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Impulsive periodic boundary value problems for fractional differential equation involving Riemann–Liouville sequential fractional derivative

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ABSTRACT

In this paper, we investigate the existence of solutions of the periodic boundary value problem for nonlinear impulsive fractional differential equation involving Riemann–Liouville sequential fractional derivative by using monotone iterative method. An example is presented to illustrate our main result.

1. Introduction

This paper deals with the existence of solutions for nonlinear impulsive fractional differential equation with periodic boundary conditions

$$\mathcal{D}^{2\alpha}u(t) = f\big(t, u, \mathcal{D}^{\alpha}u\big), \quad t \in (0,1] \setminus \{t_1, \dots, t_m\},\ 0 < \alpha \leqslant 1, \tag{1.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1), \qquad \lim_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha}u(t) = \mathcal{D}^{\alpha}u(1), \tag{1.2}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(u(t)-u(t_j)\big) = I_j\big(u(t_j)\big), \tag{1.3}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha}u(t)-\mathcal{D}^{\alpha}u(t_j)\big) = \bar{I}_j\big(u(t_j)\big), \tag{1.4}$$

where $\mathcal{D}^{\alpha}u(t) = ({}_0D_t^{\alpha}u)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}u(\tau)\,d\tau$ is the standard Riemann–Liouville fractional derivative, $\mathcal{D}^{2\alpha}u = \mathcal{D}^{\alpha}(\mathcal{D}^{\alpha}u)$ is the sequential Riemann–Liouville fractional derivative presented by Miller and Ross on p. 209 of [1], $0 < t_1 < t_2 < \cdots < t_m < 1$, $I_j, \bar{I}_j \in C(R,R)$ $(j = 1,2,\dots,m)$, f is continuous at every point $(t,u,v) \in [0,1] \times R \times R$.

Differential equation with fractional order have recently proved valuable tools in the modeling of many phenomena in various fields of science and engineering [2–5]. There has been a significant theoretical development in fractional differential equations in recent years, see [6–17]. Recently, many researchers have paid attention to existence result of solution of the initial value problem and boundary value problem for fractional differential equations (see [18–30]). For example, in [19], Belmekki et al. investigated the existence and uniqueness of solution of the following fractional differential equation with periodic boundary value condition

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$$\mathcal{D}^{\delta}u(t) - \lambda u(t) = f\big(t, u(t)\big), \quad t \in (0, 1],\ 0 < \delta < 1, \tag{1.5}$$

$$\lim_{t \to 0^+} t^{1-\delta} u(t) = u(1), \tag{1.6}$$

by using the fixed point theorem of Schaeffer and the Banach contraction principle. In [25], Wei et al. considered the existence and uniqueness of solution of the following initial value problem for fractional differential equation involving Riemann–Liouville sequential fractional derivative

$$\mathcal{D}_{0+}^{2\alpha} y(x) = f\big(x, y, \mathcal{D}_{0+}^{\alpha} y\big), \quad x \in (0, T], \tag{1.7}$$

$$x^{1-\alpha} y(x)\big|_{x=0} = y_0, \qquad x^{1-\alpha}\big(\mathcal{D}_{0+}^{\alpha} y\big)(x) = y(1) \tag{1.8}$$

by using monotone iterative method, where $\mathcal{D}_{0+}^{\alpha} = \mathcal{D}^{\alpha}$ and $\mathcal{D}_{0+}^{2\alpha} = \mathcal{D}^{2\alpha}$ are as mentioned above.

Impulsive differential equations are now recognized as an excellent source of models to simulate process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. [31,32]. Specially, periodic boundary value for impulsive differential equation has drawn much attention, for example, see [33–35].

From the viewpoint of the theoretics and practice, it is natural for mathematics to investigate the impulsive fractional differential equations. Recently Agarwal et al. [36], Benchohra and Slimani [37] have initiated the study of impulsive fractional differential equations at fixed moments. It is interesting to consider the existence of solution of impulsive fractional differential equation with periodic boundary value condition. However, to the best of the author knowledge, no one has studied the existence of solutions for BVP (1.1)–(1.4). The purpose of this paper is to fill in this gap, that is, we will study the existence and uniqueness of solution of the periodic boundary value problem for nonlinear impulsive fractional differential equation involving Riemann–Liouville sequential fractional derivative by using the method of upper and lower solutions and its associated monotone iterative method.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $C[a, b]$ ($C(a, b]$) be the Banach space of all continuous real functions defined on $[a, b]$ ($(a, b]$). For $0 < \alpha \leqslant 1$, let

$$x_{1-\alpha}(t) = \begin{cases} t^{1-\alpha} x(t), & t \in [t_0, t_1], \\ (t - t_1)^{1-\alpha} x(t), & t \in (t_1, t_2], \\ \ldots, & \ldots \\ (t - t_m)^{1-\alpha} x(t), & t \in (t_m, t_{m+1}], \end{cases}$$

and

$$\mathcal{D}^{\alpha} x_{1-\alpha}(t) = \begin{cases} t^{1-\alpha} \mathcal{D}^{\alpha} x(t), & t \in [t_0, t_1], \\ (t - t_1)^{1-\alpha} \mathcal{D}^{\alpha} x(t), & t \in (t_1, t_2], \\ \ldots, & \ldots \\ (t - t_m)^{1-\alpha} \mathcal{D}^{\alpha} x(t), & t \in (t_m, t_{m+1}], \end{cases}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$.

In order to define the solutions of (1.1)–(1.4), we consider the Banach spaces

$$PC_{1-\alpha}[0, 1] = \Big\{x\colon x_{1-\alpha}|_{[t_0, t_1]} \in C[t_0, t_1],\ x_{1-\alpha}|_{(t_j, t_{j+1}]} \in C(t_j, t_{j+1}], \\ \text{there exist } \lim_{t \to t_j^+} (t - t_j)^{1-\alpha} x(t) \text{ and } x\big(t_j^-\big) \text{ with } x\big(t_j^-\big) = x(t_j),\ j = 1, \ldots, m\Big\}$$

with norm

$$\|x\|_{PC_{1-\alpha}} = \sup\big\{t^{1-\alpha}\big|x(t)\big|\colon t \in [0, 1]\big\},$$

and

$$PC_{1-\alpha}^{\alpha}[0, 1] = \Big\{x\colon x \in PC_{1-\alpha}[0, 1],\ \mathcal{D}^{\alpha} x_{1-\alpha}\big|_{[t_0, t_1]} \in C[t_0, t_1],\ \mathcal{D}^{\alpha} x_{1-\alpha}\big|_{(t_j, t_{j+1}]} \in C(t_j, t_{j+1}], \\ \text{there exist } \lim_{t \to t_j^+} (t - t_j)^{1-\alpha} \mathcal{D}^{\alpha} x(t) \text{ and } \mathcal{D}^{\alpha} x\big(t_j^-\big) \text{ with } \mathcal{D}^{\alpha} x\big(t_j^-\big) = \mathcal{D}^{\alpha} x(t_j),\ j = 1, \ldots, m\Big\}$$

with norm

$$\|x\|_{PC_{1-\alpha}^{\alpha}} = \|x\|_{PC_{1-\alpha}} + \big\|\mathcal{D}^{\alpha} x\big\|_{PC_{1-\alpha}}.$$

Remark 2.1. If $\alpha = 1$, then the Banach space $PC_{1-\alpha}[0,1]$ reduces to the following

$$PC[0,1] = \{x\colon\ x|_{[t_0,t_1]} \in C[t_0,t_1],\ x|_{(t_j,t_{j+1}]} \in C(t_j,t_{j+1}], \text{ there exist } x(t_j^-) \text{ and } x(t_j^+) \text{ with } x(t_j^-) = x(t_j),\ j = 1,2,\dots,m\}$$

with norm

$$\|x\|_{PC} = \sup\{|x(t)|\colon\ t \in [0,1]\}.$$

And the Banach space $PC^{\alpha}_{1-\alpha}[0,1]$ reduces to the following

$$PC^1[0,1] = \{x\colon\ x \in PC[0,1],\ x|_{[t_0,t_1]} \in C^1[t_0,t_1],\ x|_{(t_j,t_{j+1}]} \in C^1(t_j,t_{j+1}], \text{ there exist } x'(t_j^+),\ x'(t_j^-) \text{ with } x'(t_j^-) = x'(t_j),\ j = 1,2,\dots,m\}$$

with norm

$$\|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}.$$

Definition 2.1. We call a function $u(t)$ a classical solution of BVP (1.1)–(1.4), if $u \in PC^{\alpha}_{1-\alpha}[0,1]$ satisfying Eq. (1.1) for every $t \in (0,1] \setminus \{t_1,\dots,t_m\}$ and the boundary condition (1.2), and at every $t_j,\ j = 1,\dots,m$, the function satisfies (1.3) and (1.4).

Lemma 2.1. *The linear impulsive boundary value problem*

$$\mathcal{D}^{\alpha} u(t) - \lambda u(t) = \sigma(t), \quad t \in (0,1] \setminus \{t_1,\dots,t_m\}, \tag{2.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) - u(1) = k, \tag{2.2}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(u(t) - u(t_j)\big) = a_j, \quad j = 1,2,\dots,m, \tag{2.3}$$

where $\lambda, k, a_j \in R$ are constants and $\sigma \in C[0,1]$, has a unique solution $u \in PC_{1-\alpha}[0,1]$ given by

$$u(t) = \frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds + \sum_{j=1}^{m} \Gamma(\alpha) G_{\lambda,\alpha}(t,t_j) a_j, \tag{2.4}$$

where

$$G_{\lambda,\alpha}(t,s) = \begin{cases} \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} + (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}), & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})t^{\alpha-1}(1-s)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}, & 0 \leqslant t < s \leqslant 1, \end{cases}$$

$$= \begin{cases} \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} e_{\alpha}(\lambda,t)e_{\alpha}(\lambda,1-s) + e_{\alpha}(\lambda,t-s), & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} e_{\alpha}(\lambda,t)e_{\alpha}(\lambda,1-s), & 0 \leqslant t < s \leqslant 1. \end{cases} \tag{2.5}$$

Here, $E_{\alpha,\alpha}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma((k+1)\alpha)}$ is Mittag-Leffler function (see [4]*), and $e_{\alpha}(\lambda,t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$.*

Proof. From (3.26) of [19], we know that the general solution of the nonhomogeneous equation (2.1) is

$$u(t) = c\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-s)^{\alpha}\big)\sigma(s)\,ds, \quad t \in (0,1] \setminus \{t_1,\dots,t_m\}, \tag{2.6}$$

where

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = c. \tag{2.7}$$

By (2.6), (2.7), and boundary condition (2.2), similar to the proof of [19], we can easily get

$$\bar{u}(t) = \frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds \tag{2.8}$$

is the unique solution of the linear problem (2.1)–(2.2). Set

$$w_j(t)=G_{\lambda,\alpha}(t,t_j)\Gamma(\alpha)a_j,\quad j=1,\dots,m,\ t\in[0,1].$$

For each $t<t_j$, we have

$$\begin{aligned}\mathcal{D}^\alpha w_j(t)&=\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\mathcal{D}^\alpha\big(t^{\alpha-1}E_{\alpha,\alpha}\big(\lambda t^\alpha\big)\big)\\&=\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\mathcal{D}^\alpha\left(\sum_{i=1}^\infty\frac{\lambda^{i-1}}{\Gamma(\alpha i)}t^{\alpha i-1}\right).\end{aligned}$$

