau)^{\xi}\|\psi_{xx}\|_{\mathrm{a},\beta}^2\,d\tau\\&\quad\leqslant C\big\|(\varphi_0,\psi_0,\varphi_{0x},\psi_{0x})\big\|_{\mathrm{a},\beta}^2 + C\xi\int_0^t (1+\tau)^{\xi-1}\big\|(\varphi,\psi,\varphi_x,\psi_x)\big\|_{\mathrm{a},\beta}^2\,d\tau\\&\qquad + C\big(N_1(t)+\delta_{\mathrm{S}}\big)M_1(t)^2.\end{aligned}\tag{2.60}$$

Finally, adding (2.56) to (2.60) and taking $N_1(t)+\delta_{\mathrm{S}}$ suitably small give the desired estimate (2.54). □

By the same inductive argument as in deriving (2.8), we can prove Proposition 2.6 which immediately yields the decay estimate (1.16).

References

[1] A.M. Il'in, O.A. Oleinik, Behavior of the solution of the Cauchy problem for certain quasilinear equations for unbounded increase of the time, Amer. Math. Soc. Transl. 42 (1964) 19–23.
[2] Y. Kagei, S. Kawashima, Stability of planar stationary solutions to the compressible Navier–Stokes equation on the half space, Comm. Math. Phys. 266 (2) (2006) 401–430.
[3] S. Kawashima, A. Matsumura, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion, Comm. Math. Phys. 101 (1985) 97–127.
[4] S. Kawashima, S. Nishibata, Stationary waves for the discrete Boltzmann equation in the half space with the reflective boundaries, Comm. Math. Phys. 211 (2000) 183–206.
[5] S. Kawashima, S. Nishibata, P. Zhu, Asymptotic stability of the stationary solution to the compressible Navier–Stokes equations in the half space, Comm. Math. Phys. 240 (2003) 483–500.
[6] T.-P. Liu, A. Matsumura, K. Nishihara, Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves, SIAM J. Math. Anal. 29 (1998) 293–308.
[7] A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system for compressible viscous gas, in: Proceedings of IMS Conference on Differential Equations from Mechanics, Hong Kong, 1999.
[8] A. Matsumura, K. Nishihara, Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity, Comm. Math. Phys. 165 (1994) 83–96.
[9] A. Matsumura, K. Nishihara, Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible gas, Comm. Math. Phys. 222 (2001) 449–474.

[10] Y. Nikkuni, S. Kawashima, Stability of stationary solutions to the half-space problem for the discrete Boltzmann equation with multiple collisions, Kyushu J. Math. 54 (2000) 233–255.
[11] M. Nishikawa, Convergence rate to the traveling wave for viscous conservation laws, Funkcial. Ekvac. 41 (1998) 107–132.