Scalings of Matrices Which Have Prespecified Row Sums and Column Sums via Optimization

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Dedicated to Alan J. Hoffman in deep appreciation of his great contributions to mathematics.

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ABSTRACT

The problem of scaling a matrix so that it has given row and column sums is transformed into a convex minimization problem. In particular, we use this transformation to characterize the existence of such scaling or corresponding approximations. We obtain new results as well as new, streamlined proofs of known results.

1. INTRODUCTION

A scaling of a nonnegative matrix A is a matrix having the form $B = XAY$ where X and Y are square diagonal matrices which have positive diagonal elements. The problem of determining a scaling of a given matrix that has

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prespecified row sums and column sums has been studied extensively over the last 20 years, beginning with the pioneering work of Sinkhorn (1964), Sinkhorn and Knopp (1967), Menon (1967), and Brualdi, Parter, and Schneider [1966] where doubly stochastic scalings are considered. A heuristic algorithm for solving the general problem has actually been suggested twenty five years earlier by Kruith of (1937), see Krupp (1979). The general problem has applications in many areas, including planning of telephone traffic and transportion, social accounting, matrix preconditioning, and algebraic image reconstruction; see Kruithof (1937), Bacharach (1970), Lamond and Steward (1981), King (1981), and references therein.

Marshall and Olkin (1968) argued that given a nonnegative matrix, the problem of determining a scaling having prespecified row sums and column sums can be solved by finding a solution to a nonlinear (nonconvex) minimization problem that is defined over an (open) subset of the set of strictly positive vectors. Of course, existence of a minimizer to such a problem is a delicate issue because of boundary problems. In particular, Marshall and Olkin showed that when the given matrix is strictly positive, the corresponding minimization problem has an optimal solution, concluding that a desired scaling exists. In this paper we show that the minimization problem identified by Marshall and Olkin is a geometric program and as such is convertible to a convex minimization problem. Further, we show that the solution of this minimization problem is, in fact, equivalent to the solution of the scaling problem. This result suggests the use of algorithms that are known to solve convex optimization problems for computing scalings which have prespecified row sums and column sums. We hope to apply our results to develop computational methods in the future. We observe that other authors have also applied results from optimization theory to matrix scaling problems—for example, Krupp (1979) and Elfving (1980), who considered entropy maximization; see also Censor (1986).

Using the reduction of the scaling problem to the solution of a convex optimization problem, we are able to apply results from the theory of convexity. We thereby obtain necessary and sufficient conditions for the existence of an optimal solution to the derived optimization problem; hence, we characterize solvability of the scaling problem. In particular, the resulting conditions reduce to nonsolvability of a certain linear system, and standard techniques from the theory of linear inequalities convert them into a finite set of combinatorial conditions. These conditions turn out to be the conditions identified by Menon and Schneider (1969).

Finally, we obtain necessary and sufficient conditions such that the above scaling problem can be solved approximately to arbitrary precision, and we obtain uniqueness results.

The following example demonstrates that the scaling problem cannot be solved over every ordered field. Let

$$A=\begin{pmatrix}1&1\\1&2\end{pmatrix},\qquad r=\begin{pmatrix}1\\1\end{pmatrix},\quad\text{and}\quad c=\begin{pmatrix}1\\1\end{pmatrix}.$$

Then the unique scaling of A whose row sums and column sums equal r and c, respectively, is the matrix

$$\begin{pmatrix}(\sqrt{2}+1)^{-1}&0\\0&(\sqrt{2}+2)^{-1}\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}\sqrt{2}&0\\0&1\end{pmatrix}=\begin{pmatrix}2-\sqrt{2}&\sqrt{2}-1\\\sqrt{2}-1&2-\sqrt{2}\end{pmatrix}.$$

Further, the pre- and postmultiplying matrices are unique up to a scalar multiple. We conclude that the above scaling problem cannot be solved over the rationals and there is no algorithm that is based on the four elementary operations that can solve the above problem.

We introduce some notation and conventions in Section 2 and state the main results in section 3. Proofs of these results are then given in Section 4. Further references to the literature will be found following the statements of our theorems in Section 3.

2. NOTATION AND CONVENTIONS

For a positive integer k, let $\langle k\rangle$ denote the set $\{1,\dots,k\}$. We will use the symbol "$\subseteq$" for set inclusion and the symbol "$\subset$" for strict set inclusion. When $I\subseteq\langle k\rangle$ we use the notation I^c to denote the complement of I within $\langle k\rangle$; the identity of the referenced integer k will always be clear from the context.

Let $A\in R^{m\times n}$. The *support of* A, denoted $\operatorname{supp}A$, is defined to be the set $\{(i,j)\in\langle m\rangle\times\langle n\rangle : A_{ij}\neq 0\}$. We say that A is nonnegative, written $A\geqslant 0$, if $A_{ij}\geqslant 0$ for all $(i,j)\in\langle m\rangle\times\langle n\rangle$. We say that A is *strictly positive*, written $A\gg 0$, if $A_{ij}>0$ for all $(i,j)\in\langle m\rangle\times\langle n\rangle$. Finally, we say that A is *semipositive*, written $A>0$, if $A\geqslant 0$ and $A\neq 0$. As usual, for $B,C\in R^{m\times n}$ we write $B\leqslant C$, $B\ll C$, or $B<C$ if $C-B\geqslant 0$, $C-B\gg 0$, or $C-B>0$, respectively. Of course, the above notation and definitions apply to vectors when either $m=1$ or $n=1$.

Again, let $A \in R^{m \times n}$. For $I \subseteq \langle m \rangle$ and $J \subseteq \langle n \rangle$ we define A_{IJ} to be the corresponding submatrix of A. In particular, when either I or J is empty, A_{IJ} denotes the empty matrix. When A is square, i.e., $m = n$, we use the abbreviated notation A_I for A_{II}. Also, if $n = 1$, i.e., A is a vector, we use the notation A_I for the corresponding subvector of A, i.e., A_I is $A_{I,\{1\}}$.

For a finite subset J of integers, let R^J denote the Euclidean space whose coordinates are numbered by the elements of J; formally, R^J is the set of functions from J into R where the image of j under such a function x is denoted by x_j. Of course, if $J = \langle n \rangle$ for a positive integer n, R^J can be identified with R^n.

Let k be a positive integer. The vector $(1, \dots, 1)^T \in R^k$ will be denoted by e. For $i = 1, \dots, k$, let e^i denote the unit vectors in R^k and for $I \subseteq \langle k \rangle$, let $e^I \equiv \sum_{i \in I} e^i$. No confusion with the exponential function e^x should occur. Incidentally, we use the notation $\exp x$ interchangeably with the notation e^x for the exponential function.