Using the identities

$$\mathcal{D}^\alpha t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}t^{\mu-\alpha}\quad(\mu>-1),\qquad \mathcal{D}^\alpha t^{\alpha-1}=0,$$

we get

$$\begin{aligned}\mathcal{D}^\alpha w_j(t)&=\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{i=2}^\infty\frac{\lambda^{i-1}}{\Gamma(\alpha(i-1))}t^{\alpha(i-1)-1}\\&=\lambda\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{i=1}^\infty\frac{\lambda^{i-1}}{\Gamma(\alpha i)}t^{\alpha i-1}\\&=\lambda\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}\sum_{i=0}^\infty\frac{(\lambda t^\alpha)^i}{\Gamma(\alpha i+\alpha)}\\&=\lambda\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\Gamma(\alpha)a_j\\&=\lambda G_{\alpha,\alpha}(t,t_j)\Gamma(\alpha)a_j=\lambda w_j(t),\quad t<t_j.\end{aligned}$$

Similarly, we can obtain that

$$\mathcal{D}^\alpha w_j(t)-\lambda w_j(t)=0,\quad t>t_j.$$

Thus, we have $\mathcal{D}^\alpha w_j(t)-\lambda w_j(t)=0$ for $t\in(0,1]\setminus\{t_j\}$. Moreover, we have

$$\begin{aligned}&\lim_{t\to0^+}t^{1-\alpha}w_j(t)-w_j(1)\\&=\lim_{t\to0^+}t^{1-\alpha}\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}\big(\lambda t^\alpha\big)\\&\quad-\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}E_{\alpha,\alpha}(\lambda)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}+(1-t_j)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(1-t_j)^\alpha\big)\right)\Gamma(\alpha)a_j\\&=\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)(1-t_j)^{\alpha-1}a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}-\frac{(1-t_j)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-t_j)^\alpha)\Gamma(\alpha)a_j}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\\&=0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(w_j(t)-w_j(t_j)\big)&=\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\cdot(t-t_j)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-t_j)^\alpha\big)\Gamma(\alpha)a_j\\&=\lim_{t\to t_j^+}E_{\alpha,\alpha}\big(\lambda(t-t_j)^\alpha\big)\Gamma(\alpha)a_j=\lim_{t\to t_j^+}\sum_{i=0}^\infty\frac{(\lambda(t-t_j)^\alpha)^i}{\Gamma(\alpha i+\alpha)}\Gamma(\alpha)a_j\\&=\frac{1}{\Gamma(\alpha)}\Gamma(\alpha)a_j=a_j.\end{aligned}$$

In consequence, we conclude that

$$u = \bar{u} + \sum_{j=1}^{m} w_j$$
$$= \frac{k\Gamma(\alpha)}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t^{\alpha-1} E_{\alpha,\alpha}\big(\lambda t^{\alpha}\big) + \int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds + \sum_{j=1}^{m} \Gamma(\alpha) G_{\lambda,\alpha}(t,t_j)a_j$$

is the solution of problem (2.1)–(2.3), and $u \in PC_{1-\alpha}[0,1]$.

Next we prove that the solution of BVP (2.1)–(2.3) is unique. Suppose that $u_1, u_2 \in PC_{1-\alpha}[0,1]$ are two solutions of BVP (2.1)–(2.3). Let $w = u_1 - u_2$, then we have

$$\mathcal{D}^{\alpha} w(t) - \lambda w(t) = 0, \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{2.9}$$
$$\lim_{t \to 0^+} t^{1-\alpha} w(t) - w(1) = 0, \tag{2.10}$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha}\big(w(t) - w(t_j)\big) = 0, \quad j = 1, 2, \ldots, m. \tag{2.11}$$

By (2.8), (2.9) and (2.10), we get that $w(t) \equiv 0$ for any $t \in (0,1] \setminus \{t_1, \ldots, t_m\}$. Since $w \in PC_{1-\alpha}[0,1]$, we have

$$\lim_{t \to t_j^-} t^{1-\alpha} w(t) = t_j^{1-\alpha} w(t_j).$$

On the other hand, $\lim_{t \to t_j^-} t^{1-\alpha} w(t) = 0$. Thus, we obtain that $w(t_j) = 0$, $j = 1, \ldots, m$. Hence, $u_1(t) = u_2(t)$ for each $t \in (0,1]$. Moreover, by (2.9), we have $\lim_{t \to 0^+} t^{1-\alpha} w(t) = w(1) = 0$, which implies that

$$\lim_{t \to 0^+} t^{1-\alpha} u_1(t) = \lim_{t \to 0^+} t^{1-\alpha} u_2(t).$$

Therefore, $u_1 \equiv u_2$. □

Remark 2.2. If $\alpha = 1$, then Lemma 2.1 reduces to the following corollary:

Corollary 2.1. *The linear impulsive boundary value problem*

$$u'(t) - \lambda u(t) = \sigma(t), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{2.12}$$
$$u(0) = u(1), \tag{2.13}$$
$$u\big(t_j^+\big) - u(t_j) = a_j, \quad j = 1, 2, \ldots, m, \tag{2.14}$$

where $\lambda, a_j \in R$ are constants and $\sigma \in C[0,1]$, has a unique solution $u \in PC[0,1]$ given by

$$u(t) = \int_0^1 g(t,s)\sigma(s)\,ds + \sum_{j=1}^{m} g(t,t_j)a_j, \tag{2.15}$$

where

$$g(t,s) = \frac{1}{1 - e^{\lambda}} \begin{cases} e^{\lambda(t-s)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ e^{\lambda(1+t-s)}, & 0 \leqslant t < s \leqslant 1. \end{cases}$$

Remark 2.3. If $J = [0,1]$ and a_j in Corollary 2.1 are replaced by $J = [0,T]$ and $I_j(u(t_j))$, respectively, then Corollary 2.1 is reduce to Lemma 2.1 in [38].

Lemma 2.2. *The linear impulsive boundary value problem*

$$\mathcal{D}^{2\alpha} u(t) + p\mathcal{D}^{\alpha} u(t) + qu(t) = \sigma(t), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{2.16}$$
$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = u(1), \qquad \lim_{t \to 0^+} t^{1-\alpha} \mathcal{D}^{\alpha} u(t) = \mathcal{D}^{\alpha} u(1), \tag{2.17}$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha}\big(u(t) - u(t_j)\big) = a_j, \quad j = 1, \ldots, m, \tag{2.18}$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha} u(t) - \mathcal{D}^{\alpha} u(t_j)\big) = b_j, \quad j = 1, \ldots, m, \tag{2.19}$$

where $p, q, a_j, b_j \in R$ are constants with $p, q > 0$ and $p^2 > 4q$ and $\sigma \in C[0,1]$, has the following representation of solution

$$u(t) = \int_0^1 G_{\lambda_2,\alpha}(t,s) \int_0^1 G_{\lambda_1,\alpha}(s,\tau)\sigma(\tau)\,d\tau + \sum_{j=1}^m \Gamma(\alpha)(b_j - \lambda_2 a_j) \int_0^1 G_{\lambda_2,\alpha}(t,s) G_{\lambda_1,\alpha}(s,t_j)\,ds + \sum_{j=1}^m \Gamma(\alpha) G_{\lambda_2,\alpha}(t,t_j) a_j, \tag{2.20}$$

where $G_{\lambda_i,\alpha}(t,s)$ $(i = 1,2)$ are as in (2.5), and

$$\lambda_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} < 0, \qquad \lambda_2 = \frac{-p - \sqrt{p^2 - 4q}}{2} < 0. \tag{2.21}$$

Proof. Let $(\mathcal{D}^\alpha - \lambda_2)u(t) = x(t)$, $t \in (0,1] \setminus \{t_1, \ldots, t_m\}$.

Then the problem (2.16)–(2.19) is equivalent to

$$\left(\mathcal{D}^\alpha - \lambda_2\right)u(t) = x(t), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{2.22}$$
$$\lim_{t \to 0^+} t^{1-\alpha} u(t) = u(1), \tag{2.23}$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha} \left(u(t) - u(t_j)\right) = a_j, \tag{2.24}$$

and

$$\left(\mathcal{D}^\alpha - \lambda_1\right)x(t) = \sigma(t), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{2.25}$$
$$\lim_{t \to 0^+} t^{1-\alpha} x(t) = x(1), \tag{2.26}$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha} \left(x(t) - x(t_j)\right) = b_j - \lambda_2 a_j. \tag{2.27}$$

By Lemma 2.1 with $k = 0$, we obtain that BVPs (2.22)–(2.24) and (2.25)–(2.27) have the following representation of solutions

$$u(t) = \int_0^1 G_{\lambda_2,\alpha}(t,s) x(s)\,ds + \sum_{j=1}^m \Gamma(\alpha) G_{\lambda_2,\alpha}(t,t_j) a_j, \tag{2.28}$$
$$x(t) = \int_0^1 G_{\lambda_1,\alpha}(t,s)\sigma(s)\,ds + \sum_{j=1}^m \Gamma(\alpha) G_{\lambda_1,\alpha}(t,t_j)(b_j - \lambda_2 a_j), \tag{2.29}$$

respectively. Substituting (2.29) into (2.28), we get (2.20). □

Lemma 2.3. *(See [30].) For $0 < \alpha \leqslant 1$, we have*

$$0 < E_{\alpha,\alpha}(x) < \frac{1}{\Gamma(\alpha)}, \quad \textit{for } x < 0.$$

The following result will play a very important role in this paper.

Lemma 2.4 *(A comparison result). If $y \in PC_{1-\alpha}[0,1] \cap L_1(0,1)$ and satisfies the relations*

$$\mathcal{D}^{2\alpha} y(t) + p\mathcal{D}^\alpha y(t) + q y(t) \geqslant 0, \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\},$$
$$\lim_{t \to 0^+} t^{1-\alpha} y(t) \geqslant y(1), \qquad \lim_{t \to 0^+} t^{1-\alpha} D^\alpha y(t) \geqslant D^\alpha y(1),$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha} \left(y(t) - y(t_j)\right) \geqslant 0,$$
$$\lim_{t \to t_j^+} (t - t_j)^{1-\alpha} \left(D^\alpha y(t) - D^\alpha y(t_j)\right) \geqslant 0,$$

where p and q are positive constants with $p^2 > 4q$, then $y(t) \geqslant 0$ for each $t \in (0,1]$.

