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Note

Conditions for Superiority of Integration Rules of the Second Kind

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Shisha [3] has introduced the concept of integration rules of the second kind for numerical integration; later Burrows [1] arrived at the same idea. In Fig. 1, where f is a continuous strictly increasing function on $[a, b]$ and $0 \leqslant a < b < \infty$ and $f(a) \geqslant 0$, it is clear by interpreting the integrals as areas that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy. \tag{1}$$

A numerical integration rule to approximate the right-hand side of (1) is called an "integration rule of the second kind" for $\int_a^b f$. We note that formulas generalizing (1) and, hence, integration rules of the second kind are available for f and $[a, b]$ more general, cf. [3, p. 225; 1, p. 152]. For the sake of simplicity we shall restrict ourselves to the setting of the sentence containing (1).

Our first observation is an extension of a well-known (cf. [2, p. 42]) "bracketing" property of the midpoint and trapezoidal rules. Let f be a strictly increasing *convex* function on $[a, b]$, $0 \leqslant a < b < \infty$, and $f(a) \geqslant 0$. If

$$I_T = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{k=1}^{n-1} f(a+kh)\right], \qquad h = \frac{b-a}{n}$$

is the compound trapezoidal rule sum applied to $\int_a^b f(x)\,dx$, if I_T' is the

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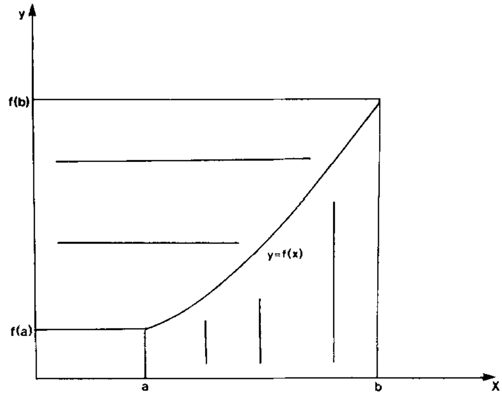

FIG. 1. Geometrical interpretation of (1).

approximation to (1) obtained by applying a compound trapezoidal rule (again with n subintervals) to $\int_{f(a)}^{f(b)} f^{-1}$, i.e., a compound trapezoidal rule of the second kind, and if I_M and I'_M are the corresponding quantities for the midpoint rule, we have

$$\max(I_M, I'_M) \leqslant \int_a^b f \leqslant \min(I_T, I'_T). \tag{2}$$

We can verify (2) as follows:

Since the trapezoidal rule sum results from integrating a polygonal function $\geqslant f$, $\int_a^b f \leqslant I_T$, as is known. Noting that $-f^{-1}$ is convex gives

$$\int_a^b f = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy \leqslant I'_T$$

from which the second inequality in (2) follows. The first inequality can be obtained similarly.

Our next observation gives conditions guaranteeing that a compound trapezoidal rule of the second kind has smaller error than the corresponding compound trapezoidal rule applied directly to $\int_a^b f$.

THEOREM 1. *Let f be a real function on $[a, b]$, $-\infty < a < b < \infty$, with $f' > 0$ and f'' continuous there and $f'(a) \neq f'(b)$. For $n = 2, 3, \ldots$, let*

$$T_n = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{k=1}^{n-1} f(a+kh)\right], \qquad h = \frac{b-a}{n}$$

be the compound trapezoidal rule applied to $\int_a^b f$ and let T'_n be the compound trapezoidal rule applied to $\int_{f(a)}^{f(b)} f^{-1}$. Suppose

$$[(f(b) - f(a))/(b-a)]^2 < f'(a)\,f'(b). \tag{3}$$

Then there exists an N such that for all $n \geqslant N$,

$$\left| \int_{f(a)}^{f(b)} f^{-1} - T_n' \right| < \left| \int_a^b f - T_n \right|. \tag{4}$$

This implies, if $a \geqslant 0$ and $f(a) \geqslant 0$, that the compound trapezoidal rule of the second kind is better for large n than the ordinary compound trapezoidal rule.