We say that a matrix B is *chainable* if $B \in R^{p \times q}$ for positive integers p and q and for every $i \in \langle p \rangle$ and $j \in \langle q \rangle$ there exist $i_1 = i, i_2, \dots, i_g$ in $\langle p \rangle$ and $j_1, \dots, j_{g-1}, j_g = j$ in $\langle q \rangle$ such that $B_{i_t j_t} > 0$ for $t = 1, \dots, g$ and $B_{i_{t+1} j_t} > 0$ for $t = 1, \dots, g$. Let m and n be positive integers and let $A \in R^{m \times n}$ be a matrix having no zero row or zero column. Then A is the direct sum of chainable blocks, i.e., there exist (unique) partitions $I(1), \dots, I(h)$ of $\langle m \rangle$ and $J(1), \dots, J(h)$ of $\langle n \rangle$ of nonempty sets such that for $r, s \in \{1, \dots, h\}$, $A_{I(r)J(s)} = 0$ if $r \neq s$ and $A_{I(r)J(s)}$ is chainable if $r = s$. In this case we call the pairs $(I(1), J(1)), \dots, (I(h), J(h))$ the *components* of A. (The proof of the above fact follows from the decomposition of the bipartite graph associated with the matrix A into its graph theoretic connected components.) We comment that it can be shown that if p and q are positive integers and B is a matrix in $R^{p \times q}$ which has no zero row or zero column, then B is chainable if and only if $B_{IJ} = 0$ for $\varnothing \neq I \subset \langle p \rangle$ and $\varnothing \neq J \subset \langle q \rangle$ implies $B_{I^c J^c} \neq 0$. This latter equivalent property is called indecomposability in Menon and Schneider (1969) and chainability in Hershkowitz, Rothblum, and Schneider (1988).

3. THE MAIN RESULTS

Throughout let A be a given $m \times n$ nonnegative matrix, and let r and c be given strictly positive vectors in R^m and R^n, respectively. A *scaling* of A is a matrix having the form $B = XAY$ where $X \in R^{m \times m}$ and $Y \in R^{n \times n}$ are diagonal matrices having positive diagonal elements. We will consider the problem of identifying a scaling B of A for which

$$Be = r \quad \text{and} \quad e^T B = c, \tag{3.1}$$

and we will refer to this problem as the *scaling problem*. In particular, a scaling B of A which satisfies (3.1) will be called an *(r,c)-scaling of A*.

Following Marshall and Olkin (1968), we will find it useful to consider the following nonlinear optimization problem in our study of (r,c)-scalings:

$$\text{Program I: } \min x^T A y$$

$$\text{s.t.} \quad \prod_{i=1}^{m} x_i^{r_i} = \prod_{j=1}^{n} y_j^{c_j} = 1,$$

$$x \geqslant 0, \qquad y \geqslant 0,$$

$$x \in R^m, \qquad y \in R^n.$$

Of course, feasibility of a pair (x,y) for Program I implies that no coordinate of x or y is zero; hence, necessarily $x \gg 0$ and $y \gg 0$. Thus, feasibility or optimality for Program I is equivalent, respectively, to feasibility or optimality for the following modified version of Program I:

$$\text{Program I}': \min x^T A y$$

$$\text{s.t.} \quad \prod_{i=1}^{m} x_1^{r_i} = \prod_{j=1}^{n} y_j^{c_j} = 1,$$

$$x \gg 0, \qquad y \gg 0,$$

$$x \in R^m, \qquad y \in R^n.$$

Unfortunately, the objective function of Program I is not necessarily a convex function, and its feasible region is not necessarily a convex set. Still, it turns out that Program I belongs to the class of optimization problems called geometric programs which are convertible into "convex" problems. Specifically, make the change of variables $s_i = \log x_i$ and $t_j = \log y_j$, and take logarithms of the objective and constraints of Program I′ to obtain:

$$\text{Program II: } \min \log\left\{ \sum_{(i,j) \in \operatorname{supp} A} \exp\left(s_i + a_{ij} + t_j\right)\right\},$$

$$\text{s.t.} \quad r^T s = c^T t = 0,$$

$$s \in R^m, \qquad t \in R^n,$$

where $a_{ij} = \log A_{ij}$ for each pair $(i,j) \in \operatorname{supp} A$.

Of course, Program I has an optimal solution if and only if Program II has an optimal solution, in which case one can obtain an optimal solution for any one of these two programs from any given solution of the other. The important virtue of Program II is that it is a convex minimization problem. Specifically, a standard result—e.g., Avriel (1976, Lemma 7.12, p. 197)—shows that the objective function of Program II is convex; of course, its two constraints are linear.

Our first result asserts that the (r, c)-scaling problem reduces to the problem of solving a nonlinear optimization problem stated as Program I, or equivalently to the "convex" optimization problem stated as Program II.

THEOREM 1 (Characterization of (r, c)-scalings).

(a) *Assume that* $e^T r = e^T c$ *and that* (x^*, y^*) *is optimal for Program I. Let* X^* *and* Y^* *be the diagonal matrices defined by*

$$X_{ii}^* = (\lambda^*)^{-1} x_i^*, \qquad i = 1, \dots, m, \tag{3.2}$$

and

$$Y_{jj}^* = y_j^*, \qquad j = 1, \dots, n, \tag{3.3}$$

where

$$\lambda^* = \frac{(x^*)^T A y^*}{e^T r}. \tag{3.4}$$

Then $B = X^* A Y^*$ *is an* (r, c)*-scaling of* A.

(b) *Assume that* X^* *and* Y^* *are diagonal matrices in* $R^{m \times m}$ *and* $R^{n \times n}$, *respectively, with positive diagonal elements, where* $B = X^* A Y^*$ *is an* (r, c)*-scaling of* A. *Then* $e^T r = e^T c$ *and the vectors* $x^* \in R^m$ *and* $y^* \in R^n$ *defined by*

$$x_i^* = \lambda^* X_{ii}^*, \qquad i = 1, \dots, m, \tag{3.5}$$

and

$$y_j^* = \mu^* Y_{jj}^*, \qquad j = 1, \dots, n, \tag{3.6}$$

where

$$\lambda^* = \prod_{i=1}^{m} (X_{ii}^*)^{-r_i/(\sum_{k=1}^{m} r_k)} \tag{3.7}$$

and

$$\mu^* = \prod_{j=1}^{n} (Y_{jj}^*)^{-c_j/(\sum_{k=1}^{n} c_k)}, \tag{3.8}$$

are optimal for Program I.

Part (a) of Theorem 1 essentially appears in Marshall and Olkin (1968, proof of Lemma 2).

THEOREM 2 (Existence of (r, c)-scalings). *The following conditions are equivalent*:

(a) *There exists a scaling B of A which satisfies*

$$Be = r \quad \text{and} \quad e^T B = c^T. \tag{3.9}$$

(b) *There exists a matrix $B \in R^{m \times n}$ with* $\operatorname{supp} B = \operatorname{supp} A$ *which satisfies* (3.9).

(c) *There exist no pair of vectors $(u, v) \in R^m \times R^n$ for which*

$$u_i + v_j \geqslant 0 \qquad \textit{for each pair} \quad (i, j) \in \operatorname{supp} A, \tag{3.10}$$

$$r^T u + c^T v \leqslant 0, \tag{3.11}$$

and

$$\textit{either} \quad u_i + v_j > 0 \quad \textit{for some} \ (i, j) \in \operatorname{supp} A \quad \textit{or} \quad r^T u + c^T v < 0. \tag{3.12}$$

(d) *There exist no pair of vectors $(u, v) \in R^m \times R^n$ for which*

$$u_i + v_j \leqslant 0 \qquad \textit{for each pair} \ (i, j) \in \operatorname{supp} A, \tag{3.13}$$

$$r^T u = c^T v = 0, \tag{3.14}$$

and

$$u_i + v_j < 0 \qquad \textit{for some pair} \quad (i,j) \in \operatorname{supp} A, \tag{3.15}$$

and, in addition, $e^T r = e^T c$.

(e) *For every* $I \subseteq \langle m \rangle$ *and* $J \subseteq \langle n \rangle$ *for which* $A_{I^c J} = 0$ *we have that*

$$\sum_{i \in I} r_i \geqslant \sum_{j \in J} c_j, \tag{3.16}$$

and equality holds in (3.16) *if and only if* $A_{IJ^c} = 0$.

