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Ricci flow on a 3-manifold with positive scalar curvature

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Abstract

In this paper we consider Hamilton's Ricci flow on a 3-manifold with a metric of positive scalar curvature. We establish several *a priori* estimates for the Ricci flow which we believe are important in understanding possible singularities of the Ricci flow. For Ricci flow with initial metric of positive scalar curvature, we obtain a sharp estimate on the norm of the Ricci curvature in terms of the scalar curvature (which is not trivial even if the initial metric has non-negative Ricci curvature, a fact which is essential in Hamilton's estimates [R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982) 255–306]), some L^2-estimates for the gradients of the Ricci curvature, and finally the Harnack type estimates for the Ricci curvature. These results are established through careful (and rather complicated and lengthy) computations, integration by parts and the maximum principles for parabolic equations.

Keywords: 3-manifolds; Ricci flow; Ricci curvature; Scalar curvature

1. Introduction

In the seminal paper [6], R. Hamilton has introduced an evolution equation for metrics on a manifold, the Ricci flow equation, in order to obtain a "better metric" by deforming metrics in a way of improving positivity of the Ricci curvature. R. Hamilton devised his heat flow type equation (originally motivated by Eells–Sampson's work [5] on the heat flow for harmonic mappings) by considering the gradient vector field of the total scalar curvature $E(g) = \int_M R_g \, d\mu_g$ on the space of metrics on a manifold M, where R_g and μ_g denote the scalar curvature and volume

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measure with respect to the metric g. Hamilton's Ricci flow is *not* the gradient flow of E, which proves ill-posed, but the normalized gradient flow of E:

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \frac{2}{n}\sigma_g g_{ij} \tag{1.1}$$

where σ_g is the average $V_g^{-1}\int_M R\,\mathrm{d}\mu_g$ of the scalar curvature. We refer to T. Aubin [1] for an excellent survey on the fundamental results on the Ricci flow. After change of the space variable scale and re-parametrization of t (1.1) is equivalent to

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij}. \tag{1.2}$$

We will refer this equation as the *Ricci flow*. The parameter t (which has no geometric significance) is suppressed from notations if no confusion may arise.

The system (1.2) has a great interest by its own from a point-view of PDE theory, it however has significance in the resolution of the Poincaré conjecture. If we run the Ricci flow (1.2) on a 3-manifold, and if we could show a limit metric exists with controlled bounds of the curvature tensor, then we could hope the limit metric must have constant Ricci curvature (see (1.1)), therefore the manifold must be a sphere. Thus Ricci flow is a very attractive approach to a possible positive answer to the Poincaré conjecture. Indeed this program, initialed by R. Hamilton, has been essentially completed by G. Perelman in [9,10]. In these papers Perelman has presented powerful and substantial new ideas to understand the singularities of solutions to (1.1). For further information, see [3,4,8], Kleiner and Lott's notes [7], the long article [2], and the further coming work by G. Besson and etc. This approach was worked out in the classical paper [6] for 3-manifolds with positive Ricci curvature by proving a series of striking *a priori* estimates for solutions of the Ricci flow.

Theorem 1.1. *(See Hamilton [6], Main Theorem 1.1, p. 255.) On any* 3*-manifold with a metric of positive Ricci curvature, there is an Einstein metric with positive scalar curvature.*

1.1. Description of main results

R. Hamilton proved Theorem 1.1 through three *a priori* estimates for the Ricci flow with initial metric of positive Ricci curvature. The essential feature in this particular case is that the scalar curvature dominates the curvature tensor. Indeed if the Ricci tensor R_{ij} is positive then

$$\frac{1}{3}R^2 \leqslant |R_{ij}|^2 \leqslant R^2. \tag{1.3}$$

In general we still have a lower bound for $|R_{ij}|$, but there is no control on $|R_{ij}|$ by the scalar curvature R. The first estimate Hamilton proved is an expected one: the Ricci flow improves positivity of the Ricci tensor. If the initial metric has positive Ricci curvature then it remains so as long as the Ricci flow alive. This conclusion is proved by using a maximum principle for solutions to tensor type parabolic equations.

One of our results shows that under Ricci flow on a 3-manifold with positive scalar curvature, the squared norm of the Ricci tensor can be controlled in terms of its scalar curvature and the initial data. Indeed we prove a comparison theorem for the quantity $|R_{ij}|^2/R^2$. We then deduce precise bounds on the scalar curvature for the Ricci flow with positive scalar curvature.

By removing the positivity assumption on the Ricci curvature, three eigenvalues of the Ricci curvature under the Ricci flow may develop into a state of dispersion: one of eigenvalues may go to $-\infty$ while another to $+\infty$ but still keep the scalar curvature R bounded. Our above result just excludes this case if the initial metric has positive scalar curvature.

The second key estimate in [6], also the most striking one, is an estimate which shows (after re-parametrization) the eigenvalues of the Ricci flow at each point approach each other. R. Hamilton achieved this by showing the variance of three eigenvalues of the Ricci tensor decay like R^κ where $\kappa \in (1,2)$ depending on the positive lower bound of the Ricci curvature of the initial metric. It is not easy to see that the curvature explodes in finite time for normalized Ricci flow (the volumes of the manifold are scaled to tend to zero), the variance of three eigenvalues of the Ricci curvature are easily seen as $|R_{ij}|^2 - R^2/3$, by (1.3) one might guess the variance $|R_{ij}|^2 - R^2/3$ has the same order as R^2. The striking fact is that indeed $(|R_{ij}|^2 - R^2/3)/R^\kappa$ is bounded for some $\kappa < 2$! That is to say, $|R_{ij}|^2 - R^2/3$ explodes much slower than R^2, so that, after re-scaling back, it shows that the variance of the eigenvalues of the Ricci tensor goes to zero. The positive Ricci curvature assumption is washed down to an elementary fact recorded in Lemma 4.2. This *core estimate* (see [6], Theorem 10.1, p. 283) is definitely false if the positivity assumption on the Ricci tensor is removed, and it seems no replacement could be easily recognized without an assumption on the Ricci curvature.

Finally in order to show a smooth metric does exist, and has constant Ricci curvature, R. Hamilton [6] established an important gradient estimate for $|\nabla R|$ in terms of R and $|R_{ij}|$ ([6], Theorem 11.1, p. 287), which in turn implies that, for the Ricci flow, $|\nabla R|$ goes to zero. This estimate was proved by using the previous crucial estimate on the variance $|R_{ij}|^2 - R^2/3$ and a clever use of the Bianchi identity: $|\nabla \operatorname{Ric}|^2 \geqslant 7|\nabla R|^2/20$, instead of the trivial one $|\nabla \operatorname{Ric}|^2 \geqslant |\nabla R|^2/3$. The key question one would ask is what kind of gradient estimates for the scalar curvature and for the Ricci curvature can we expect without the essential estimates on the variance of three eigenvalues of the Ricci tensor? This paper gives some partial answers to this question: we establish a weighted integral estimate for $|\nabla R|^2$ and an Harnack estimate for the Ricci curvature. We hope these estimates would help us to further understand the singularities in the Ricci flow on a 3-manifold with positive scalar curvature.

2. Hamilton's Ricci flow

Although most of our results will be stated only for the normalized Ricci flow, it may be useful to consider a general evolution equation. Let $\mathcal{M}(M)$ denote the space of metrics on a manifold M of dimension 3, and let $h_{ij} = R_{ij} - \frac{1}{3}(\alpha\sigma + \beta)g_{ij}$, where α, β are two constants, R_{ij} is the Ricci tensor of g_{ij}, and σ is the mean value of the scalar curvature, i.e. $\sigma = \frac{1}{V_g}\int_M R_g \,\mathrm{d}\mu_g$, here, the lower script, which will be omitted if no confusion may arise, indicates quantities which are computed with respect to a metric $g = (g_{ij})$, μ_g is the volume measure and V_g denotes the total volume. Then $\operatorname{tr}_g(h_{ij}) = R - (\alpha\sigma + \beta)$. The Ricci flow (with parameters α and β) is the following evolution equation of metrics:

$$\frac{\partial}{\partial t} g_{ij} = -2h_{ij}. \tag{2.1}$$

In what follows we assume that $(g(t)_{ij})$ (but t will be suppressed from notations, unless otherwise specified) is the maximum solution of (2.1). For simplicity we use $A(t)$ to denote $\alpha\sigma(t) + \beta$.

Let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ denote the three eigenvalues of the Ricci tensor (R_{ij}). Then $R = \sum_j \lambda_j$ and $S = \sum_j \lambda_j^2 = |R_{ij}|^2$. Set $T = \sum_j \lambda_j^3$ and $U = \sum_j \lambda_j^4$. Other symmetric functions of λ_j are given as

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 = \frac{1}{2}\big(R^2 - S\big),$$
$$\lambda_1\lambda_2\lambda_3 = \frac{1}{6}R^3 - \frac{1}{2}RS + \frac{1}{3}T.$$

Clearly

$$U = \frac{4}{3}RT - R^2S + \frac{1}{2}S^2 + \frac{1}{6}R^4.$$

The variance of λ_1, λ_2 and λ_3 is $S - \frac{1}{3}R^2$ which will be denoted by Y. Let $S_{ij} = g^{ab}R_{ib}R_{aj}$ and $T_{ij} = g^{ab}R_{ib}S_{aj}$. Then $S = \operatorname{tr}_g(S_{ij})$ and $T = \operatorname{tr}_g(T_{ij})$.