Proof. By $\lambda_1, \lambda_2 < 0$ and Lemma 2.3, we get that $0 < E_{\alpha,\alpha}(\lambda_1), E_{\alpha,\alpha}(\lambda_2) < \frac{1}{\Gamma(\alpha)}$. Thus,

$$1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1) > 0, \qquad 1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda_2) > 0. \tag{2.30}$$

For any $\sigma \in C[0,1]$ with $\sigma(t) \geqslant 0$, $t \in [0,1]$, and $k, l, a_j, b_j \geqslant 0$ being constants ($j = 1, \ldots, m$), we consider the following boundary value problem:

$$\begin{cases} \mathcal{D}^{2\alpha} y(t) + p\mathcal{D}^{\alpha} y(t) + qy(t) = \sigma(t), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \\ \lim\limits_{t\to 0^+} t^{1-\alpha} y(t) - y(1) = k, \qquad \lim\limits_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha} y(t) - \mathcal{D}^{\alpha} y(1) = l, \\ \lim\limits_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(y(t) - y(t_j)\big) = a_j, \quad j = 1, \ldots, m, \\ \lim\limits_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha} y(t) - \mathcal{D}^{\alpha} y(t_j)\big) = b_j, \quad j = 1, \ldots, m. \end{cases} \tag{2.31}$$

By Lemma 2.1, similar to the proof of Lemma 2.2, we can easily obtain that the representation of solution for BVP (2.31) is as follows

$$\begin{aligned} y(t) = {} & \int_0^1 G_{\lambda_2,\alpha}(t,s)\int_0^1 G_{\lambda_1,\alpha}(s,\tau)\,d\tau\,ds + \frac{k\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_2)} t^{\alpha-1}E_{\alpha,\alpha}\big(\lambda_2 t^{\alpha}\big) + \sum_{j=1}^{m}\Gamma(\alpha)G_{\lambda_2,\alpha}(t,t_j)a_j \\ & + \frac{(l-\lambda_2 k)\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}\int_0^1 G_{\lambda_2,\alpha}(t,s)s^{\alpha-1}E_{\alpha,\alpha}\big(\lambda_1 s^{\alpha}\big)\,ds \\ & + \sum_{j=1}^{m}\Gamma(\alpha)(b_j - \lambda_2 a_j)\int_0^1 G_{\lambda_2,\alpha}(t,s)G_{\lambda_1,\alpha}(s,t_j)\,ds. \end{aligned} \tag{2.32}$$

By (2.30), (2.32) and $k, l - \lambda_2 k, a_j, b_j - \lambda_2 a_j \geqslant 0$ ($j = 1, \ldots, m$), we get $y(t) \geqslant 0$ for $t \in [0,1]$. □

3. Main results

Definition 3.1. Let $v_0, w_0 \in PC_{1-\alpha}^{\alpha}[0,1]$. v_0 is called a lower solution of the problem (1.1)–(1.4) if it satisfies

$$\mathcal{D}^{2\alpha} v_0(t) \leqslant f\big(t, v_0, \mathcal{D}^{\alpha} v_0\big), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{3.1}$$

$$\lim_{t\to 0^+} t^{1-\alpha} v_0(t) \leqslant v_0(1), \qquad \lim_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha} v_0(t) \leqslant \mathcal{D}^{\alpha} v_0(1), \tag{3.2}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(v_0(t) - v_0(t_j)\big) \leqslant I_j\big(v_0(t_j)\big), \tag{3.3}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha} v_0(t) - \mathcal{D}^{\alpha} v_0(t_j)\big) \leqslant \bar{I}_j\big(v_0(t_j)\big). \tag{3.4}$$

And w_0 is called an upper solution of the problem (1.1)–(1.4) if it satisfies

$$\mathcal{D}^{2\alpha} w_0(t) \geqslant f\big(t, w_0, \mathcal{D}^{\alpha} w_0\big), \quad t \in (0,1] \setminus \{t_1, \ldots, t_m\}, \tag{3.5}$$

$$\lim_{t\to 0^+} t^{1-\alpha} w_0(t) \geqslant w_0(1), \qquad \lim_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha} w_0(t) \geqslant \mathcal{D}^{\alpha} w_0(1), \tag{3.6}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(w_0(t) - w_0(t_j)\big) \geqslant I_j\big(w_0(t_j)\big), \tag{3.7}$$

$$\lim_{t\to t_j^+} (t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha} w_0(t) - \mathcal{D}^{\alpha} w_0(t_j)\big) \geqslant \bar{I}_j\big(w_0(t_j)\big). \tag{3.8}$$

In the following, we assume that

$$\begin{cases} v_0(t) \leqslant w_0(t),\ t \in (0,1]: \quad \lim\limits_{t\to 0^+} t^{1-\alpha} v_0(t) \leqslant \lim\limits_{t\to 0^+} t^{1-\alpha} w_0(t), \\ \lim\limits_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha} v_0(t) \leqslant \lim\limits_{t\to 0^+} t^{1-\alpha}\mathcal{D}^{\alpha} w_0(t), \end{cases} \tag{3.9}$$

and define the order interval in space $PC_{1-\alpha}^{\alpha}[0,1]$:

$$[v_0, w_0] = \Big\{u \in PC^{\alpha}_{1-\alpha}[0,1]:\ v_0(t) \leqslant u(t) \leqslant w_0(t),\ t \in (0,1],$$
$$\lim_{t\to 0^+} t^{1-\alpha} v_0(t) \leqslant \lim_{t\to 0^+} t^{1-\alpha} u(t) \leqslant \lim_{t\to 0^+} t^{1-\alpha} w_0(t),$$
$$\lim_{t\to 0^+} t^{1-\alpha} \mathcal{D}^{\alpha} v_0(t) \leqslant \lim_{t\to 0^+} t^{1-\alpha} \mathcal{D}^{\alpha} u(t) \leqslant \lim_{t\to 0^+} t^{1-\alpha} \mathcal{D}^{\alpha} w_0(t)\Big\}.$$

Let

$$M_1(t) := \mathcal{D}^{\alpha} v_0(t) + \lambda_2\big(w_0(t) - v_0(t)\big), \qquad M_2(t) := \mathcal{D}^{\alpha} w_0(t) - \lambda_2\big(w_0(t) - v_0(t)\big).$$

For convenience, we shall assume that f satisfies the following conditions:

(H_1) there exist constants $p, q > 0$ with $p^2 > 4q$ such that

$$f\big(t, w_0, \mathcal{D}^{\alpha} w_0\big) - f\big(t, v_0, \mathcal{D}^{\alpha} v_0\big) \geqslant -p\big(\mathcal{D}^{\alpha} w_0 - \mathcal{D}^{\alpha} v_0\big) - q(w_0 - v_0),$$

where $t \in (0,1] \setminus \{t_1, \ldots, t_m\}$, $v_0, w_0 \in PC^{\alpha}_{1-\alpha}[0,1]$ are lower and upper solutions of problem (1.1)–(1.4);

(H_2) there exist positive constants p, q with $p^2 > 4q$ such that

$$f(t, x_2, y_2) - f(t, x_1, y_1) \geqslant -p(y_2 - y_1) - q(x_2 - x_1),$$

where $t \in (0,1] \setminus \{t_1, \ldots, t_m\}$, $v_0(t) \leqslant x_1 \leqslant x_2 \leqslant w_0(t)$, $M_1(t) \leqslant y_i \leqslant M_2(t)$, $i = 1, 2$;

(H_3) $I_j, \bar{I}_j \in C(R, R)$, $I_j(y) \geqslant I_j(x)$ and $\bar{I}_j(y) \geqslant \bar{I}_j(x)$, $\forall v_0(t_j) \leqslant x \leqslant y \leqslant w_0(t_j)$, $j = 1, 2, \ldots, m$.

Lemma 3.1. *Suppose that* (H_1) *and* (H_3) *hold. Then*

$$\mathcal{D}^{\alpha}\big(w_0(t) - v_0(t)\big) - \lambda_2\big(w_0(t) - v_0(t)\big) \geqslant 0, \quad t \in (0,1]. \tag{3.10}$$

Proof. Let $y(t) = \mathcal{D}^{\alpha}(w_0(t) - v_0(t)) - \lambda_2(w_0(t) - v_0(t))$, $t \in (0,1]$. Then by (H_1) and (H_3), we have

$$\begin{aligned}
\mathcal{D}^{\alpha} y(t) - \lambda_1 y(t) &= \mathcal{D}^{2\alpha}\big(w_0(t) - v_0(t)\big) + p\mathcal{D}^{\alpha}\big(w_0(t) - v_0(t)\big) + q\big(w_0(t) - v_0(t)\big) \\
&\geqslant f\big(t, w_0, \mathcal{D}^{\alpha} w_0\big) - f\big(t, v_0, \mathcal{D}^{\alpha} v_0\big) + p\mathcal{D}^{\alpha}\big(w_0(t) - v_0(t)\big) + q\big(w_0(t) - v_0(t)\big) \geqslant 0, \\
\lim_{t\to 0^+} t^{1-\alpha} y(t) - y(1) &= \lim_{t\to 0^+} t^{1-\alpha}\big(\mathcal{D}^{\alpha} w_0(t) - \mathcal{D}^{\alpha} v_0(t)\big) - \mathcal{D}^{\alpha}\big(w_0(1) - v_0(1)\big) \\
&\quad - \lambda_2 \lim_{t\to 0^+} t^{1-\alpha}\big(w_0(t) - v_0(t)\big) + \lambda_2\big(w_0(1) - v_0(1)\big) \geqslant 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to t_j^+} (t - t_j)^{1-\alpha}\big(y(t) - y(t_j)\big) &\geqslant \bar{I}_j\big(w_0(t_j)\big) - \bar{I}_j\big(v_0(t_j)\big) - \lambda_2\big[I_j\big(w_0(t_j)\big) - I_j\big(v_0(t_j)\big)\big] \\
&\geqslant 0, \quad j = 1, 2, \ldots, m.
\end{aligned}$$

By (2.4) (the representation of solution for BVP (2.1)–(2.3)), we have $y(t) \geqslant 0$ for $t \in (0,1]$. This completes the proof of Lemma 3.1. □

Remark 3.1. From Lemma 3.1, we obtain that

$$M_1(t) \leqslant D^{\alpha} v_0(t) \leqslant M_2(t), \quad t \in (0,1]. \tag{3.11}$$

Lemma 3.2. *Suppose that* (H_1) *and* (H_3) *hold. Then*

$$\Omega = \big\{\eta \in [v_0, w_0]:\ M_1(t) \leqslant \big(\mathcal{D}^{\alpha}\eta\big)(t) \leqslant M_2(t),\ t \in (0,1]\big\}$$

is a convex closed set.

For convenience, set $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$.

For $B \subset PC^{\alpha}_{1-\alpha}[0,1]$, we denote $B^{\alpha} = \{\mathcal{D}^{\alpha} u(t):\ u \in B\}$. Similar to the proof of Lemma 3 of [39], we get the following lemma.

Lemma 3.3. *If $B \subset PC^{\alpha}_{1-\alpha}[0,1]$ is bounded and the elements of $\mathcal{D}^{\alpha} B$ are equicontinuous on each J_k ($k = 0, 1, \ldots, m$), then B is a compact set.*

The Laplace convolution operator of two functions f and g, given on R^+, is defined for $t \in R^+$ by the integral

$$f * g = (f * g)(t) = \int_0^t f(t-s)g(s)\,ds.$$

Lemma 3.4. *If $\sigma \in C[0,1]$, then*

(1)
$$\mathcal{D}^\alpha\big(\big(e_\alpha(\lambda,\cdot) * \sigma\big)(t)\big) = \sigma(t) + \lambda\big(e_\alpha(\lambda,\cdot) * \sigma\big)(t), \tag{3.12}$$

where $e_\alpha(\lambda,\cdot)(t) := e_\alpha(\lambda,t)$;

(2)
$$\mathcal{D}^\alpha\left(\int_0^1 e_\alpha(\lambda,t)\sigma(s)\,ds\right) = \lambda\int_0^1 e_\alpha(\lambda,t)\sigma(s)\,ds; \tag{3.13}$$

(3)
$$\mathcal{D}^\alpha\left(\int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds\right) = \sigma(t) + \lambda\int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds. \tag{3.14}$$