(f) *Program I has an optimal solution, and* $e^T r = e^T c$.

The equivalence of conditions (a) and (b) of Theorem 2 for the chainable case appears in Menon (1968, Theorem 2), and the equivalence of (a) and (e) was established in Menon and Schneider (1969, Theorems 3.6 and 4.1). Also, the equivalence of (b) and (e) for chainable matrices appears in Brualdi (1968, Theorem 2.1). Next, the implication (f) $\Rightarrow$ (a) essentially appears in Marshall and Olkin (1968, proof of Lemma 2), and the equivalence of (b) and (d) for the case with $r = e$ and $c = e$ is given in Saunders and Schneider (1979, Theorem 3.4).

We observe that solvability of Program I or nonexistence of vectors u and v which satisfy (3.13)–(3.15) does not imply that $e^T r = e^T c$. For example, let

$$A = (1,1), \qquad r = (1), \quad \text{and} \quad c = (1,2)^T. \tag{3.17}$$

Then $e^T r = 2 \neq 3 = e^T c$, but Program I becomes the problem of minimizing $x_1(y_1 + y_2)$ subject to $x_1 = 1$ and $y_1 y_2^2 = 1$, which has an optimal solution $(x_1^*, y_1^*, y_2^*) = (1, 2^{-2/3}, 2^{1/3})$. Also, if $u \in R$ and $v \in R^2$ satisfy (3.14) and (3.13), then $u_1 = 0 = v_1 + 2v_2$, $u_1 + v_1 \leqslant 0$, and $u_1 + v_2 \leqslant 0$, implying that $v_1 = v_2 = u_1 = 0$. In particular u and v do not satisfy (3.15). So, no $(u, v) \in R^m \times R^n$ satisfy (3.13)–(3.15).

We note that in condition (e) we allow I, J and their complements to be empty, and we consider the empty matrices to be zero matrices. In particular, we have that $I = \langle m \rangle$ and $J = \langle n \rangle$ satisfy $A_{I^c J} = 0$ and $A_{IJ^c} = 0$; hence, condition (e) implies that $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. Evidently, we could have avoided the use of empty matrices by imposing the condition $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$.

THEOREM 3 (Existence of approximate (r, c)-scalings). *The following conditions are equivalent*:

(a1) *For every* $\varepsilon > 0$ *there exists a scaling* B *of* A *which satisfies*

$$\|Be - r\|_\infty \leqslant \varepsilon \quad \textit{and} \quad \|e^T B - c^T\|_\infty \leqslant \varepsilon. \tag{3.18}$$

(a2) *There exists a matrix* $A' \in R^{m \times n}$ *with* $\operatorname{supp} A' \subseteq \operatorname{supp} A$ *and* $A'_{ij} = A_{ij}$ *for every pair* $(i, j) \in \operatorname{supp} A'$, *where* A' *has a scaling* B' *which satisfies*

$$B'e = r \quad \text{and} \quad e^T B' = c^T. \tag{3.19}$$

(b1) *For every* $\varepsilon > 0$ *there exists a matrix* $B \in R^{m \times n}$ *with* $\operatorname{supp} B = \operatorname{supp} A$ *which satisfies* (3.18).

(b2) *There exists a matrix* $B \in R^{m \times n}$ *with* $\operatorname{supp} B \subseteq \operatorname{supp} A$ *which satisfies* (3.19).

(c) *There exist no pair of vectors* $(u, v) \in R^m \times R^n$ *for which*

$$u_i + v_j \geqslant 0 \qquad \text{for each pair} \quad (i, j) \in \operatorname{supp} A \tag{3.20}$$

and

$$r^T u + c^T v < 0. \tag{3.21}$$

(d) *There exist no pair of vectors* $(u, v) \in R^m \times R^n$ *for which*

$$u_i + v_j < 0 \qquad \text{for each pair} \quad (i, j) \in \operatorname{supp} A \tag{3.22}$$

and

$$r^T u = c^T v = 0 \tag{3.23}$$

and, in addition, $e^T r = c^T e$.

(e) *For every* $I \subseteq \langle m \rangle$ *and* $J \subseteq \langle n \rangle$ *for which* $A_{I^c J} = 0$ *we have that*

$$\sum_{i \in I} r_i \geqslant \sum_{j \in J} c_j. \tag{3.24}$$

(f) *The objective of Program I is bounded away from zero over the feasible set of Program I, and* $e^T r = e^T c$.

The equivalence of conditions (b2) and (e) for the case where $r = e$ and $c = e$ appears in Saunders and Schneider (1979, Theorem 3.3).

The example following Theorem 2 indicates that the assertion that the objective of Program I is bounded away from zero, or the nonexistence of vectors u and v that satisfy (3.22)–(3.23), does not imply that $e^T r = e^T c$.

THEOREM 4 (Uniqueness of (r, c)-scalings). *The matrix A has at most one* (r, c)*-scaling. Further, if* X, Y, X' *and* Y' *are diagonal matrices with* *positive diagonal elements,* $B = XAY$ *is an* (r, c)*-scaling of* A *and* $(I(1), J(1)),\dots,(J(h), J(h))$ *are the components of* A*, then* $X'AY'$ *is an* (r, c)*-scaling of* A *if and only if for some positive numbers* $\alpha_1,\dots,\alpha_h$,

$$X'_{I(t)} = \alpha_t X_{I(t)} \text{ and } Y'_{J(t)} = \alpha_t^{-1} Y_{J(t)}, \qquad t = 1,\dots,h. \tag{3.25}$$

The uniqueness of an (r, c)-scaling of a chainable matrix when such a scaling exists appears in Menon (1968, Theorem 2). The general case is implicit in Menon and Schneider (1969, Theorems 3.9 and 4.1); see also Hershkowitz, Rothblum, and Schneider (1988, Theorem 3.7).

THEOREM 5 (Uniqueness of solutions to Program I). *Suppose* $e^T r = e^T c$, *that* (x^*, y^*) *is an optimal solution for Program I and* $(I(1), J(1)),\dots,(I(h), J(h))$ *are the components of* A*. Then* (x', y') *is an optimal solution for Program I if and only if for some positive numbers* $\alpha_1,\dots,\alpha_h$ *with*

$$\prod_{t=1}^{h} \alpha_t^{\sum_{i \in I(t)} r_i} = 1 \tag{3.26}$$

we have that

$$x'_{I(t)} = \alpha_t x^*_{I(t)} \text{ and } y'_{J(t)} = \alpha_t^{-1} y^*_{J(t)}, \qquad t = 1,\dots,h. \tag{3.27}$$

In particular, if A *is chainable, there is at most one optimal solution of Program I.*

4. PROOFS

We start with the proof of Theorem 1.