It is known (see for example [6]) that

$$\left(\Delta - \frac{\partial}{\partial t}\right)R_{ij} = 6S_{ij} - 3RR_{ij} + \big(R^2 - 2S\big)g_{ij} \tag{2.2}$$

which has a form independent of α or β, and

$$\left(\Delta - \frac{\partial}{\partial t}\right)R = -2\left(S - \frac{1}{3}AR\right). \tag{2.3}$$

Furthermore one can show that

$$\left(\Delta - \frac{\partial}{\partial t}\right)S_{ij} = 10T_{ij} - 6RS_{ij} + 2\big(R^2 - 2S\big)R_{ij} + \frac{2}{3}AS_{ij} + 2g^{ab}(\nabla_k R_{ib})\big(\nabla^k R_{aj}\big) \tag{2.4}$$

so that

$$\left(\Delta - \frac{\partial}{\partial t}\right)S = 2\big(R^3 - 5RS + 4T\big) + \frac{4}{3}AS + 2|\nabla_k R_{ij}|^2. \tag{2.5}$$

One may continue the above computation to obtain

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)T_{ij} = {}& 14g^{ab}R_{ib}T_{aj} - 9RT_{ij} + 3\big(R^2 - 2S\big)S_{ij} \\ & + \frac{4}{3}AT_{ij} + 2R^{ab}\big(\nabla^k R_{ib}\big)(\nabla_k R_{aj}) \\ & + 2\big(\nabla^k R^{ab}\big)\big\{R_{ib}(\nabla_k R_{aj}) + R_{aj}(\nabla_k R_{ib})\big\}\end{aligned} \tag{2.6}$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right)T = 7RT - 9R^2S + 2R^4 + 2AT + 6g^{ab}R_{ia}\big(\nabla^k R^{ij}\big)(\nabla_k R_{bj}). \tag{2.7}$$

In what follows, we only consider a 3-manifold with a metric of positive scalar curvature. Let $(g(t)_{ij})$ be the maximum solution of the Ricci flow with initial metric $g(0)_{ij}$ of positive *constant* scalar curvature $R(0)$, unless otherwise specified. Let μ_t denote the volume measure associated with the solution metric $(g(t)_{ij})$ (at time t), and $M_t = \mathrm{d}\mu_t/\mathrm{d}\mu_0$. Then it is easy to see that

$$\frac{\partial}{\partial t}\log M_t = -\operatorname{tr}_g(h_{ij}) = -R + A.$$

3. Control the curvature tensor

For any α, β, the scalar curvature R remains positive if the initial metric possesses positive scalar curvature. To see this let us consider the function $K = e^{\frac{2}{3}\int_0^t A(s)\,ds} R$. It is easy to see that

$$\left(\Delta - \frac{\partial}{\partial t}\right) K = -2e^{\frac{2}{3}\int_0^t A(s)\,ds} S \tag{3.1}$$

so the claim follows from the maximum principle. Our next result shows that if the initial metric possesses positive scalar curvature, then the full curvature tensor of $(g(t)_{ij})$ may be controlled by its scalar curvature.

Let $V_{ij} = R_{ij} - \varepsilon R g_{ij}$ where ε is a constant. Then

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right) V_{ij} = {} & 6g^{pq} V_{pj} V_{iq} + (10\varepsilon - 3) R V_{ij} \\ & + \left(2(\varepsilon - 1) S + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2\right) g_{ij}.\end{aligned} \tag{3.2}$$

The very nice feature of this identity is that it involves α, β and σ only through the symmetric tensor V_{ij}, which takes the same form for all α, β.

Theorem 3.1. *Let M be a closed 3-manifold. If $\varepsilon \leqslant 1/3$ is a constant such that $R_{ij} - \varepsilon R g_{ij} \geqslant 0$ at $t = 0$, so does it remain. If $\varepsilon \geqslant 1$ and at $t = 0$, $R_{ij} - \varepsilon R g_{ij} \leqslant 0$, then the inequality remains to hold for all $t > 0$.*

Proof. We prove these conclusions by using the maximum principle to the tensor type parabolic equation (3.2) and (V_{ij}) (see [6], Theorem 9.1, p. 279). Let us prove the first conclusion. Suppose $\lambda \geqslant \mu \geqslant \nu$ are eigenvalues of the Ricci tensor R_{ij}. Then $S = \lambda^2 + \mu^2 + \nu^2$ and $R = \lambda + \mu + \nu$. If $\xi \neq 0$ such that $V_{ij}\xi^j = 0$ for all i. Then one of the eigenvalues of V_{ij} is zero. Since the eigenvalues of V_{ij} are $\lambda - \varepsilon R$, $\mu - \varepsilon R$, $\nu - \varepsilon R$, so we may assume that $\nu - \varepsilon R = 0$. Hence

$$\lambda + \mu = (1 - \varepsilon) R, \qquad S = \lambda^2 + \mu^2 + \varepsilon^2 R^2.$$

Since

$$\begin{aligned} & 2(\varepsilon - 1) S + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\ & \quad = 2(\varepsilon - 1)\left(\lambda^2 + \mu^2 + \varepsilon^2 R^2\right) + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\ & \quad = 2(\varepsilon - 1)\left(\lambda^2 + \mu^2\right) + 2(\varepsilon - 1)\varepsilon^2 R^2 + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\ & \quad \leqslant (\varepsilon - 1)(\lambda + \mu)^2 + 2(\varepsilon - 1)\varepsilon^2 R^2 + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\ & \quad = (\varepsilon - 1)(1 - \varepsilon)^2 R^2 + 2(\varepsilon - 1)\varepsilon^2 R^2 + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \\ & \quad = (3\varepsilon - 1)\varepsilon^2 R^2 \leqslant 0, \end{aligned}$$

where the first inequality follows from the Cauchy inequality and the last one follows our assumption $\varepsilon \leqslant 1/3$. Now the conclusion follows from Eq. (3.2) and the maximum principle.

To show the second conclusion, we apply the maximum principle to $-V_{ij}$ and use the fact that $\varepsilon \geqslant 1$. The only thing then one should notice is the following inequality

$$2(\varepsilon - 1) S + \left(4\varepsilon^2 - 3\varepsilon + 1\right) R^2 \geqslant 7R^2/16 \geqslant 0. \qquad \square$$

Corollary 3.2. *On a closed* 3*-manifold* M *with positive scalar curvature* $R(0) > 0$ (*so that* $R(t) > 0$ *for all* t).

1) *If* $R(0)_{ij} \geqslant 0$*, then* $R^2/3 \leqslant S \leqslant R^2$*. If* $\varepsilon > 0$ *such that*

$$R(0)_{ij} \geqslant -\varepsilon R(0) g(0)_{ij}$$

then

$$R^2/3 \leqslant S \leqslant \left(1 + 4\varepsilon + 6\varepsilon^2\right) R^2. \tag{3.3}$$

2) *If* $R(0)_{ij} \geqslant b R(0) g(0)_{ij}$ *for some constant* $0 \leqslant b \leqslant 1/3$*, then*

$$R^2/3 \leqslant S \leqslant \left(1 - 4b + 6b^2\right) R^2.$$

Proof. By Theorem 3.1, $R_{ij} \geqslant -\varepsilon R g_{ij}$ as long as the solution exists. Let $\lambda \geqslant \mu \geqslant \nu$ be the eigenvalues of the Ricci tensor R_{ij}. Then $\lambda, \mu, \nu \geqslant -\varepsilon R$. Suppose $\mu, \nu < 0$, then

$$\begin{aligned}\lambda^2 + \mu^2 + \nu^2 &\leqslant \left(R + |\mu| + |\nu|\right)^2 + |\mu|^2 + |\nu|^2 \\ &= (R + 2\varepsilon R)^2 + 2\varepsilon^2 R^2 \\ &= \left(1 + 4\varepsilon + 6\varepsilon^2\right) R^2,\end{aligned}$$

and similarly when $\lambda \geqslant \mu \geqslant 0$ and $\nu < 0$ we have

$$\begin{aligned}\lambda^2 + \mu^2 + \nu^2 &\leqslant (\lambda + \mu)^2 + \nu^2 \\ &= \left(R + |\nu|\right)^2 + |\nu|^2 \\ &\leqslant (R + \varepsilon R)^2 + \varepsilon^2 R^2\end{aligned}$$

and while $\nu \geqslant 0$ we then clearly have $\lambda^2 + \mu^2 + \nu^2 \leqslant R^2$. Therefore the conclusion follows. □

The estimate (3.3) shows that the scalar curvature dominates the full curvature if the initial metric has positive scalar curvature, which however is a rough estimate. We will give a sharp estimate in next section (Theorem 4.4 below).