Proof. Let $\sigma \in C[0,1]$. Then we have

$$\begin{aligned}
\mathcal{D}^\alpha\big(\big(e_\alpha(\lambda,\cdot) * \sigma\big)(t)\big) &= \mathcal{D}^\alpha\left(\int_0^t e_\alpha(\lambda,t-s)\sigma(s)\,ds\right)\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}\int_0^s e_\alpha(\lambda,s-\tau)\sigma(\tau)\,d\tau\,ds\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \sigma(\tau)\left(\int_\tau^t (t-s)^{-\alpha}e_\alpha(\lambda,s-\tau)\,ds\right)d\tau\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \sigma(\tau)\left(\int_\tau^t (t-s)^{-\alpha}\cdot(s-\tau)^{\alpha-1}\sum_{k=0}^\infty \frac{\lambda^k(s-\tau)^{\alpha k}}{\Gamma((k+1)\alpha)}\,ds\right)d\tau\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left[\int_0^t \sigma(\tau)\left(\frac{1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{-\alpha}\cdot(s-\tau)^{\alpha-1}\,ds\right)d\tau\right.\\
&\quad \left.+\int_0^t \sigma(\tau)\left(\sum_{k=1}^\infty \frac{\lambda^k}{\Gamma((k+1)\alpha)}\int_\tau^t (t-s)^{-\alpha}\cdot(s-\tau)^{\alpha(k+1)-1}\,ds\right)d\tau\right]\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left[\Gamma(1-\alpha)\int_0^t \sigma(\tau)\,d\tau + \sum_{k=1}^\infty \frac{\lambda^k\Gamma(1-\alpha)}{\Gamma(k\alpha+1)}\int_0^t \sigma(\tau)(t-\tau)^{k\alpha}\,d\tau\right]\\
&= \sigma(t) + \sum_{k=1}^\infty \frac{\lambda^k}{\Gamma(k\alpha)}\int_0^t \sigma(\tau)(t-\tau)^{k\alpha-1}\,d\tau\\
&= \sigma(t) + \lambda\int_0^t \sum_{k=0}^\infty (t-\tau)^{\alpha-1}\frac{\lambda^k(t-\tau)^{k\alpha}}{\Gamma((k+1)\alpha)}\sigma(\tau)\,d\tau\\
&= \sigma(t) + \lambda\int_0^t e_\alpha(\lambda,t-\tau)\sigma(\tau)\,d\tau.
\end{aligned}$$

Hence, we obtain (3.12). Next, we prove that (3.13) holds. In fact, we have

$$\mathcal{D}^{\alpha}\left(\int_0^1 e_{\alpha}(\lambda,t)\sigma(s)\,ds\right)=\mathcal{D}^{\alpha}\left(e_{\alpha}(\lambda,t)\int_0^1\sigma(s)\,ds\right)$$

$$=\mathcal{D}^{\alpha}\big(e_{\alpha}(\lambda,t)\big)\int_0^1\sigma(s)\,ds=\lambda e_{\alpha}(\lambda,t)\int_0^1\sigma(s)\,ds=\lambda\int_0^1 e_{\alpha}(\lambda,t)\sigma(s)\,ds.$$

Finally, by (2.5), (3.12) and (3.13), we obtain

$$\mathcal{D}^{\alpha}\left(\int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds\right)=\mathcal{D}^{\alpha}\left(\int_0^1\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}e_{\alpha}(\lambda,t)e_{\alpha}(\lambda,1-s)\sigma(s)\,ds+\int_0^t e_{\alpha}(\lambda,t-s)\sigma(s)\,ds\right)$$

$$=\lambda\int_0^1\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}e_{\alpha}(\lambda,t)e_{\alpha}(\lambda,1-s)\sigma(s)\,ds+\sigma(t)+\lambda\int_0^t e_{\alpha}(\lambda,t-s)\sigma(s)\,ds$$

$$=\sigma(t)+\lambda\int_0^1 G_{\lambda,\alpha}(t,s)\sigma(s)\,ds.$$

Thus, (3.14) is proved. □

By (2.5), it is easy to check the following lemma.

Lemma 3.5. $\mathcal{D}^{\alpha}(G_{\lambda,\alpha}(t,t_j))=\lambda G_{\lambda,\alpha}(t,t_j)$.

Now we are in the position to state our main result.

Theorem 3.6. *Assume that $v_0, w_0\in PC^{\alpha}_{1-\alpha}[0,1]$ are lower and upper solutions of problem (1.1)–(1.4), such that (3.9) holds. Suppose that $f\in C([0,1]\times R\times R)$, $I_j,\bar{I}_j\in C(R,R)$, and satisfies (H$_1$)–(H$_3$). Then there exist sequences $\{v_n\},\{w_n\}\subset PC^{\alpha}_{1-\alpha}[0,1]$ such that $\lim_{n\to\infty}v_n(t)=\rho(t)$, $\lim_{n\to\infty}w_n(t)=\gamma(t)$ on $(0,1]$ and ρ,γ are minimal and maximal solutions on the order interval $[v_0,w_0]$ for BVP (1.1)–(1.4), respectively, that is ρ,γ are two solutions of BVP (1.1)–(1.4), and for any solution u of BVP (1.1)–(1.4) such that $u\in[v_0,w_0]$, we have*

$$v_0\leqslant v_1\leqslant\cdots\leqslant v_n\leqslant\cdots\leqslant\rho\leqslant u\leqslant\gamma\leqslant\cdots\leqslant w_n\leqslant\cdots\leqslant w_1\leqslant w_0. \tag{3.15}$$

Proof. Let $\sigma(\eta)(t)=f(t,\eta(t),\mathcal{D}^{\alpha}\eta(t))+p\mathcal{D}^{\alpha}\eta(t)+q\eta(t)$, $t\in(0,1]$. For any $\eta\in\Omega$, consider the linear BVP

$$\mathcal{D}^{2\alpha}u(t)+p\mathcal{D}^{\alpha}u(t)+qu(t)=\sigma(\eta)(t),\quad t\in(0,1]\setminus\{t_1,\dots,t_m\}, \tag{3.16}$$

$$\lim_{t\to0^+}t^{1-\alpha}u(t)=u(1),\qquad \lim_{t\to0^+}t^{1-\alpha}\mathcal{D}^{\alpha}u(t)=\mathcal{D}^{\alpha}u(1), \tag{3.17}$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(u(t)-u(t_j)\big)=I_j\big(\eta(t_j)\big), \tag{3.18}$$

$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha}u(t)-\mathcal{D}^{\alpha}u(t_j)\big)=\bar{I}_j\big(\eta(t_j)\big). \tag{3.19}$$

By Lemma 2.2, BVP (3.16)–(3.19) has exactly one solution $u\in PC^{\alpha}_{1-\alpha}[0,1]$ given by

$$u(t):=(A\eta)(t)=\int_0^1 G_{\lambda_2,\alpha}(t,s)\int_0^1 G_{\lambda_1,\alpha}(s,\tau)\sigma(\eta)(\tau)\,d\tau\,ds$$
$$+\sum_{j=1}^m\Gamma(\alpha)\big(\bar{I}_j\big(\eta(t_j)\big)-\lambda_2 I_j\big(\eta(t_j)\big)\big)\int_0^1 G_{\lambda_2,\alpha}(t,s)G_{\lambda_1,\alpha}(s,t_j)\,ds+\sum_{j=1}^m\Gamma(\alpha)G_{\lambda_2,\alpha}(t,t_j)I_j\big(\eta(t_j)\big). \tag{3.20}$$

By Lemmas 3.4 and 3.5, we can obtain that

$$\big(\mathcal{D}^{\alpha}A\eta\big)(t)=\lambda_2\int_0^1 G_{\lambda_2,\alpha}(t,s)\int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(\eta)(\theta)\,d\theta\,ds+\int_0^1 G_{\lambda_1,\alpha}(t,\theta)\sigma(\eta)(\theta)\,d\theta$$
$$+\lambda_2\sum_{j=1}^{m}\Gamma(\alpha)\big(\bar{I}_j\big(\eta(t_j)\big)-\lambda_2 I_j\big(\eta(t_j)\big)\big)\int_0^1 G_{\lambda_2,\alpha}(t,s)G_{\lambda_1,\alpha}(s,t_j)\,ds$$
$$+\sum_{j=1}^{m}\Gamma(\alpha)\big(\bar{I}_j\big(\eta(t_j)\big)-\lambda_2 I_j\big(\eta(t_j)\big)\big)G_{\lambda_1,\alpha}(t,t_j)+\lambda_2\sum_{j=1}^{m}\Gamma(\alpha)G_{\lambda_2,\alpha}(t,t_j)I_j\big(\eta(t_j)\big). \tag{3.21}$$

It is easy to check that A is an operator from Ω into $PC^{\alpha}_{1-\alpha}[0,1]$ and η is a solution to BVP (1.1)–(1.4) if and only if η is a fixed point of A.

Now we prove that

$$v_0\leqslant Av_0,\qquad Aw_0\leqslant w_0. \tag{3.22}$$

Set $v_1=Av_0$, then we have by (3.16)–(3.19) that

$$\mathcal{D}^{2\alpha}v_1(t)+p\mathcal{D}^{\alpha}v_1(t)+qv_1(t)=f\big(t,v_0(t),\mathcal{D}^{\alpha}v_0(t)\big)+p\mathcal{D}^{\alpha}v_0(t)+qv_0(t),\quad t\in(0,1]\setminus\{t_1,\dots,t_m\}, \tag{3.23}$$
$$\lim_{t\to0^+}t^{1-\alpha}v_1(t)=v_1(1),\qquad \lim_{t\to0^+}t^{1-\alpha}\mathcal{D}^{\alpha}v_1(t)=\mathcal{D}^{\alpha}v_1(1), \tag{3.24}$$
$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(v_1(t)-v_1(t_j)\big)=I_j\big(v_0(t_j)\big), \tag{3.25}$$
$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha}v_1(t)-\mathcal{D}^{\alpha}v_1(t_j)\big)=\bar{I}_j\big(v_0(t_j)\big). \tag{3.26}$$

Set $\kappa(t)=v_1(t)-v_0(t)$. Since v_0 is a lower solution of problem (1.1)–(1.4), we obtain by (3.23)–(3.26) that

$$\mathcal{D}^{2\alpha}\kappa(t)+p\mathcal{D}^{\alpha}\kappa(t)+q\kappa(t)\geqslant0,\quad t\in(0,1]\setminus\{t_1,\dots,t_m\}, \tag{3.27}$$
$$\lim_{t\to0^+}t^{1-\alpha}\kappa(t)\geqslant\kappa(1),\qquad \lim_{t\to0^+}t^{1-\alpha}\mathcal{D}^{\alpha}\kappa(t)\geqslant\mathcal{D}^{\alpha}\kappa(1), \tag{3.28}$$
$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(\kappa(t)-\kappa(t_j)\big)\geqslant0,\quad j=1,\dots,m, \tag{3.29}$$
$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(\mathcal{D}^{\alpha}\kappa(t)-\mathcal{D}^{\alpha}\kappa(t_j)\big)\geqslant0,\quad j=1,\dots,m. \tag{3.30}$$

By Lemma 2.4 and (3.27)–(3.30), we obtain that $\kappa(t)\geqslant0$, $t\in(0,1]$. Thus, $v_0\leqslant Av_0$. Similarly, we can show that $Aw_0\leqslant w_0$.

Next, let $\eta_1,\eta_2\in\Omega$ such that $\eta_1\leqslant\eta_2$. By (H_2), (H_3), Lemma 2.4 and (3.20), we have

$$\sigma(\eta_1)\leqslant\sigma(\eta_2),\qquad A\eta_1\leqslant A\eta_2. \tag{3.31}$$

Similar to the proof of Lemma 3.1, for each $\eta\in\Omega$, we get that

$$\mathcal{D}^{\alpha}\big((A\eta)(t)-v_0(t)\big)-\lambda_2\big((A\eta)(t)-v_0(t)\big)\geqslant0,\quad t\in(0,1].$$

Thus,

$$\mathcal{D}^{\alpha}(A\eta)(t)\geqslant\mathcal{D}^{\alpha}v_0(t)+\lambda_2\big((A\eta)(t)-v_0(t)\big)\geqslant M_1(t),\quad t\in(0,1].$$

Similarly, we can get that $\mathcal{D}^{\alpha}(A\eta)(t)\leqslant M_2(t)$, $t\in(0,1]$, $\forall\eta\in\Omega$. Hence, $A(\Omega)\subset\Omega$.