Proof of Theorem 1. (a): Assume that $e^T r = e^T c$ and that (x^*, y^*) is optimal for Program I. The gradients of the functions defining the two equality constraints of Program I are given by

$$\nabla_{xy}\left(\prod_{i=1}^{m} x_i^{r_i} - 1\right) = \left(\prod_{i=1}^{m} x_i^{r_i}\right)\left(\frac{r_1}{x_1},\dots,\frac{r_m}{x_m},0,\dots,0\right)$$

and

$$\nabla_{xy}\left(\prod_{j=1}^{n} y_j^{c_j} - 1\right) = \left(\prod_{j=1}^{n} y_j^{c_j}\right)\left(0,\dots,0,\frac{c_1}{y_1},\dots,\frac{c_n}{y_n}\right).$$

As $r \neq 0$ and $c \neq 0$ (being strictly positive), we have that the above two gradients are linearly independent at (x^*, y^*), implying that first order conditions are satisfied at (x^*, y^*); see Avriel (1976, Theorem 2.6, p. 16). Hence, for some real numbers λ^* and μ^*

$$\frac{\partial}{\partial x_i}\left[x^T A y - \lambda^*\left(\prod_{i=1}^{m} x_i^{r_i}\right) - \mu^*\left(\prod_{j=1}^{n} y_j^{c_j}\right)\right]_{x^*, y^*} = 0, \qquad i = 1,\dots,m, \quad (4.1)$$

and

$$\frac{\partial}{\partial y_i}\left[x^T A y - \lambda^*\left(\prod_{i=1}^{m} x_i^{r_i}\right) - \mu^*\left(\prod_{j=1}^{n} y_j^{c_j}\right)\right]_{x^*, y^*} = 0, \qquad j = 1,\dots,n. \quad (4.2)$$

As (x^*, y^*) is feasible for Program I′, we have that $\prod_{i=1}^{m}(x_i^*)^{r_i} = \prod_{j=1}^{n}(y_i^*)^{c_i} = 1$, and therefore (4.1) and (4.2) imply, respectively, that

$$(Ay^*)_i = \lambda^*\left[\prod_{i=1}^{m}(x_i^*)^{r_i}\right]\frac{r_i}{x_i^*} = \lambda^*\frac{r_i}{x_i^*}, \qquad i = 1,\dots,m, \quad (4.3)$$

and

$$\left((x^*)^T A\right)_j = \mu^*\left[\prod_{j=1}^{n}(y_j^*)^{c_j}\right]\frac{c_j}{y_j^*} = \mu^*\frac{c_j}{y_j^*}, \qquad j = 1,\dots,n. \quad (4.4)$$

In particular, we have that $\lambda^* = (x^*)^T A y^* / e^T r$ and $\mu^* = (x^*)^T A y^* / e^T c$, respectively. As $e^T r = e^T c$, we conclude that $\lambda^* = \mu^*$. So λ^* satisfies (3.4). Further, (4.3) and (4.4) imply, respectively, that the diagonal matrices $X^* \in R^{m\times m}$ and $Y^* \in R^{n\times n}$ defined via (3.2) and (3.3) satisfy $X^*AY^*e = r$ and $e^T X^* A Y^* = \mu^* c^T / \lambda^* = c^T$. So, indeed, $B = X^*AY^*$ is an (r, c)-scaling of A.

(b): Assume that X^* and Y^* are diagonal matrices in $R^{m\times m}$ and $R^{n\times n}$, respectively, with positive diagonal elements, where $B = X^*AY^*$ is an (r, c)-scaling of A. Then $e^T r = e^T(Be) = (e^T B)e = c^T e = e^T c$. Also, let $s^* \in R^m$ and

$t^* \in R^n$ be defined by

$$s_i^* = \beta^* + \log X_{ii}^*, \qquad i = 1, \dots, m,$$

and

$$t_j^* = \gamma^* + \log Y_{jj}^*, \qquad j = 1, \dots, n,$$

where

$$\beta^* = -\frac{\sum_{i=1}^m r_i(\log X_{ii}^*)}{\sum_{k=1}^m r_k}$$

and

$$\gamma^* = -\frac{\sum_{j=1}^n c_j(\log Y_{jj}^*)}{\sum_{k=1}^n c_k}.$$

Evidently, these definitions assure that $\sum_{i=1}^m r_i s_i^* = 0$ and $\sum_{j=1}^n c_j t_j^* = 0$. So (s^*, t^*) is feasible for Program II. Further, if $h(\ ,\ )$ is the objective function of Program II and $\theta \equiv h(s^*, t^*)$, we have that for $i = 1, \dots, m$

$$\frac{\partial}{\partial s_i}\left[h(s,t) - \theta^{-1}e^{\beta^*}e^{\gamma^*}r^T s - \theta e^{\beta^*}e^{\gamma^*}c^T t\right]_{s^*, t^*}$$

$$= \theta^{-1} \sum_{\{j:(i,j) \in \operatorname{supp} A\}} e^{s_i^* + a_{ij} + t_j^*} - \theta^{-1}e^{\beta^*}e^{\gamma^*}r_i$$

$$= \theta^{-1} \sum_{\{j:(i,j) \in \operatorname{supp} A\}} e^{\beta^*}X_{ii}^* A_{ij} Y_{jj}^* e^{\gamma^*} - \theta^{-1}e^{\beta^*}e^{\gamma^*}r_i = 0. \tag{4.5}$$

The last equality following from the fact that $X^*AY^*e = r$. Similar arguments, using the fact that $e^T X^* A Y^* = c^T$, show that for $j = 1, \dots, n$

$$\frac{\partial}{\partial t_j}\left[h(s,t) - \theta e^{\beta^*}e^{\gamma^*}r^T s - \theta^{-1}e^{\beta^*}e^{\gamma^*}c^T t\right]_{s^* t^*} = 0. \tag{4.6}$$

It follows that (s^*, t^*) satisfy the Kuhn-Tucker conditions for Program II and therefore (s^*, t^*) is optimal for that program; see Avriel (1976, Theorem 4.38, p. 96). Evidently, $(x^*, y^*) \in R^m \times R^n$ defined by (3.5)–(3.8) satisfy $x_i^* = e^{s_i^*}$

and $y_j^* = e^{t_j^*}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$; hence this pair is, indeed, optimal for Program I. ∎

Before establishing Theorem 2 we need some auxillary results. Let $h(\, , \,)$ be the objective function of Program II, i.e., for each $(s, t) \in R^m \times R^n$,

$$h(s,t) = \log\left\{ \sum_{(i,j) \in \operatorname{supp} A} \exp(s_i + a_{ij} + t_j) \right\}. \tag{4.7}$$

We have already observed that this function is convex. A direction of recession of $h(\, , \,)$ is a pair $(u, v) \in R^m \times R^n$ for which

$$\sup\{ h(s+u, t+v) - h(s,t) : (s,t) \in R^m \times R^n \} \leqslant 0;$$

see Rockafellar (1970, pp. 66–68). Also, a *direction along which* $h(\, , \,)$ *is constant* is a pair $(u, v) \in R^m \times R^n$ for which

$$h(s+u, t+v) = h(s,t) \qquad \text{for all} \quad (s,t) \in R^{m \times n}.$$

We next characterize directions of recession of $h(\, , \,)$ and directions along which $h(\, , \,)$ is constant.