4. Control the norm of the Ricci tensor

In this section we work with the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ on a 3-manifold with initial metric such that $R(0) > 0$. Thus $R > 0$ for all time t. The variance of three eigenvalues of the Ricci tensor (R_{ij}), $Y \equiv S - \frac{1}{3}R^2$ is non-negative. The third variable in our mind is not T whose sign is difficulty to determined. Motivated by the fundamental work [6], we consider the polynomial of the eigenvalues of (R_{ij}):

$$P = S^2 + \frac{1}{2}R^4 - \frac{5}{2}R^2 S + 2RT \tag{4.1}$$

as the third independent variable, which is non-negative, as showed in [6].

Let us begin with a geometric explanation about the polynomial P. In terms of the three eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ of the Ricci tensor (R_{ij}) ([6], Lemma 10.6, p. 285)

$$P = (\lambda_1 - \lambda_2)^2\left[\lambda_1^2 + (\lambda_1 + \lambda_2)(\lambda_2 - \lambda_3)\right] + \lambda_3^2(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3). \tag{4.2}$$

Hence we establish the following

Lemma 4.1. *$P \geqslant 0$ and $P = 0$ if and only if 1) $\lambda_1 = \lambda_2 = \lambda_3$, or 2) one of eigenvalues λ_i is zero, the other two are equal number.*

Together with this lemma, the following lemma explains why the assumption of positive Ricci curvature is special.

Lemma 4.2. *Suppose that $\lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$ and*

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \varepsilon(\lambda_1 + \lambda_2 + \lambda_3) \tag{4.3}$$

for some $\varepsilon \in [0, 1/3]$, then

$$P \geqslant \varepsilon^2 S\big(S - R^2/3\big). \tag{4.4}$$

Therefore all evolution equations should be written in terms of R, Y and P. For example

$$\left(\Delta - \frac{\partial}{\partial t}\right) Y = 2R^3 - \frac{26}{3} RS + 8T + 2|\nabla_k R_{ij}|^2 - \frac{2}{3}|\nabla R|^2.$$

We need the following lemma, which is proved in ([6], Theorem 10.1, p. 283), though in a different form.

Lemma 4.3. *For any constant κ, set $L_\kappa = \Delta + 2(\kappa - 1)\nabla \log R.\nabla$. Then*

$$R^\kappa \left(L_\kappa - \frac{\partial}{\partial t}\right) \frac{Y}{R^\kappa} = \left[P - \frac{2-\kappa}{2} SY\right] \frac{4}{R} + (2-\kappa)(\kappa - 1)\frac{Y|\nabla R|^2}{R} + 2\left|\nabla_k R_{ij} - \frac{R_{ij}}{R}\nabla_k R\right|^2. \tag{4.5}$$

In particular for any $\kappa \in [1, 2]$, the following differential inequality holds

$$R^\kappa \left(L_\kappa - \frac{\partial}{\partial t}\right) \frac{Y}{R^\kappa} \geqslant \left(P - \frac{2-\kappa}{2} SY\right) \frac{4}{R}. \tag{4.6}$$

Our only contribution here is that, if the scalar curvature is positive, then (4.6) allows us to derive a sharp estimate on the scalar function $|R_{ij}|^2$. Indeed, choose $\kappa = 2$, then

$$R^2 \left(L_2 - \frac{\partial}{\partial t}\right) \frac{Y}{R^2} \geqslant \frac{4}{R} P \geqslant 0$$

hence we have

Theorem 4.4. *Under the Ricci flow, and suppose $R(0) > 0$. Then as long as the Ricci flow exists we always have*

$$\frac{S}{R^2} \leqslant \max_M \frac{S(0)}{R(0)^2}. \tag{4.7}$$

The estimate in this theorem is sharp.

4.1. Gradient estimate for scalar curvature

Next we want to treat the gradient of the Ricci tensor, begin with the scalar curvature R.

Lemma 4.5. *Let $B_F = (\Delta - \frac{\partial}{\partial t})F$ where F is a scalar function, $\Delta = \Delta_{g(t)}$, and $g(t)$ evolves according to the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$. Then*

$$\left(\Delta - \frac{\partial}{\partial t}\right)\frac{|\nabla F|^2}{R} = \frac{2}{R}\langle \nabla B_F, \nabla F\rangle + \frac{2S}{R^2}|\nabla F|^2 + \frac{2}{R}\left|\nabla_k\nabla_l F - R^{-1}(\nabla_k R)(\nabla_l F)\right|^2. \tag{4.8}$$

Proof. According to the Bochner identity and the Ricci flow

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla F|^2 = 2|\nabla\nabla F|^2 + 2\langle \nabla B_F, \nabla F\rangle, \tag{4.9}$$

together with the chain rule

$$\left(\Delta - \frac{\partial}{\partial t}\right)\frac{|\nabla F|^2}{R} = \frac{2}{R}\langle \nabla B_F, \nabla F\rangle + \left\{\frac{2S}{R^2} + \frac{2}{R^3}|\nabla R|^2\right\}|\nabla F|^2 + \frac{2}{R}|\nabla\nabla F|^2 - \frac{2}{R^2}\left\langle \nabla R, \nabla|\nabla F|^2\right\rangle. \tag{4.10}$$

In order to use the hessian term which is non-negative, we observe

$$\nabla_k |\nabla F|^2 = 2g^{ab}(\nabla_a F)(\nabla_k \nabla_b F)$$

so that

$$\left\langle \nabla R, \nabla|\nabla F|^2\right\rangle = 2\left(\nabla^k R\right)(\nabla^l F)(\nabla_k\nabla_l F).$$

It follows that

$$|\nabla\nabla F|^2 = \left|\nabla_k\nabla_l F - a(\nabla_k R)(\nabla_l F)\right|^2 - a^2|\nabla R|^2|\nabla F|^2 + a\left\langle \nabla R, \nabla|\nabla F|^2\right\rangle.$$

Therefore

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)\frac{|\nabla F|^2}{R} = {}& \frac{2}{R}\langle \nabla B_F, \nabla F\rangle + \left\{\frac{2S}{R^2} + \frac{2}{R^3}|\nabla R|^2\right\}|\nabla F|^2 \\ &+ \frac{2}{R}\left|\nabla_k\nabla_l F - a(\nabla_k R)(\nabla_l F)\right|^2 \\ &- \frac{2a^2}{R}|\nabla R|^2|\nabla F|^2 + a\frac{2}{R}\left\langle \nabla R, \nabla|\nabla F|^2\right\rangle \\ &- \frac{2}{R^2}\left\langle \nabla R, \nabla|\nabla F|^2\right\rangle.\end{aligned} \tag{4.11}$$

In particular, to eliminate the term $\langle \nabla R, \nabla|\nabla F|^2\rangle$, choose $a = 1/R$ and the conclusion follows. □

Let $(X_t, \mathbb{P}^{s,x})$ be the diffusion process associated with the time-dependent elliptic operator $\frac{1}{2}\Delta$. Then we have

Theorem 4.6. *Under the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ with initial metric of positive scalar curvature $R(0) > 0$. Then*

$$H(t,x) \leqslant \mathbb{P}^{0,x}\left(H(0,X_s)\mathrm{e}^{2\delta\int_0^t R(s,X_s)\,\mathrm{d}s}\right), \tag{4.12}$$

where $H = \frac{|\nabla R|^2}{R} + 84S$. In particular if $R(t,\cdot) \leqslant \theta(t)$ on M, then

$$H(t,x) \leqslant \mathrm{e}^{2\delta\int_0^t \theta(s)\,\mathrm{d}s} \max_M H(0,\cdot).$$

Proof. In particular, if we apply the formula to the scalar curvature R, we establish

$$\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)\frac{|\nabla R|^2}{R} = -\frac{2}{R}\langle \nabla S, \nabla R\rangle + \frac{S}{R^2}|\nabla R|^2 + \frac{1}{R}\left|\nabla_k\nabla_l R - R^{-1}(\nabla_k R)(\nabla_l R)\right|^2. \tag{4.13}$$

Together with the evolution equations for R^2 and Y one deduces that

$$\begin{aligned}\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)H = {} & -\frac{2}{R}\langle \nabla S, \nabla R\rangle + \frac{1}{R}\left|\nabla_k\nabla_l R - R^{-1}(\nabla_k R)(\nabla_l R)\right|^2 \\ & + \left[2\xi\frac{P}{R} - \frac{2Y}{R}\left(\xi S + \eta R^2\right) - \frac{2}{3}\eta R^3\right] \\ & + \left\{\xi|\nabla_k R_{ij}|^2 - \frac{1}{3}\xi|\nabla R|^2 + \eta|\nabla R|^2 + \frac{S}{R^2}|\nabla R|^2\right\}\end{aligned}$$

where

$$H = \frac{|\nabla R|^2}{R} + \xi Y + \eta R^2$$

where ξ and η are two constants. In this formula, the only term we have to deal is $\langle \nabla S, \nabla R\rangle$, which we handle as the following.

$$|b\nabla_k R_{ij} - aR_{ij}\nabla_k R|^2 = b^2|\nabla_k R_{ij}|^2 + a^2 S|\nabla R|^2 - ab\langle \nabla S, \nabla R\rangle$$

in which we set $ab = \frac{2}{R}$, i.e. $a = \frac{2}{bR}$, and thus

$$-\frac{2}{R}\langle \nabla S, \nabla R\rangle = \left|b\nabla_k R_{ij} - \frac{2}{bR}R_{ij}\nabla_k R\right|^2 - b^2|\nabla_k R_{ij}|^2 - \frac{4}{b^2}\frac{S}{R^2}|\nabla R|^2.$$