Proof. It is known (cf. [2, pp. 42, 58]) that

$$\lim_{n\to\infty} n^2 \left| \int_a^b f - T_n \right| = \frac{(b-a)^2}{12} |f'(b) - f'(a)|. \tag{5}$$

Noting that, on $[f(a), f(b)]$, $(f^{-1})'(y) = 1/f'(f^{-1}(y))$, we have similarly

$$\begin{aligned} \lim_{n\to\infty} n^2 \left| \int_{f(a)}^{f(b)} f^{-1} - T_n' \right| &= \frac{[f(b) - f(a)]^2}{12} |(f^{-1})'(f(b)) - (f^{-1})'(f(a))| \\ &= \frac{[f(b) - f(a)]^2}{12} \left| \frac{f'(b) - f'(a)}{f'(a) f'(b)} \right|. \end{aligned} \tag{6}$$

Now (4) follows from (3), (5), and (6).

Remarks. (1) If (3) holds with the inequality reversed, then so does (4). In this case the compound trapezoidal rule of the second kind for $\int_a^b f$ would be worse. However, therefore the compound trapezoidal rule of the second kind for $\int_{f(a)}^{f(b)} f^{-1}$ would be *better* than the compound trapezoidal rule for that integral.

(2) The hypotheses of Theorem 1 ensure that, for large n, the compound *midpoint* rule of the second kind for $\int_a^b f$ is better than the (ordinary) compound midpoint formula M_n for $\int_a^b f$. The proof is virtually identical to the proof of Theorem 1 since (cf. [2, pp. 42, 58])

$$\lim_{n\to\infty} n^2 \left| \int_a^b f - M_n \right| = \frac{(b-a)^2}{24} |f'(b) - f'(a)|.$$

(3) Let R be a (simple) integration rule with an error of the form

$$\int_\alpha^\beta f - R = c(\beta - \alpha)^s f^{(k)}(\xi), \qquad \alpha < \xi < \beta, \tag{7}$$

where s and k are positive integers and c is independent of α, β, and f. Then results about the superiority, for large n, of the compound version of R of the second kind can be obtained as above, using a slight generalization of the second theorem on page 58 in [2]. (For example, the Newton–Cotes and

TABLE I

Absolute Value of Error in Using a Compound Trapezoidal Rule

Integral	$n = 16$	$n = 128$	$n = 512$
$\int_1^e \log x\, dx$	0.00060725	0.00000923	0.00000077
$\int_0^1 e^y\, dy$	0.00055921	0.00000882	0.00000023
$\int_0^1 \sqrt{x}\, dx$	0.00308555	0.00014096	0.00001788
$\int_0^1 y^2\, dy$	0.00065103	0.00001016	0.00000008

Gauss–Legendre rules satisfy (7).) However, the condition analogous to (3) will involve higher derivatives of f^{-1} and will not be as nice as (3).

Table I shows the absolute value of the error when the compound trapezoidal rule is used for $f(x) \equiv \log x$, $a = 1$, $b = e$, and for $f^{-1}(y) \equiv e^y$, $f(a) = 0$, $f(b) = 1$. It is easily verified that f satisfies the hypotheses of Theorem 1 and hence the compound trapezoidal rule of the second kind is superior for n large. However, both $\int_a^b f - T_n$ and $\int_{f(a)}^{f(b)} f^{-1} - T_n'$ are $O(n^{-2})$ (cf. (5), (6)) and the improvement due to using the rule of the second kind is slight. Table I also includes numerical results for $\int_a^b f(x)\, dx = \int_0^1 \sqrt{x}\, dx$ and $\int_{f(a)}^{f(b)} f^{-1}(y)\, dy = \int_0^1 y^2\, dy$. Here the hypotheses of Theorem 1 are not satisfied (since $f'(0)$ does not exist) but the compound trapezoidal rule of the second kind is markedly superior. In fact $\int_0^1 y^2\, dy - T_n'$ is $O(n^{-2})$ whereas it can be shown that $\int_0^1 \sqrt{x}\, dx - T_n \geqslant c/n^{3/2}$, $c > 0$, for all n.

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