LEMMA 1. *A pair* $(u, v) \in R^m \times R^n$ *is a direction of recession of* $h(\, , \,)$ *if and only if*

$$u_i + v_j \leqslant 0 \qquad \textit{for every pair} \quad (i,j) \in \operatorname{supp} A. \tag{4.8}$$

Proof. A pair $(u, v) \in R^m \times R^n$ is a direction of recession of $h(\, , \,)$ if and only if for every pair $(s, t) \in R^m \times R^n$

$$\sum_{(i,j) \in \operatorname{supp} A} \left[\exp(s_i + a_{ij} + t_j)\right] \left[\exp(u_i + v_j)\right]$$

$$\leqslant \sum_{(i,j) \in \operatorname{supp} A} \exp(s_i + a_{ij} + t_j). \tag{4.9}$$

Trivially, (4.8) implies that (4.9) holds for every pair $(s, t) \in R^m \times R^n$. To see the reverse implication, assume that $(u, v) \in R^m \times R^n$ does not satisfy (4.8), and we will construct a pair of vectors (s, t) for which (4.9) does not hold. As

(u,v) do not satisfy (4.8), we have that for some $(i^*, j^*) \in \operatorname{supp} A$, $u_{i^*} + v_{j^*} > 0$. Let $\beta \equiv \exp(u_{i^*} + v_{j^*}) > 1$, and for each $M > 0$ let $s(M) = Me^{i^*}$ $(\in R^m)$ and $t(M) = Me^{j^*}$ $(\in R^n)$. It then follows that

$$\sum_{(i,j) \in \operatorname{supp} A} \left\{ \exp\left[s(M)_i + a_{ij} + t(M)_j \right] \right\} \left[\exp(u_i + v_j) \right]$$

$$\geqslant \beta \exp\left(a_{i^*j^*} + 2M \right) = \beta A_{i^*j^*} e^{2M}. \tag{4.10}$$

Also, as $s(M)_i + a_{ij} + t(M)_j \leqslant a_{ij} + M$ for every pair $(i,j) \in (\operatorname{supp} A) \setminus (i^*, j^*)$, we have that

$$\sum_{(i,j) \in \operatorname{supp} A} \exp\left[s(M)_i + a_{ij} + t(M)_j \right]$$

$$\leqslant \sum_{(i,j) \in \operatorname{supp} A} A_{ij} e^M + A_{i^*j^*}(e^{2M} - e^M). \tag{4.11}$$

Since $\beta > 1$, it immediately follows that for sufficiently large M the right-hand side of (4.10) dominates the right-hand side of (4.11). For such M, (4.9) need not hold for $s = s(M)$ and $t = t(M)$, establishing a contradiction and thereby completing our proof. ■

COROLLARY 1. *A pair* $(u,v) \in R^m \times R^n$ *is a direction along which* $h(\ ,\ )$ *is constant if and only if* $u_i + v_j = 0$ *for every pair* $(i,j) \in \operatorname{supp} A$.

Proof. Trivially, (u,v) is a direction along which h is constant if and only if both (u,v) and $(-u,-v)$ are directions of recession of $h(\ ,\ )$. Hence, the conclusion of our corollary is immediate from Lemma 1. ■

We are now ready to establish Theorem 2.

Proof of Theorem 2. In this proof we refer to conditions found in the statement of Theorem 2. We will establish Theorem 2 by showing the sequence of implications: (a) ⇒ (b), (b) ⇔ (c), (c) ⇒ (d), (d) ⇒ (f), (f) ⇒ (a), (e) ⇒ (c), and (c) ⇒ (e).

(a) ⇒ (b): This implication is trivial, as each scaling of A has the same support as A.

(b) $\Leftrightarrow$ (c): The assertion in (b) means that there exists a solution to the linear system

$$\sum_{\substack{j \in \{1,\ldots,n\} \\ (i,j) \in \operatorname{supp} A}} z_{ij} = r_i, \qquad i = 1,\ldots,m, \tag{4.12}$$

$$\sum_{\substack{i = \{1,\ldots,m\} \\ (i,j) \in \operatorname{supp} A}} z_{ij} = c_j, \qquad j = 1,\ldots,m \tag{4.13}$$

$$z \gg 0, \tag{4.14}$$

where z is the (variable) vector given by

$$z = \left(z_{ij}\right)_{(i,j) \in \operatorname{supp} A}. \tag{4.15}$$

It now follows from the theorem of the alternative stated in Theorem A.2 of the Appendix that this system has no solution if and only if there is no pair of vectors $(u, v) \in R^m \times R^n$ satisfying (3.10), (3.11), and (3.12).

(c) $\Rightarrow$ (d): Assume that (c) holds. We first establish that $e^T r = e^T c$. Suppose that this is not the case and $e^T r \neq e^T c$. Let $u = e \in R^m$ and $v = -e \in R^n$ if $e^T r < e^T c$, and let $u = -e \in R^m$ and $v = e \in R^n$ if $e^T r > e^T c$. In either case $u_i + v_j = 0$ for all $(i, j) \in \operatorname{supp} A$ and $r^T u + c^T v = -|r^T e - c^T e| < 0$, contradicting (c). So, indeed, $e^T r = e^T c$. Finally, if (u, v) satisfy (3.13)–(3.15), then, trivially, $(-u, -v)$ satisfy (3.10)–(3.12), implying that (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (f): Evidently, Program I has an optimal solution if and only if Program II does. Hence, we consider only Program II. Let $C = \{(s,t) \in R^m \times R^n : r^T s = c^T t = 0\}$. By Lemma 1 and Corollary 1, condition (d) asserts that every direction of recession of h which is in C is a direction along which h is constant. Thus, we conclude from Rockafellar (1970, Theorem 27.3, p. 267) that Program II attains a minimum.

(f) $\Leftrightarrow$ (a): This equivalence is immediate from Theorem 1.

(e) $\Rightarrow$ (c): Suppose that (e) holds, i.e., for $I \subseteq \langle m \rangle$ and $J \subseteq \langle n \rangle$

$$A_{I^c J} = 0 \quad \Rightarrow \quad \sum_{i \in I} r_i \geqslant \sum_{j \in J} c_j \text{ with equality holding if and only if } A_{IJ^c} = 0. \tag{4.16}$$

In particular, we have that

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j; \tag{4.17}$$

see the third comment following the statement of Theorem 2. We first establish that there exists no triplet $(u, v, s) \in R^m \times R^n \times R^{\operatorname{supp} A}$ satisfying

$$s_{ij} = u_i + v_j \qquad \text{for every pair} \quad (i, j) \in \operatorname{supp} A, \tag{4.18}$$

$$u \geqslant 0, \qquad v \leqslant 0, \tag{4.19}$$

$$s \geqslant 0, \tag{4.20}$$

and either

$$r^T u + c^T v < 0 \tag{4.21}$$

or

$$r^T u + c^T v = 0 \quad \text{and} \quad \sum_{(i,j) \in \operatorname{supp} A} s_{ij} > 0. \tag{4.22}$$

Of course, (c) asserts the nonexistence of triplets (u, v, s) satisfying (4.18), (4.20), and either (4.21) or (4.22) without requiring that (4.19) be satisfied. Hence nonexistence of (u, v, s) satisfying (4.18)–(4.20) and either (4.21) or (4.22) is a weaker assertion than (c). Suppose that the set of triplets (u, v, s) satisfying (4.18)–(4.20) and either (4.21) or (4.22) is not empty. Let (u^*, v^*, s^*) be a triplet in that set which minimizes $\nu(u, v, s) \equiv |\operatorname{supp} u| + |\operatorname{supp} v| + |\operatorname{supp} s|$ among such triplets (where absolute value signs are used to denote cardinalities of sets). Of course, both (4.21) and (4.22) assure that (u^*, v^*) is nonzero. Let $I \equiv \{i = 1, \dots, m : u_i^* > 0\}$ and $J \equiv \{j = 1, \dots, n : v_j^* < 0\}$. In particular, either I or J is nonempty. Also, (4.18)–(4.20) imply that if $i \in I^c$ and $j \in J$ then $(i, j) \notin \operatorname{supp} A$, i.e., $A_{I^c J} = 0$. Hence, (4.16) implies that

$$\sum_{i \in I} r_i \geqslant \sum_{j \in J} c_j. \tag{4.23}$$

For $\theta > 0$ let $(u(\theta), v(\theta), s(\theta)) \in R^m \times R^n \times R^{\operatorname{supp} A}$ be defined by $u(\theta) = u^* - \theta e^I$, $v(\theta) = v^* + \theta e^J$, and $s(\theta)_{ij} = s_{ij}^* - \theta[(e^I)_i - (e^J)_j]$ for each