Therefore

$$\begin{aligned}\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)H = {} & \frac{1}{R}\left|\nabla_k\nabla_l R - R^{-1}(\nabla_k R)(\nabla_l R)\right|^2 + \left|b\nabla_k R_{ij} - \frac{2}{bR}R_{ij}\nabla_k R\right|^2 \\ & + \left[2\xi\frac{P}{R} - \frac{2Y}{R}\left(\xi S + \eta R^2\right) - \frac{2}{3}\eta R^3\right] + \left(\xi - b^2\right)|\nabla_k R_{ij}|^2 \\ & - \frac{1}{3}\xi|\nabla R|^2 + \eta|\nabla R|^2 + \left(1 - \frac{4}{b^2}\right)\frac{S}{R^2}|\nabla R|^2.\end{aligned} \tag{4.14}$$

We next need to decide the signs of three constants ξ, η and b. It is suggested that $\xi \geqslant b^2$ (otherwise we can not control $|\nabla_k R_{ij}|^2$). Under this choice, and we are expected with the good choices of these three constants, we will lose nothing from the first two terms on the right-hand side of Eq. (4.14), we thus simply drop these two, and use the inequality ([6], Lemma 11.6, p. 288) $|\nabla_k R_{ij}|^2 \geqslant \frac{7}{20}|\nabla R|^2$, so that

$$\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)H \geqslant 2\xi\frac{P}{R} - \frac{2Y}{R}\left(\xi S + \eta R^2\right) - \frac{2}{3}\eta R^3 + \left(\frac{1}{60}\xi - \frac{7}{20}b^2 + \eta\right)|\nabla R|^2 + \left(1 - \frac{4}{b^2}\right)\frac{S}{R^2}|\nabla R|^2. \tag{4.15}$$

Obviously a simple choice for b is $b = 2$ so that the last term in (4.15) is dropped. Thus

$$\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)H \geqslant 2\xi\frac{P}{R} - \frac{2}{3}\eta R^3 - 2R\left(\frac{S}{R^2} + \frac{\eta}{\xi}\right)\xi Y + \left(\frac{1}{60}\xi - \frac{7}{5} + \eta\right)|\nabla R|^2 \tag{4.16}$$

(by the way, this inequality is enough to prove Hamilton's estimate [6], Theorem 11.1, p. 287). Substituting Y by H one obtains

$$\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)H \geqslant 2\xi\frac{P}{R} + 2\left(\delta + \frac{\eta}{\xi} - \frac{1}{3}\right)\eta R^3 - 2R\left(\delta + \frac{\eta}{\xi}\right)H + \left(\frac{1}{60}\xi + 2\delta + 2\frac{\eta}{\xi} - \frac{7}{5} + \eta\right)|\nabla R|^2 \tag{4.17}$$

in which we choose $\eta = 0$ and $\xi = 84$ to obtain

$$\frac{1}{2}\left(\Delta - \frac{\partial}{\partial t}\right)H \geqslant -2\delta RH. \tag{4.18}$$

The estimate in the theorem follows from the Kac formula. □

5. Estimates on the scalar curvature

Thanks to the resolution of the Yamabe problem (e.g., for a 3-manifold with a metric of positive scalar curvature, we may run the general Ricci flow with an initial metric $(g(0)_{ij})$ such that $R(0)$ is a positive constant. Under such an initial metric, and if we run the Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2\left(R_{ij} - \frac{1}{3}\sigma g_{ij}\right)$$

(i.e. for the case $\alpha = 1$ and $\beta = 0$), then all volume measures μ_t associated with the Ricci flow $(g(t)_{ij})$ have the same volume. We may choose a non-negative constant ε such that $R(0)_{ij} \geqslant -\varepsilon R(0)g(0)_{ij}$, which is possible since $R(0) > 0$. We will of course choose the least one $\varepsilon \geqslant 0$ for a given initial data. Then $R > 0$ as long as the Ricci flow exists, and by Theorem 3.1 $R(t)_{ij} \geqslant -\varepsilon R(t)g(t)_{ij}$ for all t. Furthermore there is a constant $\delta \geqslant \frac{1}{3}$ such that

$$R(t)^2/3 \leqslant S(t) \leqslant \delta R(t)^2. \tag{5.1}$$

For example, any $\delta \geqslant 1 + 4\varepsilon + 6\varepsilon^2$ will do. In the case of the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$, the optimal choice for δ is the maximum of $S(0)/R(0)^2$.

We may deduce several elementary estimates on the scalar curvature R. Define $\rho(t) = \exp(-\frac{3}{2}\int_0^t A(s)\,\mathrm{d}s)$.

Theorem 5.1. *Under the general Ricci flow with initial metric having positive constant scalar curvature* $R(0)$ *and*

$$\rho(t) = \exp\left(-\frac{3}{2}\int_0^t A(s)\,\mathrm{d}s\right).$$

Then

$$\int_0^t \rho(s)\,\mathrm{d}s < \frac{3}{2}\frac{1}{R(0)}$$

and

$$\frac{3R(0)\rho(t)}{3-2R(0)\int_0^t \rho(s)\,\mathrm{d}s} \leqslant R(t) \leqslant \frac{R(0)\rho(t)}{1-2\delta\int_0^t \rho(s)\,\mathrm{d}s}$$

as long as the Ricci flow $(g(t)_{ij})$ *exists.*

Proof. Let $\psi(r) = \mathrm{e}^{-\frac{3}{2}\frac{1}{r}}$ and set

$$F = \psi(K)\exp\left\{-\int_0^t \rho(s)\,\mathrm{d}s\right\},$$

where K is given as in (3.1). Then, one can show that

$$\left(\Delta - \frac{\psi''(K)}{\psi'(K)}\nabla K - \frac{\partial}{\partial t}\right)F \leqslant 0. \tag{5.2}$$

By the maximum principle applying to (5.2) and the fact that $R(0)$ is constant, we have

$$\frac{3}{2}\frac{\rho(t)}{R(t)} + \int_0^t \rho(s)\,\mathrm{d}s \leqslant \frac{3}{2}\frac{1}{R(0)}.$$

Since we always have $\frac{\rho(t)}{R(t)} > 0$ so that

$$\int_0^t \rho(s)\,\mathrm{d}s \leqslant \frac{3}{2}\frac{1}{R(0)}$$

for all t and

$$R(t) \geqslant \frac{3R(0)\rho(t)}{3-2R(0)\int_0^t \rho(s)\,\mathrm{d}s}$$

for all t provides the Ricci flow exists.

Similarly, since $S \leqslant \delta R^2$ we consider $\psi(r) = \mathrm{e}^{-\frac{1}{2\delta}\frac{1}{r}}$ and

$$F = \psi(K)\exp\left\{-\int_0^t \rho(s)\,\mathrm{d}s\right\}.$$

Then

$$\left(\Delta - \frac{\psi''(K)}{\psi'(K)}\nabla K - \frac{\partial}{\partial t}\right)F \geqslant 0, \tag{5.3}$$

so that the maximum principle then implies that

$$\frac{\rho(t)}{2\delta R} + \int_0^t \rho(s)\,\mathrm{d}s \geqslant \frac{1}{2\delta R(0)}$$

and therefore

$$R(t) \leqslant \frac{R(0)\rho(t)}{1 - 2\delta\int_0^t \rho(s)\,\mathrm{d}s}. \qquad \square$$

6. L^2-estimates for scalar curvature

Let us begin with an elementary lemma.

Lemma 6.1. *Let* (g_{ij}) *be the maximal solution of the evolution equation*: $\frac{\partial}{\partial t}g_{ij} = -2h_{ij}$ *where* $h_{ij} = R_{ij} - \frac{1}{3}(\alpha\sigma + \beta)g_{ij}$. *Then for any* $f \in C^{2,1}(\mathbb{R}_+ \times M)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M f(t,\cdot)\,\mathrm{d}\mu_t &= -\int_M \left(\Delta - \frac{\partial}{\partial t}\right) f(t,\cdot)\,\mathrm{d}\mu_t \\ &\quad - \int_M Rf(t,\cdot)\,\mathrm{d}\mu_t + A\int_M f(t,\cdot)\,\mathrm{d}\mu_t \end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2\,\mathrm{d}\mu_t &= -2\int_M |\nabla\nabla f|^2\,\mathrm{d}\mu_t - \int_M R|\nabla f|^2\,\mathrm{d}\mu_t \\ &\quad + \frac{1}{3}A\int_M |\nabla f|^2\,\mathrm{d}\mu_t + 2\int_M (\Delta f)\left(\Delta - \frac{\partial}{\partial t}\right) f\,\mathrm{d}\mu_t. \end{aligned} \tag{6.2}$$

Proof. The first equation is easy and follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M f(t,\cdot)\,\mathrm{d}\mu_t &= \int_M \frac{\partial}{\partial t} f(t,\cdot)\,\mathrm{d}\mu_t + \int_M f(t,\cdot)\frac{\partial}{\partial t}(\log M_t)\,\mathrm{d}\mu_t \\ &= \int_M \frac{\partial}{\partial t} f(t,\cdot)\,\mathrm{d}\mu_t - \int_M \operatorname{tr}_g(h_{ij}) f(t,\cdot)\,\mathrm{d}\mu_t. \end{aligned} \tag{6.3}$$