Let $v_n=Av_{n-1}$, $w_n=Aw_{n-1}$ ($n=1,2,\dots$). By (3.22) and (3.31), we have

$$v_0\leqslant v_1\leqslant\cdots\leqslant v_n\leqslant\cdots\leqslant w_n\leqslant\cdots\leqslant w_1\leqslant w_0, \tag{3.32}$$
$$M_1(t)\leqslant\mathcal{D}^{\alpha}v_n(t),\mathcal{D}^{\alpha}w_n(t)\leqslant M_2(t),\quad n=1,2,\dots. \tag{3.33}$$

By (3.32), we see that the upper sequence $\{w_n\}$ is monotone nonincreasing and is bounded from below and that the lower sequence $\{v_n\}$ is monotone nondecreasing and is bounded from above. Moreover, we have by (3.33) that $\mathcal{D}^{\alpha}v_n(t),\mathcal{D}^{\alpha}w_n(t)\in[M_1(t),M_2(t)]$. Let $B=\{v_n\colon n=1,2,\dots\}$. In the following, we will show that B is a relatively compact set in $PC^{\alpha}_{1-\alpha}[0,1]$.

By (H_1) and Definition 3.1, we have for each $\eta \in [v_0, w_0]$ that

$$\begin{aligned}\mathcal{D}^{2\alpha} v_0(t) + p\mathcal{D}^{\alpha} v_0(t) + qv_0(t) &\leqslant f\big(t, v_0(t), \mathcal{D}^{\alpha} v_0(t)\big) + p\mathcal{D}^{\alpha} v_0(t) + qv_0(t)\\ &\leqslant f\big(t, \eta(t), \mathcal{D}^{\alpha} \eta(t)\big) + p\mathcal{D}^{\alpha} \eta(t) + q\eta(t)\\ &\leqslant f\big(t, w_0(t), \mathcal{D}^{\alpha} w_0(t)\big) + p\mathcal{D}^{\alpha} w_0(t) + qw_0(t)\\ &\leqslant \mathcal{D}^{2\alpha} w_0(t) + p\mathcal{D}^{\alpha} w_0(t) + qw_0(t), \quad t \in (0, 1].\end{aligned}$$

Since $B, \Omega \subset PC^{\alpha}_{1-\alpha}[0,1]$ are bounded sets, therefore, $\{\sigma(v_n)(t) = f(t, v_n, \mathcal{D}^{\alpha} v_n) + p\mathcal{D}^{\alpha} v_n(t) + qv_n(t) \mid v_n \in [v_0, w_0]\}$, $\{I_j(v_n(t)) \mid v_n \in [v_0, w_0]\}$ and $\{\bar{I}_j(v_n(t)) \mid v_n \in [v_0, w_0]\}$ are bounded sets. Hence, there exist constants $N > 0$, N_j and $\bar{N}_j$ ($j = 1, \dots, m$) such that

$$\big\|\sigma(v_n)\big\|_{PC_{1-\alpha}} = \sup_{0\leqslant t\leqslant 1} t^{1-\alpha}\big|\sigma(v_n)(t)\big| \leqslant N \iff \big|\sigma(v_n)(t)\big| \leqslant Nt^{\alpha-1}, \quad \forall t \in (0, 1], \tag{3.34}$$

$$\big\|I_j(v_n)\big\|_{PC_{1-\alpha}} \leqslant N_j \iff \big|I_j\big(v_n(t_j)\big)\big| \leqslant N_j t_j^{\alpha-1}, \tag{3.35}$$

$$\big\|\bar{I}_j(v_n)\big\|_{PC_{1-\alpha}} \leqslant \bar{N}_j \iff \big|\bar{I}_j\big(v_n(t_j)\big)\big| \leqslant \bar{N}_j t_j^{\alpha-1}, \tag{3.36}$$

where $n = 1, 2, \dots$ and $j = 1, 2, \dots, m$.

From $v_n = Av_{n-1}$, (3.20) and (3.21), we have

$$\begin{aligned} v_n(t) &= \int_0^1 G_{\lambda_2,\alpha}(t,s) \int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(v_{n-1})(\theta)\,d\theta\,ds\\ &\quad + \sum_{j=1}^m \Gamma(\alpha)\big(\bar{I}_j\big(v_{n-1}(t_j)\big) - \lambda_2 I_j\big(v_{n-1}(t_j)\big)\big) \int_0^1 G_{\lambda_2,\alpha}(t,s) G_{\lambda_1,\alpha}(s,t_j)\,ds\\ &\quad + \sum_{j=1}^m \Gamma(\alpha) G_{\lambda_2,\alpha}(t,t_j) I_j\big(v_{n-1}(t_j)\big), \end{aligned} \tag{3.37}$$

$$\begin{aligned} \mathcal{D}^{\alpha} v_n(t) &= \lambda_2 \int_0^1 G_{\lambda_2,\alpha}(t,s) \int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(v_{n-1})(\theta)\,d\theta\,ds + \int_0^1 G_{\lambda_1,\alpha}(t,\theta)\sigma(v_{n-1})(\theta)\,d\theta\\ &\quad + \lambda_2 \sum_{j=1}^m \Gamma(\alpha)\big(\bar{I}_j\big(v_{n-1}(t_j)\big) - \lambda_2 I_j\big(v_{n-1}(t_j)\big)\big) \int_0^1 G_{\lambda_2,\alpha}(t,s) G_{\lambda_1,\alpha}(s,t_j)\,ds\\ &\quad + \sum_{j=1}^m \Gamma(\alpha)\big(\bar{I}_j\big(v_{n-1}(t_j)\big) - \lambda_2 I_j\big(v_{n-1}(t_j)\big)\big) G_{\lambda_1,\alpha}(t,t_j)\\ &\quad + \lambda_2 \sum_{j=1}^m \Gamma(\alpha) G_{\lambda_2,\alpha}(t,t_j) I_j\big(v_{n-1}(t_j)\big). \end{aligned} \tag{3.38}$$

For any $\tau_1, \tau_2 \in J_k$ ($k = 0, 1, \dots, m$) with $\tau_1 < \tau_2$ (we here consider the case of $\tau_1 > 0$, the case of $\tau_1 = 0$ can be considered similarly), we have by (3.38) that

$$\begin{aligned} &\big|\tau_1^{1-\alpha}\mathcal{D}^{\alpha} v_n(\tau_1) - \tau_2^{1-\alpha}\mathcal{D}^{\alpha} v_n(\tau_2)\big|\\ &\quad\leqslant \left|\lambda_2 \int_0^1 \big[\tau_1^{1-\alpha} G_{\lambda_2,\alpha}(\tau_1,s) - \tau_2^{1-\alpha} G_{\lambda_2,\alpha}(\tau_2,s)\big] \int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(v_{n-1})(\theta)\,d\theta\,ds\right|\\ &\qquad + \left|\int_0^1 \big(\tau_1^{1-\alpha} G_{\lambda_1,\alpha}(\tau_1,\theta) - \tau_2^{1-\alpha} G_{\lambda_1,\alpha}(\tau_2,\theta)\big)\sigma(v_{n-1})(\theta)\,d\theta\right|\\ &\qquad + \left|\lambda_2 \Gamma(\alpha) \sum_{j=1}^m \big(\bar{I}_j\big(v_{n-1}(t_j)\big) - \lambda_2 I_j\big(v_{n-1}(t_j)\big)\big) \int_0^1 \big(\tau_1^{1-\alpha} G_{\lambda_2,\alpha}(\tau_1,s) - \tau_2^{1-\alpha} G_{\lambda_2,\alpha}(\tau_2,s)\big) G_{\lambda_1,\alpha}(s,t_j)\,ds\right| \end{aligned}$$

$$+\left|\Gamma(\alpha)\sum_{j=1}^{m}\big(\bar{I}_j\big(v_{n-1}(t_j)\big)-\lambda_2 I_j\big(v_{n-1}(t_j)\big)\big)\big(\tau_1^{1-\alpha}G_{\lambda_1,\alpha}(\tau_1,t_j)-\tau_2^{1-\alpha}G_{\lambda_1,\alpha}(\tau_2,t_j)\big)\right|$$
$$+\left|\lambda_2\Gamma(\alpha)\sum_{j=1}^{m}I_j\big(v_{n-1}(t_j)\big)\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,t_j)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,t_j)\big)\right|. \tag{3.39}$$

In the following, we shall prove that (3.39) tends to zeros as $\tau_2\to\tau_1$. For convenience, we first prove the following claims.

Claim 1.

$$\left|\int_0^1\big(\tau_1^{1-\alpha}G_{\lambda,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda,\alpha}(\tau_2,s)\big)s^{\alpha-1}\,ds\right|\to 0,\quad \text{as } \tau_2\to\tau_1, \tag{3.40}$$

where $\lambda=\lambda_1$ *or* λ_2.

In fact, by the representation of $G_{\lambda,\alpha}(t,s)$, we get

$$\begin{aligned}
&\left|\int_0^{\tau_1}\big(\tau_1^{1-\alpha}G_{\lambda,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda,\alpha}(\tau_2,s)\big)s^{\alpha-1}\,ds\right|\\
&=\left|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda\tau_1^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^{\tau_1}E_{\alpha,\alpha}\big(\lambda(1-s)^{\alpha}\big)(1-s)^{\alpha-1}s^{\alpha-1}\,ds+\int_0^{\tau_1}\tau_1^{1-\alpha}E_{\alpha,\alpha}\big(\lambda(\tau_1-s)^{\alpha}\big)(\tau_1-s)^{\alpha-1}s^{\alpha-1}\,ds\right.\\
&\quad\left.-\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda\tau_2^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^{\tau_1}E_{\alpha,\alpha}\big(\lambda(1-s)^{\alpha}\big)(1-s)^{\alpha-1}s^{\alpha-1}\,ds-\int_0^{\tau_1}\tau_2^{1-\alpha}E_{\alpha,\alpha}\big(\lambda(\tau_2-s)^{\alpha}\big)(\tau_2-s)^{\alpha-1}s^{\alpha-1}\,ds\right|\\
&\leqslant\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\big|E_{\alpha,\alpha}\big(\lambda\tau_1^{\alpha}\big)-E_{\alpha,\alpha}\big(\lambda\tau_2^{\alpha}\big)\big|\int_0^{\tau_1}E_{\alpha,\alpha}\big(\lambda(1-s)^{\alpha}\big)(1-s)^{\alpha-1}s^{\alpha-1}\,ds\\
&\quad+\int_0^{\tau_1}\big|\tau_1^{1-\alpha}(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_1-s)^{\alpha}\big)-\tau_2^{1-\alpha}(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_2-s)^{\alpha}\big)\big|s^{\alpha-1}\,ds\\
&\leqslant\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\tau_1^{\alpha}(1-\tau_1)^{\alpha-1}}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda))\alpha}\big|E_{\alpha,\alpha}\big(\lambda\tau_1^{\alpha}\big)-E_{\alpha,\alpha}\big(\lambda\tau_2^{\alpha}\big)\big|\\
&\quad+\int_0^{\tau_1}\big|\tau_1^{1-\alpha}(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_1-s)^{\alpha}\big)-\tau_2^{1-\alpha}(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_2-s)^{\alpha}\big)\big|s^{\alpha-1}\,ds.
\end{aligned} \tag{3.41}$$