$(i,j)\in \operatorname{supp} A$. Then $(u(\theta), v(\theta), s(\theta))$ satisfies (4.18) for all $\theta > 0$. Now, if $u_i^* = 0$ then $i \notin I$, implying that $u(\theta)_i = u_i^* = 0$; hence $\operatorname{supp} u(\theta) \subseteq \operatorname{supp} u^*$. Similarly, $\operatorname{supp} v(\theta) \subseteq \operatorname{supp} v^*$. Also, if $s_{ij}^* = 0$ for $(i,j) \in \operatorname{supp} A$, then either $u_i^* = v_j^* = 0$ or $u_i^* > 0$ and $v_j^* < 0$, and in either case $(e^I)_i - (e^J)_j = 0$; hence $\operatorname{supp} s(\theta) \subseteq \operatorname{supp} s^*$. Further, for

$$\theta^* = \min\left\{ \min_{i\in I} u_i^*, \min_{j\in J} |v_j^*|, \min_{\substack{(i,j)\in \operatorname{supp} A \\ u_i^* + v_j^* > 0}} \frac{s_{ij}^*}{u_i^* + v_j^*} \right\} > 0 \tag{4.24}$$

we have that $u(\theta^*) \geqslant 0$, $v(\theta^*) \leqslant 0$, $s(\theta^*) \geqslant 0$ and $\nu(u(\theta^*), v(\theta^*), s(\theta^*)) < \nu(u^*, v^*, s^*)$. In particular, $(u(\theta^*), v(\theta^*), s(\theta^*))$ satisfies (4.18)–(4.20). We continue by showing that $(u(\theta^*), v(\theta^*), s(\theta^*))$ satisfies either (4.21) or (4.22). First observe that (4.23) implies that

$$\begin{aligned} r^T u(\theta^*) + c^T v(\theta^*) &= r^T u^* + c^T v^* - \theta^* \left(\sum_{i\in I} r_i - \sum_{j\in J} c_j \right) \\ &\leqslant r^T u^* + c^T v^*. \end{aligned} \tag{4.25}$$

Now, if (u^*, v^*, s^*) satisfies (4.21), i.e., $r^T u^* + c^T v^* < 0$, or if $\sum_{i\in I} r_i > \sum_{j\in J} c_j$, then (4.25) implies that $(u(\theta^*), v(\theta^*), s(\theta^*))$ satisfies (4.21). It remains to consider the case where (u^*, v^*, s^*) satisfies (4.22), i.e., $r^T u^* + c^T v^* = 0$ and $\sum_{(i,j)\in \operatorname{supp} A} s_{ij}^* > 0$, and where $\sum_{i\in I} r_i = \sum_{j\in J} c_j$. In this case we have that

$$\begin{aligned} r^T u(\theta^*) + c^T v(\theta^*) &= r^T u^* + c^T v^* - \theta^* \left(\sum_{i\in I} r_i - \sum_{j\in J} c_j \right) \\ &= 0 + 0 = 0. \end{aligned}$$

Also, (4.16) implies that $A_{II^c} = 0$. As we have already seen that $A_{I^cJ} = 0$, we have that $(e^I)_i - (e^J)_j = 0$ for all $(i,j) \in \operatorname{supp} A$. So

$$\begin{aligned} \sum_{(i,j)\in \operatorname{supp} A} s(\theta^*)_{ij} &= \sum_{(i,j)\in \operatorname{supp} A} \left\{ u_i^* + v_j^* - \theta^* \left[(e^I)_i - (e^J)_j \right] \right\} \\ &= \sum_{(i,j)\in \operatorname{supp} A} s_{ij}^* > 0, \end{aligned}$$

and we see that $(u(\theta^*), v(\theta^*), s(\theta^*))$ satisfies (4.22). So $(u(\theta^*), v(\theta^*), s(\theta^*))$ satisfies (4.18)–(4.20) and either (4.21) or (4.22). As $\nu(u(\theta^*), v(\theta^*), s(\theta^*)) < \nu(u^*, v^*, s^*)$, we get a contradiction, thereby showing that no triplet (u, v, s) satisfies (4.18)–(4.20) and either (4.21) or (4.22).

We finally establish that there exists no triplet (u, v, s) satisfying (4.18), (4.20), and either (4.21) or (4.22), thereby establishing (c). Suppose (u, v, s) satisfies the above conditions, and let $\bar{u} = u + Me$ and $\bar{v} = v - Me$ for sufficiently large M so that $\bar{u} \geqslant 0$ and $\bar{v} \leqslant 0$. Then $(\bar{u}, \bar{v}, s)$ satisfies (4.18), (4.19), and (4.20). Further, (4.17) assures that

$$r^T\bar{u} + c^T\bar{v} = r^Tu + c^Tv + M\left(\sum_{i=1}^{m} r_i - \sum_{j=1}^{n} c_j\right) = r^Tu + c^Tv.$$

As (u, v, s) satisfies (4.21) or (4.22), we immediately conclude that $(\bar{u}, \bar{v}, s)$ satisfies the same condition, respectively. So, $\bar{u}, \bar{v}, s$ satisfies (4.18)–(4.20) and either (4.21) or (4.22), a contradiction.

(c) $\Rightarrow$ (e): Assume that (c) holds. To establish (e) let $I \subseteq \langle m \rangle$ and $J \subseteq \langle n \rangle$ satisfy $A_{I^cJ} = 0$. Consider the vectors $u = e^I \in R^m$ and $v = -e^J \in R^n$. As $A_{I^cJ} = 0$, we have that $u_i + v_j \geqslant 0$ for each pair $(i, j) \in \operatorname{supp} A$. Thus, we conclude from (c) that $r^Tu + c^Tv \geqslant 0$, i.e.,

$$\sum_{i \in I} r_i - \sum_{j \in J} c_j = r^Tu + c^Tv \geqslant 0. \tag{4.26}$$

Further, if equality holds in (4.26) then (c) implies that $u_i + v_j = 0$ for all $(i, j) \in \operatorname{supp} A$, assuring that no pair $(i, j) \in I^c \times J$ is in $\operatorname{supp} A$, i.e., $A_{IJ^c} = 0$.

We finally show that if $A_{IJ^c} = 0$ and $A_{I^cJ} = 0$ then $\sum_{i \in I} r_i = \sum_{j \in J} c_j$. First observe that as $(u, v) \equiv (-e, e) \in R^m \times R^n$ satisfy $u_i + v_j = 0$ for every $(i, j) \in \operatorname{supp} A$ we have that

$$-\sum_{i=1}^{m} r_i + \sum_{j=1}^{n} c_j = r^Tu + c^Tv \geqslant 0. \tag{4.27}$$

Similarly, by considering $(u, v) = (e, -e) \in R^m \times R^n$ we obtain the reverse inequality. So, necessarily,

$$\sum_{i=1}^{m} r_i - \sum_{j=1}^{n} c_j = 0. \tag{4.28}$$

Now, if $A_{IJ^c} = 0$ and $A_{I^cJ} = 0$, our earlier arguments show that

$$\sum_{i \in I} r_i - \sum_{j \in J} c_j \geqslant 0 \tag{4.29}$$

and

$$\sum_{i \in I^c} r_i - \sum_{j \in J^c} c_j \geqslant 0. \tag{4.30}$$

Adding the left hand sides of (4.29) and (4.30) yields the expression $\sum_{i=1}^{m} r_i - \sum_{j=1}^{n} c_j$, and therefore we conclude from (4.29), (4.30), and (4.28) that equality holds in (4.29) and (4.30), as asserted. ■

Proof of Theorem 3. In this proof we refer to the conditions of the statement of Theorem 4.