According to the Bochner identity

$$\begin{aligned}\Delta|\nabla f|^2 &= 2\Gamma_2(f) + 2\langle\nabla\Delta f, \nabla f\rangle \\ &= 2|\nabla\nabla f|^2 + 2\operatorname{Ric}(\nabla f, \nabla f) + 2\langle\nabla\Delta f, \nabla f\rangle, \end{aligned}$$

together with the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2h_{ij}$ we have

$$\frac{\partial}{\partial t}|\nabla f|^2 = \left(\frac{\partial}{\partial t} g^{ij}\right)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} + 2\left\langle \nabla\left(\frac{\partial}{\partial t} f\right), \nabla f\right\rangle = 2h^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} + 2\left\langle \nabla\left(\frac{\partial}{\partial t} f\right), \nabla f\right\rangle,$$

so that

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2\left(R^{ij} - h^{ij}\right)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} + 2\left\langle \nabla\left(\Delta - \frac{\partial}{\partial t}\right) f, \nabla f\right\rangle. \tag{6.4}$$

Since

$$R^{ij} - h^{ij} = \frac{1}{3} A g^{ij}$$

where $A(t) = \alpha\sigma(t) + \beta$, we thus have

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla f|^2 = 2|\nabla\nabla f|^2 + \frac{2}{3} A|\nabla f|^2 + 2\left\langle \nabla\left(\Delta - \frac{\partial}{\partial t}\right) f, \nabla f\right\rangle. \tag{6.5}$$

Applying (6.3) to $|\nabla f|^2$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2\,\mathrm{d}\mu_t = -2\int_M |\nabla\nabla f|^2\,\mathrm{d}\mu_t + \frac{1}{3} A\int_M |\nabla f|^2\,\mathrm{d}\mu_t - \int_M R|\nabla f|^2\,\mathrm{d}\mu_t - 2\int_M \left\langle \nabla\left(\Delta - \frac{\partial}{\partial t}\right) f, \nabla f\right\rangle \mathrm{d}\mu_t,$$

and integration by parts we obtain (7.6). □

Theorem 6.2. *Under the Ricci flow* (2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2\,\mathrm{d}\mu_t \leqslant -\int_M R|\nabla f|^2\,\mathrm{d}\mu_t + \frac{1}{3} A\int_M |\nabla f|^2\,\mathrm{d}\mu_t + \frac{3}{2}\int_M \left[\left(\Delta - \frac{\partial}{\partial t}\right) f\right]^2 \mathrm{d}\mu_t. \tag{6.6}$$

Proof. On 3-manifolds $|\nabla\nabla f|^2 \geqslant \frac{1}{3}(\Delta f)^2$ so that by (6.2)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2\,\mathrm{d}\mu_t &\leqslant -\int_M R|\nabla f|^2\,\mathrm{d}\mu_t + \frac{1}{3} A\int_M |\nabla f|^2\,\mathrm{d}\mu_t \\ &\quad - \frac{2}{3}\int_M (\Delta f)^2\,\mathrm{d}\mu_t + 2\int_M (\Delta f)\left(\Delta - \frac{\partial}{\partial t}\right) f\,\mathrm{d}\mu_t \\ &\leqslant -\int_M R|\nabla f|^2\,\mathrm{d}\mu_t + \frac{1}{3} A\int_M |\nabla f|^2\,\mathrm{d}\mu_t \\ &\quad + \frac{3}{2}\int_M \left[\left(\Delta - \frac{\partial}{\partial t}\right) f\right]^2 \mathrm{d}\mu_t. \qquad \square\end{aligned}$$

Together with the evolution for the scalar curvature R one can establish the following

Corollary 6.3. *Under the Ricci flow* (2.1) *we have the following energy estimate for the scalar curvature R*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla R|^2\,\mathrm{d}\mu_t \leqslant{}& -\int_M R|\nabla R|^2\,\mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla R|^2\,\mathrm{d}\mu_t \\ &+ 6\int_M S^2\,\mathrm{d}\mu_t + \frac{2}{3}A^2\int_M R^2\,\mathrm{d}\mu_t \\ &- 4A\int_M RS\,\mathrm{d}\mu_t. \end{aligned}\tag{6.7}$$

In particular if $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ with initial metric $g(0)_{ij}$ such that $R(0)$ is a positive constant, then

$$\int_M |\nabla R|^2\,\mathrm{d}\mu_t \leqslant 6\int_0^t \left\{\mathrm{e}^{-R(0)(t-s)}\int_M S^2\,\mathrm{d}\mu_s\right\}.\tag{6.8}$$

6.1. Some applications

Another application of Eq. (6.2) is to obtain information on spectral gaps.

Theorem 6.4. *Let $(g(t)_{ij})$ be the solution to the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ on the* 3*-manifold M.*

1) *If $\lambda(t)$ denotes the first non-negative eigenvalue of $(M, g(t)_{ij})$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = 2\lambda^2 + \lambda\int_M Rf^2\,\mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2\,\mathrm{d}\mu_t - \int_M R|\nabla f|^2\,\mathrm{d}\mu_t \tag{6.9}$$

where f is an eigenvector: $\Delta f = -\lambda f$ such that $\int_M f^2\,\mathrm{d}\mu_t = 1$.

2) *If $\lambda(t)$ denotes the first non-negative eigenvalue of $\Delta + \frac{1}{4}R$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda \leqslant \frac{19}{8}\lambda^2.\tag{6.10}$$

Proof. Let us consider an eigenvector f with eigenvalue λ (both depending smoothly on t):

$$(\Delta + V)f = -\lambda f; \qquad \int_M f^2\,\mathrm{d}\mu_t = 1$$

where V is some potential (depending on t as well) to be chosen later. Then, by Eq. (6.2)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2\,\mathrm{d}\mu_t ={}& \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\,\mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2\,\mathrm{d}\mu_t - \int_M R|\nabla f|^2\,\mathrm{d}\mu_t \\ &+ 2\int_M (V+\lambda)^2 f^2\,\mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2\,\mathrm{d}\mu_t. \end{aligned}\tag{6.11}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla f|^2\,\mathrm{d}\mu_t = \int_M f^2\frac{\mathrm{d}}{\mathrm{d}t}(V+\lambda)\,\mathrm{d}\mu_t + \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\,\mathrm{d}\mu_t + \int_M (V+\lambda)f^2\left(\frac{\partial}{\partial t}\log M_t\right)\mathrm{d}\mu_t$$
$$= \int_M f^2\frac{\mathrm{d}}{\mathrm{d}t}(V+\lambda)\,\mathrm{d}\mu_t + \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\,\mathrm{d}\mu_t - \int_M (R-A)(V+\lambda)f^2\,\mathrm{d}\mu_t,$$

combining with (6.11), $\int_M f^2\,\mathrm{d}\mu_t = 1$ and

$$\int_M |\nabla f|^2\,\mathrm{d}\mu_t = \int_M (\lambda+V)f^2\,\mathrm{d}\mu_t,$$

we deduce the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = \int_M (R-A)(V+\lambda)f^2\,\mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2\,\mathrm{d}\mu_t - \int_M R|\nabla f|^2\,\mathrm{d}\mu_t + 2\int_M (V+\lambda)^2 f^2\,\mathrm{d}\mu_t + \frac{1}{3}A\int_M |\nabla f|^2\,\mathrm{d}\mu_t - \int_M f^2\frac{\partial V}{\partial t}\,\mathrm{d}\mu_t$$
$$= -2\int_M |\nabla\nabla f|^2\,\mathrm{d}\mu_t - \int_M R|\nabla f|^2\,\mathrm{d}\mu_t + 2\int_M (V+\lambda)^2 f^2\,\mathrm{d}\mu_t - \int_M f^2\frac{\partial V}{\partial t}\,\mathrm{d}\mu_t + \int_M \left(R-\frac{2}{3}A\right)(V+\lambda)f^2\,\mathrm{d}\mu_t.$$

Choose $V = -\varphi(t)R(t,\cdot)$ where φ depends on t only. Then

$$\frac{\partial V}{\partial t} = -\varphi' R - \varphi\frac{\partial}{\partial t}R$$

so that

$$\int_M f^2\frac{\partial V}{\partial t}\,\mathrm{d}\mu_t = -\int_M \varphi' R f^2\,\mathrm{d}\mu_t - \varphi\int_M f^2\frac{\partial}{\partial t}R\,\mathrm{d}\mu_t.$$

To treat the term $\int_M f^2\frac{\partial}{\partial t}R\,\mathrm{d}\mu_t$ we use integration by parts again, thus

$$\int_M R|\nabla f|^2\,\mathrm{d}\mu_t = \int_M R(V+\lambda)f^2\,\mathrm{d}\mu_t + \frac{1}{2}\int_M f^2(\Delta R)\,\mathrm{d}\mu_t$$
$$= \int_M R(V+\lambda)f^2\,\mathrm{d}\mu_t + \frac{1}{2}\int_M f^2\frac{\partial R}{\partial t}\,\mathrm{d}\mu_t + \int_M f^2\left(-S+\frac{1}{3}AR\right)\mathrm{d}\mu_t.$$