The estimate of the second term of (3.41) is as follows:

$$\begin{aligned}
&\int_0^{\tau_1}\big|\tau_1^{1-\alpha}(\tau_1-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_1-s)^{\alpha}\big)-\tau_2^{1-\alpha}(\tau_2-s)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_2-s)^{\alpha}\big)\big|s^{\alpha-1}\,ds\\
&=\int_0^{\tau_1}\left|\tau_1^{1-\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k(\tau_1-s)^{(k+1)\alpha-1}s^{\alpha-1}}{\Gamma((k+1)\alpha)}-\tau_2^{1-\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k(\tau_2-s)^{(k+1)\alpha-1}s^{\alpha-1}}{\Gamma((k+1)\alpha)}\right|ds\\
&\leqslant\int_0^{\tau_1}\big|\tau_1^{1-\alpha}-\tau_2^{1-\alpha}\big|\sum_{k=0}^{\infty}\frac{|\lambda|^k(\tau_1-s)^{(k+1)\alpha-1}s^{\alpha-1}}{\Gamma((k+1)\alpha)}\,ds\\
&\quad+\int_0^{\tau_1}\tau_2^{1-\alpha}\sum_{k=0}^{\infty}\frac{|\lambda|^k|(\tau_1-s)^{(k+1)\alpha-1}-(\tau_2-s)^{(k+1)\alpha-1}|s^{\alpha-1}}{\Gamma((k+1)\alpha)}\,ds
\end{aligned}$$

$$\leqslant \left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| \sum_{k=0}^{\infty} \frac{|\lambda|^k}{\Gamma((k+1)\alpha)} \int_0^{\tau_1} (\tau_1 - s)^{(k+1)\alpha-1} s^{\alpha-1}\, ds$$

$$+ \tau_2^{1-\alpha} \sum_{k=0}^{\infty} \frac{|\lambda|^k}{\Gamma((k+1)\alpha)} \int_0^{\tau_1} \left|(\tau_1 - s)^{(k+1)\alpha-1} - (\tau_2 - s)^{(k+1)\alpha-1}\right| s^{\alpha-1}\, ds$$

$$\leqslant \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \tau_1^{2\alpha-1} E_{\alpha,\alpha}\left(|\lambda|\tau_1^{\alpha}\right)\left|\tau_1^{1-\alpha} - \tau_2^{1-\alpha}\right| + \tau_2^{1-\alpha} \sum_{k=0}^{\infty} \frac{|\lambda|^k}{\Gamma((k+1)\alpha)} R_k, \tag{3.42}$$

where $R_k = \int_0^{\tau_1} |(\tau_1 - s)^{(k+1)\alpha-1} - (\tau_2 - s)^{(k+1)\alpha-1}| s^{\alpha-1}\, ds$. Concerning R_k, we distinguish the following two cases.

Case 1. k is such that $(k+1)\alpha < 1$. Then we have

$$R_k = \int_0^{\tau_1} \left((\tau_1 - s)^{(k+1)\alpha-1} - (\tau_2 - s)^{(k+1)\alpha-1}\right) s^{\alpha-1}\, ds$$

$$= \int_0^{\tau_1} (\tau_1 - s)^{(k+1)\alpha-1} s^{\alpha-1}\, ds - \int_0^{\tau_2} (\tau_2 - s)^{(k+1)\alpha-1} s^{\alpha-1}\, ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{(k+1)\alpha-1} s^{\alpha-1}\, ds$$

$$= \frac{\Gamma(\alpha)\Gamma((k+1)\alpha)}{\Gamma((k+2)\alpha)} \left(\tau_1^{(k+2)\alpha-1} - \tau_2^{(k+2)\alpha-1}\right) + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{(k+1)\alpha-1} s^{\alpha-1}\, ds$$

$$\leqslant \frac{\Gamma(\alpha)\Gamma((k+1)\alpha)}{\Gamma((k+2)\alpha)} \left(\tau_1^{(k+2)\alpha-1} - \tau_2^{(k+2)\alpha-1}\right) + \frac{1}{(k+1)\alpha} (\tau_2 - \tau_1)^{(k+1)\alpha} \tau_1^{\alpha-1}.$$

Case 2. k is such that $(k+1)\alpha \geqslant 1$. Without loss of generality, we assume that there exists $n \in N$ such that $2^{n-1} \leqslant (k+1)\alpha - 1 < 2^n$. Then

$$R_k = \int_0^{\tau_1} \left[(\tau_2 - s)^{(k+1)\alpha-1} - (\tau_1 - s)^{(k+1)\alpha-1}\right] s^{\alpha-1}\, ds$$

$$= \int_0^{\tau_1} \Big[\left((\tau_2 - s)^{\frac{1}{2}((k+1)\alpha-1)} + (\tau_1 - s)^{\frac{1}{2}((k+1)\alpha-1)}\right)\left((\tau_2 - s)^{\frac{1}{2^2}((k+1)\alpha-1)} + (\tau_1 - s)^{\frac{1}{2^2}((k+1)\alpha-1)}\right) \cdots$$

$$\times \left((\tau_2 - s)^{\frac{1}{2^n}((k+1)\alpha-1)} + (\tau_1 - s)^{\frac{1}{2^n}((k+1)\alpha-1)}\right)\left((\tau_2 - s)^{\frac{1}{2^n}((k+1)\alpha-1)} - (\tau_1 - s)^{\frac{1}{2^n}((k+1)\alpha-1)}\right)\Big] s^{\alpha-1}\, ds$$

$$\leqslant 2\tau_2^{\frac{1}{2}((k+1)\alpha-1)} \cdot 2\tau_2^{\frac{1}{2^2}((k+1)\alpha-1)} \cdots 2\tau_2^{\frac{1}{2^n}((k+1)\alpha-1)} \int_0^{\tau_1} \left((\tau_2 - s)^{\frac{1}{2^n}((k+1)\alpha-1)} - (\tau_1 - s)^{\frac{1}{2^n}((k+1)\alpha-1)}\right) s^{\alpha-1}\, ds$$

$$< 2^n \tau_2^{\frac{1}{2}((k+1)\alpha-1)} \int_0^{\tau_1} (\tau_2 - \tau_1)^{\frac{1}{2^n}((k+1)\alpha-1)} s^{\alpha-1}\, ds$$

$$\leqslant 2\left((k+1)\alpha - 1\right) \tau_2^{\frac{1}{2}((k+1)\alpha-1)} \int_0^{\tau_1} (\tau_2 - \tau_1)^{\frac{1}{2}} s^{\alpha-1}\, ds$$

$$= \frac{2}{\alpha}\left((k+1)\alpha - 1\right) \tau_1^{\alpha} \tau_2^{\frac{1}{2}((k+1)\alpha-1)} (\tau_2 - \tau_1)^{\frac{1}{2}}$$

by using of the following inequality

$$(x+y)^{\beta} - x^{\beta} \leqslant y^{\beta}, \quad x, y \geqslant 0,\ 0 \leqslant \beta < 1.$$

Thus,

$$\begin{aligned}
&\tau_2^{1-\alpha}\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma((k+1)\alpha)}R_k\\
&=\tau_2^{1-\alpha}\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma((k+1)\alpha)}\int_0^{\tau_1}\left|(\tau_1-s)^{(k+1)\alpha-1}-(\tau_2-s)^{(k+1)\alpha-1}\right|s^{\alpha-1}\,ds\\
&\leqslant\tau_2^{1-\alpha}\sum_{(k+1)\alpha<1}\frac{|\lambda|^k}{\Gamma((k+1)\alpha)}\left[\frac{\Gamma(\alpha)\Gamma((k+1)\alpha)}{\Gamma((k+2)\alpha)}\left(\tau_1^{(k+2)\alpha-1}-\tau_2^{(k+2)\alpha-1}\right)+\frac{(\tau_2-\tau_1)^{(k+1)\alpha}\tau_1^{\alpha-1}}{(k+1)\alpha}\right]\\
&\quad+\tau_2^{1-\alpha}\sum_{(k+1)\alpha\geqslant1}\frac{|\lambda|^k}{\Gamma((k+1)\alpha-1)}\frac{2}{\alpha}\tau_1^{\alpha}\tau_2^{\frac12((k+1)\alpha-1)}(\tau_2-\tau_1)^{\frac12}.
\end{aligned}$$

The first term (finite sum) in the right-hand side of the previous inequality clearly tends to zero as $\tau_2\to\tau_1$. Moreover, for the second term in the right-hand side of the previous inequality, let k_0 be a smallest positive integer which satisfies $(k+1)\alpha\geqslant1$, then we have

$$\begin{aligned}
&\tau_2^{1-\alpha}\sum_{(k+1)\alpha\geqslant1}\frac{|\lambda|^k}{\Gamma((k+1)\alpha-1)}\frac{2}{\alpha}\tau_1^{\alpha}\tau_2^{\frac12((k+1)\alpha-1)}(\tau_2-\tau_1)^{\frac12}\\
&=\frac{2}{\alpha}\tau_2^{\frac12(1-\alpha)}\tau_1^{\alpha}(\tau_2-\tau_1)^{\frac12}\sum_{k=k_0}^{\infty}\frac{|\lambda|^k}{\Gamma((k+1)\alpha-1)}\tau_2^{\frac12k\alpha}\\
&=\frac{2|\lambda|^{k_0}}{\alpha}\tau_2^{\frac12(1+(k_0-1)\alpha)}\tau_1^{\alpha}(\tau_2-\tau_1)^{\frac12}\sum_{j=0}^{\infty}\frac{|\lambda|^j\tau_2^{\frac12j\alpha}}{\Gamma(j\alpha+(k_0+1)\alpha-1)}\quad(j=k-k_0)\\
&=\frac{2|\lambda|^{k_0}}{\alpha}\tau_2^{\frac12(1+(k_0-1)\alpha)}\tau_1^{\alpha}E_{\alpha,(k_0+1)\alpha-1}\left(|\lambda|\tau_2^{\frac{\alpha}{2}}\right)(\tau_2-\tau_1)^{\frac12}\\
&\to0,\quad\text{as }\tau_2\to\tau_1,
\end{aligned}$$

which implies that (3.42) tends zero as $\tau_2\to\tau_1$. Hence, (3.41) tends zero as $\tau_2\to\tau_1$.