(a1) ⇒ (b1) and (a2) ⇒ (b2): These two implications are trivial and follow from the fact that a scaling of a matrix has the same support as the given matrix.

(b1) ⇒ (a1): Suppose that (b1) holds. Let $\varepsilon > 0$ be given, and suppose that $B \in R^{m \times n}$ has the same support as A and satisfies (3.18). Let $r' \equiv Be$ and $c' = B^T e$. Then the equivalence of (a) and (b) in Theorem 2 assures the existence of a scaling B' of A for which $B'e = r' = Be$ and $e^T B' = (c')^T = c^T B$. In particular, $\|B'e - r\|_\infty = \|Be - r\|_\infty \leqslant \varepsilon$ and $\|e^T B' - c^T\|_\infty = \|e^T B - c^T\|_\infty \leqslant \varepsilon$, establishing (a1).

(b2) ⇒ (a2): Suppose that (b2) holds and $B \in R^{m \times n}$ satisfies $\operatorname{supp} B \subseteq \operatorname{supp} A$, $Be = r$, and $e^T B = c^T$. Let A' be the $m \times n$ matrix defined by $A'_{ij} = A_{ij}$ if $(i, j) \in \operatorname{supp} B$ and $A'_{ij} = 0$ otherwise. As B is a matrix having the same support as A' for which $Be = r$ and $e^T B = c^T$, we conclude from the equivalence of conditions (a) and (b) of Theorem 2 that A' has a scaling B' for which $B'e = r$ and $e^T B' = c^T$, establishing (a2).

(b1) ⇔ (b2) ⇔ (c): Assertion (b2) means that there exists a solution to the linear system

$$\sum_{\substack{j \in \{1,\dots,n\} \\ (i,j) \in \operatorname{supp} A}} z_{ij} = r_i, \qquad i = 1, \dots, m,$$

$$\sum_{\substack{i \in \{1,\dots,m\} \\ (i,j) \in \operatorname{supp} A}} z_{ij} = c_j, \qquad j = 1, \dots, n,$$

$$z \geqslant 0,$$

where the (variable) vector z is given by $z=(z_{ij})_{(i,j)\in \operatorname{supp}A}$. Also, the assertion in (b1) means that for every $\varepsilon>0$ there exists a solution to the linear system

$$\left|\sum_{\substack{j\in\{1,\dots,n\}\\(i,j)\in \operatorname{supp}A}} z_{ij}-r_i\right|\leqslant\varepsilon,\qquad i=1,\dots,m,$$

$$\left|\sum_{\substack{i\in\{1,\dots,m\}\\(i,j)\in \operatorname{supp}A}} z_{ij}-c_j\right|\leqslant\varepsilon,\qquad j=1,\dots,n,$$

$$z\gg 0,$$

where, again, $z=(z_{ij})_{(i,j)\in \operatorname{supp}A}$. The equivalence of the solvability of these two systems now follows from a standard result about linear systems; see Theorem A.4 of the Appendix. Also, it follows from the theorem of the alternative stated in Theorem A.3 of the Appendix that the first of the above two systems has a solution if and only if there is no pair of vectors $(u,v)\in R^m\times R^n$ satisfying (3.20) and (3.21).

(c) $\Rightarrow$ (d): Assume that (c) holds. The arguments used to establish the implication (c) $\Rightarrow$ (d) in Theorem 2 show that in this case necessarily $e^Tr=e^Tc$. Now, assume that $(u,v)\in R^m\times R^n$ satisfies (3.22) and (3.23). Then for some $\varepsilon>0$, $u_i+v_j+\varepsilon r_i\leqslant 0$ for all $(i,j)\in\operatorname{supp}A$. Let $u'=-u-\varepsilon r$ and $v'=-v$. Then $u_i'+v_j'=-(u_i+v_j)-\varepsilon r_i\geqslant 0$ for all $(i,j)\in\operatorname{supp}A$ and $r^Tu'+c^Tv'=-r^Tu-c^Tv-\varepsilon r^Tr=-\varepsilon r^Tr<0$. So (u',v') satisfies (3.20) and (3.21), which contradicts (c). This completes the proof that (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (c): Suppose that (d) holds and $(u,v)\in R^m\times R^n$ satisfies (3.20) and (3.21). Let $\gamma\equiv r^Te=c^Te$, $u'\equiv -u+\gamma^{-1}(r^Tu)e$, and $v'\equiv -v+\gamma^{-1}(c^Tv)e$. Then for every $(i,j)\in\operatorname{supp}A$, (3.20) and (3.21) imply that

$$u_i'+v_j'=-(u_i+v_j)+\gamma^{-1}(r^Tu+c^Tv)<-(u_i+v_j)\leqslant 0.$$

Also, $r^Tu'=c^Tv'=0$. So (u',v') satisfies (3.22)–(3.23), which contradicts (d). This completes the proof that (d) $\Rightarrow$ (c).

(f) $\Rightarrow$ (d): Assume that (f) holds and that $(u,v)\in R^m\times R^n$ satisfies (3.22) and (3.23). Let $\delta\equiv\max\{u_i+v_j:(i,j)\in\operatorname{supp}A\}$. In particular, (3.22) assures that $\delta<0$. Now, for every $M>0$ let $x(M)$ and $y(M)$ be defined by $x(M)_i=\exp Mu_i$, $i=1,\dots,m$, and $y(M)_j=\exp Mv_j$, $j=1,\dots,n$. Evidently, for every $M>0$, we have $x(M)\gg 0$, $y(M)\gg 0$. Also, by (3.23), $\prod_{i=1}^m x(M)_i^{r_i}=\exp(\sum_{i=1}^m Mr_iu_i)=\exp 0=1$ and $\prod_{j=1}^n y(M)_j^{c_j}=\exp(\sum_{j=1}^n Mc_jv_j)=\exp 0$

$= 1$. Thus $(x(M), y(M))$ is feasible for Program I for each $M > 0$. In addition, we have that $x(M)_i y(M)_j = \exp M(u_i + v_j) \leqslant \exp \delta M$ for each $(i, j) \in \operatorname{supp} A$, implying that $0 < x(M)^T A y(M) \leqslant (e^T A e)(\exp \delta M)$.

As $\delta < 0$, we conclude that $x(M)^T A y(M) \to 0$ as $M \to \infty$, implying that the objective of Program I is not bounded away from zero, contradicting (f).

(d) $\Rightarrow$ (f): Suppose that (d) holds and that the objective of Program I is not bounded away from zero. Let $\alpha \equiv \min\{A_{ij} : (i, j) \in \operatorname{supp} A\}$. Then there exists a pair (x, y) that is feasible for Program I, and $x^T A y < \alpha$. In particular, for every $(i, j) \in \operatorname{supp} A$

$$x_i y_j \leqslant \frac{x_i A_{ij} y_j}{\alpha} \leqslant \frac{x^T A y}{\alpha} < 1. \tag{4.31}$$

Let $u \in R^m$ and $v \in R^n$ be defined by $u_i = \log x_i$, $i = 1, \ldots, m$ and $v_j = \log y_j$, $j = 1, \ldots, n$. Then (4.31) implies (3.22), and the feasibility of (x, y) for Program I implies (3.23), contradicting (d).