In other words

$$\int_M f^2 \frac{\partial R}{\partial t}\, \mathrm{d}\mu_t = 2\int_M f^2\left(S - \frac{1}{3}AR\right) \mathrm{d}\mu_t + 2\int_M R|\nabla f|^2\, \mathrm{d}\mu_t \\ - 2\int_M R(V+\lambda) f^2\, \mathrm{d}\mu_t.$$

Therefore

$$-\int_M f^2 \frac{\partial V}{\partial t}\, \mathrm{d}\mu_t = \int_M \varphi' R f^2\, \mathrm{d}\mu_t + 2\varphi \int_M f^2\left(S - \frac{1}{3}AR\right) \mathrm{d}\mu_t \\ - 2\varphi \int_M R|\nabla f|^2\, \mathrm{d}\mu_t + 2\varphi \int_M R(\varphi R + \lambda) f^2\, \mathrm{d}\mu_t.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = -2\int_M |\nabla\nabla f|^2\, \mathrm{d}\mu_t - (1+2\varphi)\int_M R|\nabla f|^2\, \mathrm{d}\mu_t \\ + \int_M \left(2\varphi S - \frac{2}{3}\varphi AR + \varphi' R\right) f^2\, \mathrm{d}\mu_t \\ + \varphi \int_M R\left(R + 4\varphi R - \frac{2}{3}A + 2\lambda\right) f^2\, \mathrm{d}\mu_t \\ + \lambda \int_M \left(R + 4\varphi R - \frac{2}{3}A + 2\lambda\right) f^2\, \mathrm{d}\mu_t.$$

In particular, under the Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, then $A = 0$, and if choose φ to be a constant, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = -2\int_M |\nabla\nabla f|^2\, \mathrm{d}\mu_t - (1+2\varphi)\int_M R|\nabla f|^2\, \mathrm{d}\mu_t \\ + 2\varphi \int_M S f^2\, \mathrm{d}\mu_t + \varphi \int_M R(R + 4\varphi R + 2\lambda) f^2\, \mathrm{d}\mu_t \\ + \lambda \int_M (R + 4\varphi R + 2\lambda) f^2\, \mathrm{d}\mu_t. \tag{6.12}$$

In particular, if we choose $\varphi = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda = 2\lambda^2 + \lambda \int_M R f^2\, \mathrm{d}\mu_t - 2\int_M |\nabla\nabla f|^2\, \mathrm{d}\mu_t - \int_M R|\nabla f|^2\, \mathrm{d}\mu_t. \tag{6.13}$$

However if we choose $\varphi = -1/4$ then

$$\frac{d}{dt}\lambda = -2\int_M |\nabla\nabla f|^2 \, d\mu_t - \frac{1}{2}\int_M R|\nabla f|^2 \, d\mu_t + 2\lambda^2 - \frac{1}{2}\int_M Sf^2 \, d\mu_t - \frac{1}{2}\lambda\int_M Rf^2 \, d\mu_t.$$

While $S \geqslant \frac{1}{3}R^2$ so that

$$\begin{aligned}\frac{d}{dt}\lambda &= -2\int_M |\nabla\nabla f|^2 \, d\mu_t - \frac{1}{2}\int_M R|\nabla f|^2 \, d\mu_t \\ &\quad + 2\lambda^2 - \frac{1}{6}\int_M \big(R^2 - 3\lambda R\big) f^2 \, d\mu_t \\ &\leqslant \frac{19}{8}\lambda^2.\end{aligned}$$

Let us come back to Eq. (6.12). By the Bochner identity

$$\Gamma_2(f) = |\nabla\nabla f|^2 + \mathrm{Ric}(\nabla f, \nabla f)$$

so that

$$\int_M \Gamma_2(f) \, d\mu_t = \int_M |\nabla\nabla f|^2 \, d\mu_t + \int_M \mathrm{Ric}(\nabla f, \nabla f) \, d\mu_t,$$

while by integration by parts

$$\begin{aligned}\int_M \Gamma_2(f) \, d\mu_t &= -\int_M \langle \nabla\Delta f, \nabla f\rangle \, d\mu_t \\ &= \int_M (\Delta f)^2 \, d\mu_t \\ &= \int_M (\varphi R + \lambda)^2 f^2\end{aligned}$$

hence

$$-\int_M |\nabla\nabla f|^2 \, d\mu_t = -\int_M (\varphi R + \lambda)^2 f^2 + \int_M \mathrm{Ric}(\nabla f, \nabla f) \, d\mu_t.$$

Inserting this fact into (6.12) we obtain, after simplification,

$$\begin{aligned}\frac{d}{dt}\lambda &= (1+2\varphi)\lambda\int_M Rf^2 \, d\mu_t - (1+2\varphi)\int_M R|\nabla f|^2 \, d\mu_t \\ &\quad + \big(2\varphi^2 + \varphi\big)\int_M R^2 f^2 \, d\mu_t + 2\varphi\int_M Sf^2 \, d\mu_t \\ &\quad + 2\int_M \mathrm{Ric}(\nabla f, \nabla f) \, d\mu_t.\end{aligned} \tag{6.14}$$

While under the Ricci flow

$$\mathrm{Ric}(\nabla f,\nabla f)\geqslant -\varepsilon R|\nabla f|^2$$

where $\varepsilon\in[0,1/3]$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda\geqslant(1+2\varphi)\lambda\int_M Rf^2\,\mathrm{d}\mu_t-(1+2\varphi+2\varepsilon)\int_M R|\nabla f|^2\,\mathrm{d}\mu_t + \left(2\varphi^2+\varphi\right)\int_M R^2f^2\,\mathrm{d}\mu_t+2\varphi\int_M Sf^2\,\mathrm{d}\mu_t.$$

In particular by choosing $\varphi=0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda=\lambda\int_M Rf^2\,\mathrm{d}\mu_t-\int_M R|\nabla f|^2\,\mathrm{d}\mu_t+2\int_M \mathrm{Ric}(\nabla f,\nabla f)\,\mathrm{d}\mu_t$$

which yields (6.9). □

6.2. L^2-estimates for the Ricci tensor

Our next goal is to improve these L^2-estimates to weighted forms.

Theorem 6.5. *Under the Ricci flow* (2.1) *with initial metric of positive scalar curvature, and* $\gamma\in[0,2/5]$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla R|^2R^\gamma\,\mathrm{d}\mu_t\leqslant-\int_M\left[8\gamma\frac{S}{R}+\left(\frac{2}{3}\gamma+1\right)A+R\right]|\nabla R|^2R^\gamma\,\mathrm{d}\mu_t + 6\int_M S^2R^\gamma\,\mathrm{d}\nu_t.$$

In particular if $A=0$ and $R(0)$ is a positive constant, then

$$\int_M |\nabla R|^2R^\gamma\,\mathrm{d}\mu_t\leqslant 6\int_0^t\left\{\mathrm{e}^{-(\frac{8\gamma}{3}+1)R(0)(t-s)}\int_M S^2R^\gamma\,\mathrm{d}\mu_s\right\}. \tag{6.15}$$

Proof. By elementary computations

$$\left(\Delta-\frac{\partial}{\partial t}\right)|\nabla R|^2=2|\nabla\nabla R|^2-4\langle\nabla S,\nabla R\rangle+2A|\nabla R|^2, \tag{6.16}$$

$$\begin{aligned}\left(\Delta-\frac{\partial}{\partial t}\right)|\nabla S|^2={}&2|\nabla\nabla S|^2+\frac{10}{3}A|\nabla S|^2-20R|\nabla S|^2\\&+4\left\{3R^2-5S\right\}\langle\nabla R,\nabla S\rangle\\&+16\langle\nabla T,\nabla S\rangle+4\left\langle\nabla|\nabla_kR_{ij}|^2,\nabla S\right\rangle,\end{aligned} \tag{6.17}$$

and

$$\left(\Delta-\frac{\partial}{\partial t}\right)|\nabla T|^2=2|\nabla\nabla T|^2+14\left(\frac{1}{3}A+R\right)|\nabla T|^2-18R^2\langle\nabla S,\nabla T\rangle$$
$$+\left\{14T-36RS+16R^3\right\}\langle\nabla R,\nabla T\rangle$$
$$+12\big\langle\nabla\left\{g^{ab}R_{ia}\left(\nabla^k R^{ij}\right)(\nabla_k R_{bj})\right\},\nabla T\big\rangle. \tag{6.18}$$

By chain rule, (6.16) and the evolution equation for the scalar curvature we deduce easily the following

$$\left(\Delta-\frac{\partial}{\partial t}\right)\left(R^\gamma|\nabla R|^2\right)=\left(-2\gamma\frac{S}{R}+\left(\frac{2\gamma}{3}+2\right)A\right)|\nabla R|^2R^\gamma$$
$$+\gamma(\gamma-1)R^{\gamma-2}|\nabla R|^4+2|\nabla\nabla R|^2R^\gamma$$
$$-4\langle\nabla S,\nabla R\rangle R^\gamma+2\big\langle\nabla R^\gamma,\nabla|\nabla R|^2\big\rangle.$$