Furthermore, we have

$$\begin{aligned}
&\left|\int_{\tau_1}^{\tau_2}\left(\tau_1^{1-\alpha}G_{\lambda,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda,\alpha}(\tau_2,s)\right)s^{\alpha-1}\,ds\right|\\
&\leqslant\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\left|E_{\alpha,\alpha}\left(\lambda\tau_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda\tau_2^{\alpha}\right)\right|\int_{\tau_1}^{\tau_2}E_{\alpha,\alpha}\left(\lambda(1-s)^{\alpha}\right)(1-s)^{\alpha-1}s^{\alpha-1}\,ds\\
&\quad+\tau_2^{1-\alpha}\int_{\tau_1}^{\tau_2}E_{\alpha,\alpha}\left(\lambda(\tau_2-s)^{\alpha}\right)(\tau_2-s)^{\alpha-1}s^{\alpha-1}\,ds\\
&\leqslant\frac{\Gamma(\alpha)\tau_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|(1-\tau_1)^{\alpha})(1-\tau_2)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\left|E_{\alpha,\alpha}\left(\lambda\tau_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda\tau_2^{\alpha}\right)\right|(\tau_2-\tau_1)\\
&\quad+\frac{1}{\alpha}\tau_2^{1-\alpha}\tau_1^{\alpha-1}E_{\alpha,\alpha}\left(|\lambda|(\tau_2-\tau_1)^{\alpha}\right)(\tau_2-\tau_1)^{\alpha}\\
&\to0,\quad\text{as }\tau_2\to\tau_1,
\end{aligned}\tag{3.43}$$

and

$$\begin{aligned}
&\left|\int_{\tau_2}^{1}\left(\tau_1^{1-\alpha}G_{\lambda,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda,\alpha}(\tau_2,s)\right)s^{\alpha-1}\,ds\right|\\
&\leqslant\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\left|E_{\alpha,\alpha}\left(\lambda\tau_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda\tau_2^{\alpha}\right)\right|\int_{\tau_2}^{1}E_{\alpha,\alpha}\left(\lambda(1-s)^{\alpha}\right)(1-s)^{\alpha-1}s^{\alpha-1}\,ds
\end{aligned}$$

$$\leqslant \frac{\Gamma(\alpha)\tau_2^{\alpha-1}E_{\alpha,\alpha}(|\lambda|(1-\tau_2)^\alpha)(1-\tau_2)^\alpha}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda))\alpha}\left|E_{\alpha,\alpha}\left(\lambda\tau_1^\alpha\right)-E_{\alpha,\alpha}\left(\lambda\tau_2^\alpha\right)\right|$$
$$\to 0, \quad \text{as } \tau_2 \to \tau_1. \tag{3.44}$$

Combining (3.41) (tends to 0, as $\tau_2 \to \tau_1$), (3.43) and (3.44), we obtain that Claim 1 ((3.40)) holds.

Claim 2.

$$\left|\int_0^1 \left(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\right)\int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(v_{n-1})(\theta)\,d\theta\,ds\right| \to 0, \quad \text{as } \tau_2 \to \tau_1. \tag{3.45}$$

We first estimate the integral $\int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(v_{n-1})(\theta)\,d\theta$. By (3.34), we have

$$\begin{aligned}
&\int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma(v_{n-1})(\theta)\,d\theta \\
&\leqslant N\int_0^1 G_{\lambda_1,\alpha}(s,\theta)\theta^{\alpha-1}\,d\theta \\
&\leqslant N\int_0^s\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1 s^\alpha)E_{\alpha,\alpha}(\lambda_1(1-\theta)^\alpha)s^{\alpha-1}(1-\theta)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}+(s-\theta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_1(s-\theta)^\alpha\right)\right)\theta^{\alpha-1}\,d\theta \\
&\quad + N\int_s^1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1 s^\alpha)E_{\alpha,\alpha}(\lambda_1(1-\theta)^\alpha)s^{\alpha-1}(1-\theta)^{\alpha-1}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}\theta^{\alpha-1}\,d\theta \\
&\leqslant \frac{N\Gamma(\alpha)E_{\alpha,\alpha}^2(|\lambda_1|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}s^{\alpha-1}\int_0^s(1-\theta)^{\alpha-1}\theta^{\alpha-1}\,d\theta + NE_{\alpha,\alpha}\left(|\lambda_1|\right)\int_0^s(s-\theta)^{\alpha-1}\theta^{\alpha-1}\,d\theta \\
&\quad + \frac{N\Gamma(\alpha)E_{\alpha,\alpha}^2(|\lambda_1|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}s^{\alpha-1}\int_s^1(1-\theta)^{\alpha-1}\theta^{\alpha-1}\,d\theta \\
&= \frac{N\Gamma(\alpha)E_{\alpha,\alpha}^2(|\lambda_1|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}s^{\alpha-1}\int_0^1(1-\theta)^{\alpha-1}\theta^{\alpha-1}\,d\theta + NE_{\alpha,\alpha}\left(|\lambda_1|\right)\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}s^{2\alpha-1} \\
&= \frac{N\Gamma^3(\alpha)E_{\alpha,\alpha}^2(|\lambda_1|)}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1))\Gamma(2\alpha)}s^{\alpha-1} + NE_{\alpha,\alpha}\left(|\lambda_1|\right)\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}s^{2\alpha-1} \\
&\leqslant C_1 s^{\alpha-1},
\end{aligned} \tag{3.46}$$

where

$$C_1 = NE_{\alpha,\alpha}\left(|\lambda_1|\right)\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} + \frac{N\Gamma^3(\alpha)E_{\alpha,\alpha}^2(|\lambda_1|)}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1))\Gamma(2\alpha)}.$$

Combine (3.40) ($\lambda = \lambda_2$) and (3.46), we can infer that Claim 2 holds.

Claim 3.

$$\left|\int_0^1 \left(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\right)G_{\lambda_1,\alpha}(s,t_j)\,ds\right| \to 0, \quad \text{as } \tau_2 \to \tau_1. \tag{3.47}$$

We here only consider the case of $j < k$, that is $t_j < \tau_1$. The other cases of $j = k$ and $j > k$, we can considered similarly.

Similar to the proof of (3.41), we obtain

$$\begin{aligned}&\left|\int_0^{t_j}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)G_{\lambda_1,\alpha}(s,t_j)\,ds\right|\\&\quad\leqslant\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}E_{\alpha,\alpha}\big(\lambda_1 t_j^{\alpha}\big)E_{\alpha,\alpha}\big(\lambda_1(1-t_j)^{\alpha}\big)(1-t_j)^{\alpha-1}\\&\qquad\times\left|\int_0^{t_j}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)s^{\alpha-1}\,ds\right|\\&\quad\to 0,\quad \text{as } \tau_2\to\tau_1.\end{aligned}\tag{3.48}$$

Moreover, similar to the proof of (3.41), we get

$$\begin{aligned}&\left|\int_{t_j}^{\tau_1}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)G_{\lambda_1,\alpha}(s,t_j)\,ds\right|\\&\quad\leqslant\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda_1)}E_{\alpha,\alpha}\big(\lambda_1 \tau_1^{\alpha}\big)E_{\alpha,\alpha}\big(\lambda_1(1-t_j)^{\alpha}\big)(1-t_j)^{\alpha-1}\\&\qquad\times\left|\int_{t_j}^{\tau_1}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)s^{\alpha-1}\,ds\right|\\&\qquad+E_{\alpha,\alpha}\big(\lambda_1(\tau_1-t_j)^{\alpha}\big)\left|\int_{t_j}^{\tau_1}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)(s-t_j)^{\alpha-1}\,ds\right|\\&\quad\to 0,\quad \text{as } \tau_2\to\tau_1.\end{aligned}\tag{3.49}$$

Thus, we get by (3.48) and (3.49) that

$$\left|\int_0^{\tau_1}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)G_{\lambda_1,\alpha}(s,t_j)\,ds\right|\to 0,\quad \text{as } \tau_2\to\tau_1.\tag{3.50}$$

Similarly, we have

$$\left|\int_{\tau_1}^{\tau_2}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)G_{\lambda_1,\alpha}(s,t_j)\,ds\right|\to 0,\quad \text{as } \tau_2\to\tau_1,\tag{3.51}$$

and

$$\left|\int_{\tau_2}^{1}\big(\tau_1^{1-\alpha}G_{\lambda_2,\alpha}(\tau_1,s)-\tau_2^{1-\alpha}G_{\lambda_2,\alpha}(\tau_2,s)\big)G_{\lambda_1,\alpha}(s,t_j)\,ds\right|\to 0,\quad \text{as } \tau_2\to\tau_1.\tag{3.52}$$

From (3.49)–(3.51), we conclude that Claim 3 holds.

Claim 4.

$$\big|\big(\tau_1^{1-\alpha}G_{\lambda,\alpha}(\tau_1,t_j)-\tau_2^{1-\alpha}G_{\lambda,\alpha}(\tau_2,t_j)\big)t_j^{\alpha-1}\big|\to 0,\quad \textit{as } \tau_2\to\tau_1,\tag{3.53}$$

where $\lambda=\lambda_1$ *or* λ_2.

Equally, we here only consider the case of $j<k$, that is $t_j<\tau_1$. The other cases of $j=k$ and $j>k$, we can considered similarly. From (2.5), we have

$$\begin{aligned}&\big|\big(\tau_1^{1-\alpha}G_{\lambda,\alpha}(\tau_1,t_j)-\tau_2^{1-\alpha}G_{\lambda,\alpha}(\tau_2,t_j)\big)t_j^{\alpha-1}\big|\\&\quad\leqslant\left|\frac{\Gamma(\alpha)t_j^{1-\alpha}}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}\big(\lambda(1-t_j)^{\alpha}\big)(1-t_j)^{\alpha-1}\big(E_{\alpha,\alpha}\big(\lambda\tau_1^{\alpha}\big)-E_{\alpha,\alpha}\big(\lambda\tau_2^{\alpha}\big)\big)\right|\\&\qquad+t_j^{\alpha-1}\big|\tau_1^{1-\alpha}(\tau_1-t_j)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_1-t_j)^{\alpha}\big)-\tau_2^{1-\alpha}(\tau_2-t_j)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(\tau_2-t_j)^{\alpha}\big)\big|,\end{aligned}$$

which implies that (3.53) holds. Substituting (3.40), (3.45), (3.47) and (3.53) into (3.39), we get

$$\left|\tau_1^{1-\alpha}\mathcal{D}^{\alpha}v_n(\tau_1)-\tau_2^{1-\alpha}\mathcal{D}^{\alpha}v_n(\tau_2)\right|\to 0,\quad \text{as } \tau_2\to\tau_1.$$

This means the elements of $\mathcal{D}^{\alpha}B$ are equicontinuous on each J_k $(k=0,1,\dots,m)$. By Lemma 3.3 we get that $B=\{v_n(t)\}$ is a relatively compact set of $PC_{1-\alpha}^{\alpha}[0,1]$. Similarly, we can also show that $\{w_n(t)\}$ $(w_n(t)=Aw_{n-1}(t))$ is a relatively compact set of $PC_{1-\alpha}^{\alpha}[0,1]$. Thus, the sequences $\{v_n\}$, $\{w_n\}$ converge uniformly and monotonically to ρ, γ respectively as $n\to\infty$. Moreover, by (3.32) and (3.33), the limits ρ, γ satisfy

$$v_0(t)\leqslant v_1(t)\leqslant\cdots\leqslant v_n(t)\leqslant\cdots\leqslant\rho(t)\leqslant\gamma(t)\leqslant\cdots\leqslant w_n(t)\leqslant\cdots\leqslant w_1(t)\leqslant w_0(t),$$
$$M_1(t)\leqslant\mathcal{D}^{\alpha}\rho(t),\qquad \mathcal{D}^{\alpha}\gamma(t)\leqslant M_2(t),\quad t\in(0,1].$$