Then we apply (6.1) to function $R^\gamma|\nabla R|^2$, after simplification we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M|\nabla R|^2R^\gamma\,\mathrm{d}\mu_t=\int_M\left[2\gamma\frac{S}{R}-\left(\frac{2\gamma}{3}+1\right)A-R\right]|\nabla R|^2R^\gamma\,\mathrm{d}\mu_t$$
$$-\gamma(\gamma-1)\int_M R^{\gamma-2}|\nabla R|^4\,\mathrm{d}\mu_t-2\int_M|\nabla\nabla R|^2R^\gamma\,\mathrm{d}\mu_t$$
$$+4\int_M\langle\nabla S,\nabla R\rangle R^\gamma\,\mathrm{d}\mu_t-2\int_M\big\langle\nabla R^\gamma,\nabla|\nabla R|^2\big\rangle\mathrm{d}\mu_t. \tag{6.19}$$

Using integration by parts in the last two integrals we then deduce

$$4\int_M\langle\nabla S,\nabla R\rangle R^\gamma\,\mathrm{d}\mu_t=-4\int_M(S\Delta R)\,\mathrm{d}\nu_t-4\gamma\int_M\frac{S}{R}|\nabla R|^2\,\mathrm{d}\nu_t$$

and

$$-2\int_M\big\langle\nabla R^\gamma,\nabla|\nabla R|^2\big\rangle\mathrm{d}\mu_t=2\gamma\int_M R^{-1}(\Delta R)|\nabla R|^2\,\mathrm{d}\nu_t+2\gamma(\gamma-1)\int_M R^{-2}|\nabla R|^4\,\mathrm{d}\nu_t.$$

Let $\mathrm{d}\nu_t=R^\gamma\,\mathrm{d}\mu_t$, and use the elementary inequality $|\nabla\nabla R|^2\geqslant\frac{1}{3}(\Delta R)^2$ one deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M|\nabla R|^2\,\mathrm{d}\nu_t\leqslant-\int_M\left[2\gamma\frac{S}{R}+\left(\frac{2\gamma}{3}+1\right)A+R\right]|\nabla R|^2\,\mathrm{d}\nu_t$$
$$+\gamma(\gamma-1)\int_M R^{-2}|\nabla R|^4\,\mathrm{d}\nu_t$$
$$-\frac{2}{3}\int_M\left[(\Delta R)^2-2\left(\frac{3\gamma}{2R}|\nabla R|^2-3S\right)(\Delta R)\right]\mathrm{d}\nu_t$$
$$\leqslant-\int_M\left[8\gamma\frac{S}{R}+\left(\frac{2\gamma}{3}+1\right)A+R\right]|\nabla R|^2\,\mathrm{d}\nu_t$$
$$-\gamma\left(1-\frac{5}{2}\gamma\right)\int_M\frac{|\nabla R|^4}{R^2}\,\mathrm{d}\nu_t+6\int_M S^2\,\mathrm{d}\nu_t$$

which allows us to establish an estimate for $\int_M |\nabla R|^2 \, d\nu_t$. Setting $\gamma = 2/5$ which seems to be the best choice, one proves the theorem. □

7. Harnack estimate for Ricci flow

We in this section prove Harnack estimates for the Ricci curvature under the Ricci flow $\frac{\partial g_{ij}}{\partial t} = -2h_{ij}$ on a closed 3-manifold.

7.1. Some formulae about Ricci tensors

Let (V_{ij}) be a symmetric tensor depending smoothly in t. Then

$$\frac{\partial}{\partial t} \operatorname{tr}_g(V_{ij}) = 2h^{ij} V_{ij} + \operatorname{tr}_g\left(\frac{\partial V_{ab}}{\partial t}\right). \tag{7.1}$$

We then deduce the following evolution equations:

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla_b V_{ij}) = \nabla_b \frac{\partial V_{ij}}{\partial t} &+ g^{pq} V_{qj}(\nabla_i h_{pb} + \nabla_b h_{pi} - \nabla_p h_{ib}) \\ &+ V_{iq} g^{pq}(\nabla_j h_{pb} + \nabla_b h_{pj} - \nabla_p h_{jb}),\end{aligned} \tag{7.2}$$

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_a \nabla_b V_{ij} = \nabla_a \nabla_b \frac{\partial V_{ij}}{\partial t} &+ V_{pj} g^{pq}(\nabla_a \nabla_i h_{qb} + \nabla_a \nabla_b h_{qi} - \nabla_a \nabla_q h_{ib}) \\ &+ V_{ip} g^{pq}(\nabla_a \nabla_b h_{qj} + \nabla_a \nabla_j h_{qb} - \nabla_a \nabla_q h_{bj}) \\ &+ g^{pq}(\nabla_a V_{pj})(\nabla_i h_{qb} + \nabla_b h_{qi} - \nabla_q h_{ib}) \\ &+ g^{pq}(\nabla_a V_{ip})(\nabla_b h_{qj} + \nabla_j h_{qb} - \nabla_q h_{bj}) \\ &+ g^{pq}(\nabla_p V_{ij})(\nabla_b h_{qa} + \nabla_a h_{qb} - \nabla_q h_{ba}) \\ &+ g^{pq}(\nabla_b V_{pj})(\nabla_i h_{qa} + \nabla_a h_{qi} - \nabla_q h_{ia}) \\ &+ g^{pq}(\nabla_b V_{ip})(\nabla_a h_{qj} + \nabla_j h_{qa} - \nabla_q h_{aj})\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta V_{ij}) = \Delta \frac{\partial V_{ij}}{\partial t} &+ 2h_{ab} \nabla^a \nabla^b V_{ij} - V_{bj} \nabla^a \nabla^b h_{ia} - V_{ib} \nabla^a \nabla^b h_{aj} \\ &+ V_{ip} g^{pq}\big(\nabla^a \nabla_a h_{qj} + \nabla^a \nabla_j h_{qa}\big) \\ &+ V_{pj} g^{pq}\big(\nabla^a \nabla_i h_{qa} + \nabla^a \nabla_a h_{qi}\big) \\ &+ \big(\nabla^b V_{ij}\big)\big(2\nabla^a h_{ab} - \nabla_b \operatorname{tr}_g(h_{kl})\big) \\ &+ 2g^{pq}\big(\nabla^a V_{pj}\big)(\nabla_i h_{qa} + \nabla_a h_{qi}) - 2\big(\nabla^a V_{bj}\big)\big(\nabla^b h_{ia}\big) \\ &+ 2g^{pq}\big(\nabla^a V_{ip}\big)(\nabla_a h_{qj} + \nabla_j h_{qa}) - 2\big(\nabla^a V_{ib}\big)\big(\nabla^b h_{aj}\big).\end{aligned}$$

7.2. Harnack inequality for Ricci curvature

In this section we assume that $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ with initial $(g(0)_{ij})$ such that the scalar curvature $R(0)$ is a positive constant. Then $R \geqslant R(0)$ and

$$\frac{1}{3} \leqslant \frac{S}{R^2} \leqslant \max_M \frac{S(0)}{R(0)^2}$$

as long as the solution flow to (1.2) exists.

Calculating in a normal coordinate system

$$\left(\Delta - \frac{\partial}{\partial t}\right)|\nabla_k R_{ij}|^2 = 2\left(\nabla^k R^{ij}\right)\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R_{ij}) - 2R^{ka}(\nabla_k R_{ij})\left(\nabla_a R^{ij}\right) - 2g^{jq}R^{ip}(\nabla_k R_{ij})\left(\nabla^k R_{pq}\right) - 2g^{ip}R^{jq}(\nabla_k R_{ij})\left(\nabla^k R_{pq}\right) + 2\left\langle \nabla(\nabla_k R_{ij}), \nabla\left(\nabla^k R^{ij}\right)\right\rangle,$$

and applying the Ricci identity for symmetric tensor (T_{ij}) on 3-manifolds:

$$\begin{aligned} &\nabla_c(\Delta T_{ij}) - \Delta(\nabla_c T_{ij}) \\ &\quad = T_{aj}\left(\nabla^a R_{ci}\right) + T_{ia}\left(\nabla^a R_{cj}\right) - g^{ab}T_{ia}(\nabla_j R_{cb}) - g^{ab}T_{aj}(\nabla_i R_{cb}) \\ &\qquad + \left(\nabla^a T_{ai}\right)(2R_{cj} - Rg_{cj}) + \left(\nabla^a T_{aj}\right)(2R_{ci} - Rg_{ci}) - \left(\nabla^a T_{ij}\right)R_{ca} \\ &\qquad - 2g^{ab}R_{cb}\left\{(\nabla_i T_{aj}) + (\nabla_j T_{ai})\right\} + (\nabla_j T_{ci} + \nabla_i T_{cj})R \\ &\qquad + 2g^{pq}\left(\nabla^a T_{pi}\right)R_{aq}g_{cj} - 2\left(\nabla^a T_{ci}\right)R_{aj} \\ &\qquad + 2g^{pq}\left(\nabla^a T_{pj}\right)R_{aq}g_{ci} - 2\left(\nabla^a T_{cj}\right)R_{ai}. \end{aligned} \tag{7.3}$$

Applying (7.3) to the Ricci symmetric tensor R_{ij} together with the Bianchi identity $\nabla^a R_{aj} = \frac{1}{2}\nabla_j R$, through a long but complete elementary computation one can establish the following evolution equation for $\nabla_k R_{ij}$ under the Ricci flow $\frac{\partial}{\partial t}g_{ij} = -2R_{ij}$ on 3-manifolds:

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R_{ij}) &= g^{ab}R_{aj}\left\{5(\nabla_k R_{bi}) + 2(\nabla_b R_{ki})\right\} \\ &\quad + g^{ab}R_{ia}\left\{5(\nabla_k R_{bj}) + 2(\nabla_b R_{kj})\right\} \\ &\quad + g^{ab}R_{ka}\left\{2(\nabla_i R_{bj}) + 2(\nabla_j R_{bi}) + (\nabla_b R_{ij})\right\} \\ &\quad - 3R_{ij}(\nabla_k R) + 2Rg_{ij}(\nabla_k R) - 2g_{ij}(\nabla_k S) \\ &\quad - R\left\{(\nabla_j R_{ki}) + (\nabla_i R_{kj}) + 3(\nabla_k R_{ij})\right\} \\ &\quad - (\nabla_i R)\left\{R_{kj} - \frac{1}{2}Rg_{kj}\right\} - (\nabla_j R)\left\{R_{ki} - \frac{1}{2}Rg_{ki}\right\} \\ &\quad - 2g_{kj}R^{ab}(\nabla_a R_{bi}) - 2g_{ki}R^{ab}(\nabla_a R_{bj}). \end{aligned} \tag{7.4}$$

Taking trace in Eq. (7.4) to obtain

$$\left(\Delta - \frac{\partial}{\partial t}\right)(\nabla_k R) = -2(\nabla_k S) + g^{ij}R_{kj}(\nabla_i R). \tag{7.5}$$

It follows the evolution equation for $|\nabla_k R_{ij}|^2$:

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)|\nabla_k R_{ij}|^2 &= -6R|\nabla_k R_{ij}|^2 - 4R\left(\nabla^k R^{ij}\right)(\nabla_i R_{kj}) \\ &\quad - 7\left(\nabla^k R\right)(\nabla_k S) + 5R|\nabla R|^2 - 8R_{kj}\left(\nabla^k R^{ij}\right)(\nabla_i R) \\ &\quad + 16g^{ab}R_{aj}\left(\nabla^k R^{ij}\right)\left\{(\nabla_k R_{bi}) + (\nabla_b R_{ki})\right\} \\ &\quad + 2\left\langle \nabla(\nabla_k R_{ij}), \nabla\left(\nabla^k R^{ij}\right)\right\rangle, \end{aligned} \tag{7.6}$$

so that

$$\left(\Delta - \frac{\partial}{\partial t}\right)\frac{|\nabla_k R_{ij}|^2}{R} = \left(\frac{2S}{R^2} - 6\right)|\nabla_k R_{ij}|^2 + 5|\nabla R|^2 - \frac{7}{R}\langle \nabla S, \nabla R\rangle$$
$$- 4\left(\nabla^k R^{ij}\right)(\nabla_i R_{kj}) - \frac{8}{R}R_{kj}\left(\nabla^k R^{ij}\right)(\nabla_i R)$$
$$+ \frac{16}{R}g^{ab}R_{ai}\left(\nabla^k R^{ij}\right)\left\{(\nabla_b R_{jk}) + (\nabla_k R_{jb})\right\}$$
$$+ \frac{2}{R}\left\langle \nabla(\nabla_k R_{ij}), \nabla\left(\nabla^k R^{ij}\right)\right\rangle - 2\left\langle \nabla \log R, \nabla \frac{|\nabla_k R_{ij}|^2}{R}\right\rangle. \tag{7.7}$$

We are now in a position to prove the following

Theorem 7.1. *Under the Ricci flow* (1.2) *on the* 3*-manifold with initial metric with positive constant scalar curvature $R(0)$, such that $R(0)_{ij} \geqslant -\varepsilon R(0)g(0)_{ij}$ for some $\varepsilon \in [0, 1/3]$. Then*

$$\left(L - \frac{\partial}{\partial t}\right)F \geqslant -C(\varepsilon)RF$$

where $L = \Delta + 2\nabla \log R$, $F = |\nabla_k R_{ij}|^2/R$ and

$$C(\varepsilon) = \frac{28}{3} + 16\sqrt{3\delta} + 8\sqrt{\delta} + \frac{49}{5}\delta + 16\varepsilon.$$

In particular, suppose $R(t,\cdot) \leqslant \theta(t)$ then

$$\left(L - \frac{\partial}{\partial t}\right)F \geqslant -C(\varepsilon)\theta(t)F$$

so that

$$\frac{|\nabla_k R_{ij}|^2}{R} \leqslant e^{C(\varepsilon)\int_0^t \theta(s)\,ds}\frac{|\nabla_k R(0)_{ij}|^2}{R(0)}. \tag{7.8}$$

Proof. Use a normal coordinate which diagonalizes the Ricci curvature (R_{ij}), and set $X_{kij} = \nabla_k R_{ij}$. Then Eq. (7.7) may be written as the following

$$\left(\Delta - \frac{\partial}{\partial t}\right)F = \left(\frac{2S}{R^2} - 6\right)RF + 5|\nabla R|^2 - \frac{7}{R}\langle \nabla S, \nabla R\rangle$$
$$+ \frac{16}{R}\lambda_i X_{kij}^2 + \frac{16}{R}\lambda_i X_{kij}X_{ikj} - \frac{8}{R}\lambda_i X_{iji}X_{jaa} - 4X_{kij}X_{ikj}$$
$$+ \frac{2}{R}\left\langle \nabla(\nabla_k R_{ij}), \nabla\left(\nabla^k R^{ij}\right)\right\rangle - 2\langle \nabla \log R, \nabla F\rangle. \tag{7.9}$$

We then use the following identity, for any constant $\xi \neq 0$

$$-\langle \nabla S, \nabla R\rangle = \frac{1}{\xi}|\nabla_k R_{ij} - \xi R_{ij}\nabla_k R|^2 - \frac{1}{\xi}|\nabla_k R_{ij}|^2 - \xi S|\nabla R|^2 \tag{7.10}$$

so that, for $\xi > 0$ we have

$$-\frac{7}{R}\langle \nabla S, \nabla R\rangle = \frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij} - \xi R_{ij}\nabla_k R|^2 - \frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}|^2 - 7\xi\frac{S}{R}|\nabla R|^2$$
$$\geqslant -\frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}|^2 - 7\xi\frac{S}{R}|\nabla R|^2$$
$$\geqslant -\frac{1}{\xi}\frac{7}{R}|\nabla_k R_{ij}|^2 - 7\xi\delta R|\nabla R|^2.$$

By choosing $7\xi\delta R = 5$, i.e. $\xi = \frac{5}{7\delta R}$ we deduce that

$$\left(\Delta - \frac{\partial}{\partial t}\right)F \geqslant \left(\frac{2S}{R^2} - 6 - \frac{49}{5}\delta\right)RF + \frac{16}{R}\lambda_i X_{kij}^2 + \frac{16}{R}\lambda_i X_{kij}X_{ikj}$$
$$- \frac{8}{R}\lambda_i X_{iji}X_{jaa} - 4X_{kij}X_{ikj} - 2\langle\nabla \log R, \nabla F\rangle.$$

Since

$$|\lambda_i X_{kij}X_{ikj}| \leqslant \sqrt{S}\sqrt{\sum_i\Big(\sum_{j,k}X_{kij}X_{ikj}\Big)^2}$$
$$\leqslant \sqrt{S}\sqrt{\sum_i\sqrt{\sum_{j,k}X_{kij}^2}\sqrt{\sum_{j,k}X_{ikj}^2}}$$
$$\leqslant \sqrt{3S}|\nabla_k R_{ij}|^2$$

and

$$|\lambda_i X_{iji}X_{jaa}| = \Big|\sum_i \lambda_i\Big(\sum_{j,a}X_{iji}X_{jaa}\Big)\Big|$$
$$\leqslant \sqrt{S}\sqrt{\sum_i\Big(\sum_{j,a}X_{iji}X_{jaa}\Big)^2}$$
$$\leqslant \sqrt{S}\sqrt{\sum_{j,a}X_{jaa}^2\Big(\sum_i\sum_{i,j,a}X_{iji}^2\Big)}$$
$$\leqslant \sqrt{S}|\nabla_k R_{ij}|^2.$$

Therefore, as $\lambda_i \geqslant -\varepsilon R$,

$$\left(\Delta - \frac{\partial}{\partial t}\right)F \geqslant \left(\frac{2S}{R^2} - 6 - \frac{49}{5}\delta\right)RF - 16\varepsilon RF - \frac{16}{R}\sqrt{3S}|\nabla_k R_{ij}|^2$$
$$- \frac{8}{R}\sqrt{S}|\nabla_k R_{ij}|^2 - 4|\nabla_k R_{ij}|^2 - 2\langle\nabla \log R, \nabla F\rangle$$
$$\geqslant \left(\frac{2}{3} - 10 - 16\sqrt{3\delta} - 8\sqrt{\delta} - \frac{49}{5}\delta - 16\varepsilon\right)RF$$
$$- 2\langle\nabla \log R, \nabla F\rangle. \qquad \square$$

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