By the monotone convergence of v_n to ρ and the assumption of functions f, $\bar{I}_j$, I_j implies that $\sigma(v_n)(t)$ convergence to $\sigma(\rho)(t)$, $t\in(0,1]$. Let $n\to\infty$ in (3.37) and apply the dominated convergence theorem, ρ satisfies the following equation

$$\rho(t)=\int_0^1 G_{\lambda_2,\alpha}(t,s)\int_0^1 G_{\lambda_1,\alpha}(s,\theta)\sigma\big(\rho(\theta)\big)\,d\theta\,ds$$
$$+\sum_{j=1}^{m}\Gamma(\alpha)\big(\bar{I}_j\big(\rho(t_j)\big)-\lambda_2 I_j\big(\rho(t_j)\big)\big)\int_0^1 G_{\lambda_2,\alpha}(t,s)G_{\lambda_1,\alpha}(s,t_j)\,ds+\sum_{j=1}^{m}\Gamma(\alpha)G_{\lambda_2,\alpha}(t,t_j)I_j\big(\rho(t_j)\big),$$

which implies that $\rho(t)$ is an integral representation of the solution to problem (1.1)–(1.4). By the assumptions of f, I_j and $\bar{I}_j$ $(j=1,\dots,m)$, ρ is a classical solution of BVP (1.1)–(1.4). This proves that the sequence $\{v_n\}$ converges to a solution ρ of BVP (1.1)–(1.4). Similarly, we can prove that sequence $\{w_n\}$ converges to a solution γ of BVP (1.1)–(1.4), and satisfies relation $\rho(t)\leqslant\gamma(t)$, $M_1(t)\leqslant\mathcal{D}^{\alpha}\rho(t)$, $\mathcal{D}^{\alpha}\gamma(t)\leqslant M_2(t)$, $t\in(0,1]$. By using standard arguments, we can easily prove that (3.15) holds and $\rho(t)$ and $\gamma(t)$ are minimal and maximal solutions of BVP (1.1)–(1.4) on the order interval $[v_0,w_0]$, respectively. □

Remark 3.2. In [30], Wei et al. investigated the existence and uniqueness of solution of the periodic boundary value problem for fractional differential equation as follows:

$$\mathcal{D}^{\alpha}u(t)=f\big(t,u(t)\big),\quad t\in(0,T],$$
$$t^{1-\alpha}u(t)\big|_{t=0}=t^{1-\alpha}u(t)\big|_{t=T},$$

where $0<\alpha\leqslant 1$ and $0<T<+\infty$.

Combine our main result Theorem 3.6 and Theorem 3.1 in [30], we can easy to obtain the existence of solution of the following impulsive periodic boundary value problem for fractional differential equation

$$\mathcal{D}^{\alpha}u(t)=f\big(t,u(t)\big),\quad t\in(0,1]\setminus\{t_1,\dots,t_m\},\ 0<\alpha\leqslant 1,\tag{3.54}$$
$$\lim_{t\to 0^+}t^{1-\alpha}u(t)=u(1),\tag{3.55}$$
$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(u(t)-u(t_j)\big)=I_j\big(u(t_j)\big).\tag{3.56}$$

Let $v_0,w_0\in PC_{1-\alpha}[0,1]$. We say that the function v_0 is a lower solution for problem (3.54)–(3.56) if

$$\mathcal{D}^{\alpha}v_0(t)\leqslant f(t,v_0),\quad t\in(0,1]\setminus\{t_1,\dots,t_m\},$$
$$\lim_{t\to 0^+}t^{1-\alpha}v_0(t)\leqslant v_0(1),$$
$$\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}\big(v_0(t)-v_0(t_j)\big)\leqslant I_j\big(v_0(t_j)\big).$$

Analogously, w_0 is an upper solution for problem (3.54)–(3.56) if it verifies similar conditions for the inequalities reversed.

Theorem 3.7. *Suppose that $v_0,w_0\in PC_{1-\alpha}[0,1]$ are lower and upper solutions of problem* (3.54)–(3.56), *such that $v_0(t)\leqslant w_0(t)$ for $t\in(0,1]$ and $\lim_{t\to 0^+}t^{1-\alpha}v_0(t)\leqslant\lim_{t\to 0^+}t^{1-\alpha}w_0(t)$. Let the following conditions hold*:

(H_4) *There exists a constant $M > 0$ such that the function $f \in C([0,1] \times R)$ satisfies*

$$f(t,u_2) - f(t,u_1) \geqslant -M(u_2 - u_1), \quad v_0 \leqslant u_1 \leqslant u_2 \leqslant w_0;$$

(H_5) $I_j \in C(R,R)$, $I_j(y) \geqslant I_j(x)$, $\forall v_0(t_j) \leqslant x \leqslant y \leqslant w_0(t_j)$, $j = 1,2,\ldots,m$.

Then there exist sequences $\{v_n\}, \{w_n\} \subset PC_{1-\alpha}[0,1]$ such that $\lim_{n\to\infty} v_n(t) = \rho(t)$, $\lim_{n\to\infty} w_n(t) = \gamma(t)$ on $(0,1]$ and ρ, γ are minimal and maximal solutions on the order interval $[v_0, w_0]$ for BVP (3.54)–(3.56)*, respectively, that is ρ, γ are two solutions of BVP* (3.54)–(3.56)*, and for any solution u of BVP* (3.54)–(3.56) *such that $u \in [v_0, w_0]$, we have*

$$v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant \rho \leqslant u \leqslant \gamma \leqslant \cdots \leqslant w_n \leqslant \cdots \leqslant w_1 \leqslant w_0.$$

Example 3.1. Consider the following BVP

$$\mathcal{D}^\alpha u(t) = f\big(t, u(t)\big), \quad t \in (0,1] \setminus \{0.4\}, \tag{3.57}$$

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = u(1), \tag{3.58}$$

$$\lim_{t\to 0.4^+} (t-0.4)^{1-\alpha}\big(u(t) - u(0.4)\big) = I\big(u(0.4)\big), \tag{3.59}$$

where $\alpha = 0.2$, $f(t,u) = -\sin^2((1+t^2)u) + \frac{25}{2}(1 - \sqrt{1+|u|})$, $I(x) = \frac{1}{5}x$.

Consider the functions

$$v_0(t) = 0, \quad \forall t \in (0,1], \quad \text{and} \quad w_0(t) = \begin{cases} t^{-0.8}, & 0 < t \leqslant 0.4, \\ \frac{1}{2}(t-0.4)^{-0.8}, & 0.4 < t \leqslant 1. \end{cases}$$

It is easy to check that $v_0, w_0 \in PC_\alpha[0,1]$. Since

$$\lim_{t\to 0^+} t^{0.8} v_0(t) = 0 = v_0(1),$$
$$\lim_{t\to 0.4^+} (t-0.4)^{0.8}\big(v_0(t) - v_0(0.4)\big) = 0 = I\big(v_0(0.4)\big),$$
$$\mathcal{D}^{0.2} v_0(t) = 0 = f\big(t, v_0(t)\big), \quad 0 < t \leqslant 1,$$

we have that $v_0(t)$ is a lower solution for BVP (3.57)–(3.59). In the following, we will prove that $w_0(t)$ is an upper solution for BVP (3.57)–(3.59). Indeed,

$$\lim_{t\to 0^+} t^{0.8} w_0(t) = 1 > 0.7524 = \frac{1}{2} 0.6^{-0.8} = w_0(1), \tag{3.60}$$

$$\lim_{t\to 0.4^+} (t-0.4)^{0.8}\big(w_0(t) - w_0(0.4)\big) = \frac{1}{2} > 0.4163 = \frac{1}{5} 0.4^{-0.8} = I\big(w_0(0.4)\big). \tag{3.61}$$

Moreover, if $0 < t \leqslant 0.4$, then

$$\begin{aligned} \mathcal{D}^{0.2} w_0(t) &= \frac{1}{\Gamma(0.8)} \frac{d}{dt} \int_0^t (t-\tau)^{-0.2} \tau^{-0.8}\, d\tau = 0 \\ &> -\sin^2\big((1+t^2)t^{-0.8}\big) + \frac{25}{2}\big(1 - \sqrt{1+t^{-0.8}}\big) = f\big(t, w_0(t)\big); \end{aligned} \tag{3.62}$$

and, if $0.4 < t \leqslant 1$, then

$$\begin{aligned} \mathcal{D}^{0.2} w_0(t) &= \frac{1}{\Gamma(0.8)} \frac{d}{dt} \left[\int_0^{0.4} (t-\tau)^{-0.2} \tau^{-0.8}\, d\tau + \frac{1}{2} \int_{0.4}^t (t-\tau)^{-0.2} (\tau - 0.4)^{-0.8}\, d\tau \right] \\ &= \frac{1}{\Gamma(0.8)} \frac{d}{dt} \int_0^{0.4} (t-\tau)^{-0.2} \tau^{-0.8}\, d\tau = -\frac{0.2}{\Gamma(0.8)} \int_0^{0.4} (t-\tau)^{-1.2} \tau^{-0.8}\, d\tau \\ &= -\frac{0.2}{\Gamma(0.8)} (t-0.4)^{-0.4} \int_0^{0.4} \left(\frac{t-0.4}{t-\tau} \right)^{0.4} \cdot (t-\tau)^{-0.8} \tau^{-0.8}\, d\tau \end{aligned}$$

$$
\begin{aligned}
&> -\frac{0.2}{\Gamma(0.8)}(t-0.4)^{-0.4}\int_0^{0.4}(t-\tau)^{-0.8}\tau^{-0.8}\,d\tau \\
&> -\frac{0.2}{\Gamma(0.8)}(t-0.4)^{-0.4}\int_0^{0.4}(0.4-\tau)^{-0.8}\tau^{-0.8}\,d\tau \\
&= -\frac{0.2\cdot 0.4^{-0.6}\cdot\Gamma(0.2)^2}{\Gamma(0.8)\Gamma(0.4)}(t-0.4)^{-0.4} = -2.8284(t-0.4)^{-0.4} \\
&> -2.9463(t-0.4)^{-0.4} = -\frac{25}{2}\cdot\frac{1}{3}\sqrt{w_0} \geqslant \frac{25}{2}\left(1-\sqrt{1+w_0}\right) \\
&\geqslant f\bigl(t, w_0(t)\bigr), \qquad (3.63)
\end{aligned}
$$

since

$$1-\sqrt{1+x} \leqslant -\frac{1}{3}\sqrt{x}, \quad x \geqslant 0.5625,$$

and

$$w_0(t) \geqslant 0.6^{-0.8} = 0.7524, \quad 0.4 < t \leqslant 1.$$

Combine (3.60)–(3.63), we obtain that w_0 is an upper solution for BVP (3.57)–(3.59). Obviously, I satisfies the condition (H_5), v_0 and w_0 satisfy $v_0(t) \leqslant w_0(t)$ for $t \in (0,1]$ and $\lim_{t\to 0^+} t^{0.8}v_0(t) < \lim_{t\to 0^+} t^{0.8}w_0(t)$. Finally, we have

$$
\begin{aligned}
f(t,u_2)-f(t,u_1) &= -\left(\sin^2\bigl((1+t^2)u_2\bigr)-\sin^2\bigl((1+t^2)u_1\bigr)\right)-\frac{25}{2}\left(\sqrt{1+u_2}-\sqrt{1+u_1}\right) \\
&\geqslant -2\left(\sin\bigl((1+t^2)u_2\bigr)-\sin\bigl((1+t^2)u_1\bigr)\right)-\frac{25}{2}\cdot\frac{u_2-u_1}{\sqrt{1+u_2}+\sqrt{1+u_1}} \\
&\geqslant -2\bigl(1+t^2\bigr)(u_2-u_1)-\frac{25}{4}(u_2-u_1) \\
&\geqslant -\frac{41}{4}(u_2-u_1)
\end{aligned}
$$

for $v_0(t) \leqslant u_1 \leqslant u_2 \leqslant w_0(t)$, which implies that the condition (H_4) is satisfied. Hence, by Theorem 3.7, BVP (3.57)–(3.59) has a solution in $[v_0, w_0]$.

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