(e) $\Rightarrow$ (c): Assume that (e) holds. Then arguments similar to those used in the proof of the corresponding implication in Theorem 2 show that there exists no triplet $(u, v, s) \in R^m \times R^n \times R^{\operatorname{supp} A}$ which satisfies (4.18)–(4.21). Next, further arguments used in the proof of Theorem 2 show that nonexistence of (u, v, s) satisfying (4.18)–(4.21) implies nonexistence of (u, v, s) satisfying (4.18), (4.20), and (4.21). The latter is equivalent to (c).

(c) $\Rightarrow$ (e): Assume that (c) holds. To establish (e) let $I \subseteq \langle m \rangle$ and $J \subseteq \langle n \rangle$ satisfy $A_{I^c J} = 0$. Consider the vectors $u \equiv e^I \in R^m$ and $v \equiv -e^J \in R^n$. As $A_{I^c J} = 0$, we have that $u_i + v_j \geqslant 0$ for each pair $(i, j) \in \operatorname{supp} A$. Thus, we conclude from (c) that $r^T u + c^T v \geqslant 0$, i.e.,

$$\sum_{i \in I} r_i - \sum_{j \in J} c_j = r^T u + c^T v \geqslant 0. \quad \blacksquare \tag{4.32}$$

Proof of Theorem 4. If A has a zero row or zero column, then A has no (r, c)-scaling. Hence assume that A has no zero row or column. Now suppose that A has two scalings $B = XAY$ and $B' = X'AY'$ for which

$$Be = B'e = r \quad \text{and} \quad e^T B = e^T B' = c^T, \tag{4.33}$$

where X, X', Y and Y' are diagonal matrices of appropriate size having positive diagonal elements. Let $\bar{X} = X'X^{-1}$ and $\bar{Y} = Y^{-1}Y'$. Then $B' = \bar{X}B\bar{Y}$

and $B=(\overline{X})^{-1}B'(\overline{Y})^{-1}$. In particular, as $\sum_{j=1}^{n}B_{ij}=r_i$, we get from the inequality between the arithmetic and geometric mean that for $i=1,\dots,m$

$$1=\frac{r_i}{r_i}=\frac{(B'e)_i}{r_i}=\frac{\left(\sum_{j=1}^{n}\overline{X}_{ii}B_{ij}\overline{Y}_{jj}\right)}{r_i}$$
$$=\overline{X}_{ii}\sum_{j=1}^{n}\frac{B_{ij}}{r_i}\overline{Y}_{jj}\geqslant \overline{X}_{ii}\left(\prod_{j=1}^{n}\overline{Y}_{jj}^{B_{ij}/r_i}\right). \tag{4.34}$$

Raising the above inequalities to the r_ith power and taking products, we get that

$$1\geqslant\prod_{i=1}^{m}\left[\overline{X}_{ii}\left(\prod_{j=1}^{n}\overline{Y}_{jj}^{B_{ij}/r_i}\right)\right]^{r_i}=\left(\prod_{i=1}^{m}\overline{X}_{ii}^{r_i}\right)\left(\prod_{j=1}^{n}\overline{Y}_{jj}^{\sum_{i=1}^{m}B_{ij}}\right)$$
$$=\left(\prod_{i=1}^{m}\overline{X}_{ii}^{r_i}\right)\left(\prod_{j=1}^{n}\overline{Y}_{jj}^{c_j}\right). \tag{4.35}$$

Exchanging B and B' and replacing $\overline{X}$ by $(\overline{X})^{-1}$ and $\overline{Y}$ by $(\overline{Y})^{-1}$, we also get that

$$1\geqslant\left[\prod_{i=1}^{m}\left(\overline{X}_{ii}^{1}\right)^{-r_i}\right]\left[\prod_{j=1}^{n}\left(\overline{Y}_{jj}^{1}\right)^{-c_j}\right]=\left[\left(\prod_{i=1}^{m}\overline{X}_{ii}\right)\left(\prod_{j=1}^{n}\overline{Y}_{jj}\right)\right]^{-1}. \tag{4.36}$$

It follows from (4.35) and (4.36) that these two inequalities must hold as equalities, implying that each of the inequalities in (4.34) must hold as an equality. As the inequality between the arithmetic and geometric mean holds as equality if and only if the terms averaged with positive weights are all equal to the corresponding average, we get that the equalities in (4.34) imply that for $i=1,\dots,m$.

$$\overline{X}_{ii}\overline{Y}_{jj}=1 \qquad \text{for all} \quad j=1,\dots,n \quad \text{with } B_{ij}>0. \tag{4.37}$$

Thus, $\overline{X}_{ii}\overline{Y}_{jj}=1$ for all $(i,j)\in \operatorname{supp}B=\operatorname{supp}A$, i.e., $\overline{X}_{ii}B_{ij}\overline{Y}_{jj}=B_{ij}$. Hence,

$$B'=\overline{X}B\overline{Y}=B, \tag{4.38}$$

which proves that A has at most one (r,c)-scaling.

We next conclude from (4.37) that

$$(i, j_1) \in \operatorname{supp} A \text{ and } (i, j_2) \in \operatorname{supp} A \;\Rightarrow\; \overline{Y}_{j_1 j_1} = \overline{Y}_{j_2 j_2} \tag{4.39}$$

and

$$(i_1, j) \in \operatorname{supp} A \text{ and } (i_2 j) \in \operatorname{supp} A \;\Rightarrow\; \overline{X}_{i_1 i_1} = \overline{X}_{i_2 i_2}. \tag{4.40}$$

Hence it easily follows from the definition of chainability that if $I \subseteq \langle m \rangle$ and $J \subseteq \langle n \rangle$ such that A_{IJ} is chainable, then $\overline{X}_{ii}$ coincide for all $i \in I$ and $\overline{Y}_{jj}$ coincide for all $j \in J$. In particular, if α is the common value of the $\overline{X}_{ii}$'s when i ranges over I, then (4.37) shows that α^{-1} is the common value of the $\overline{Y}_{jj}$'s when j ranges over J. Thus, $\alpha = \overline{X}_{ii} = X'_{ii} X_{ii}^{-1}$ for each $i \in I$ and $\alpha^{-1} = \overline{Y}_{jj} = Y'_{jj} Y_{jj}^{-1}$ for each $j \in J$, implying that $X'_I = \alpha X_I$ and $Y'_J = \alpha^{-1} Y_J$. As $(I(1), J(1)), \ldots, (J(h), J(h))$ are the components of A, each $A_{I(t)J(t)}$ is chainable and (3.25) follows for corresponding numbers $\alpha_1, \ldots, \alpha_h$.

Finally, assume that X' and Y' satisfy (3.25). Then for $t = 1, \ldots, h$, $X'_{ii} Y'_{jj} = X_{ii} Y_{jj}$ for every $i \in I(t)$ and $j \in I(t)$, implying that

$$\begin{aligned}(X'AY')_{I(t)J(t)} &= X'_{I(t)} A_{I(t)J(t)} Y'_{J(t)} \\ &= X_{I(t)} A_{I(t)J(t)} Y_{J(t)} = (XAY)_{I(t)J(t)}.\end{aligned} \tag{4.41}$$

Also, as $A_{I(t)J(s)} = 0$ for $t \neq s$, we have that for such t and s

$$\begin{aligned}(X'AY')_{I(t)J(s)} &= X'_{I(t)} A_{I(t)J(s)} Y'_{J(s)} = 0 \\ &= X_{I(t)} A_{I(t)J(s)} Y_{J(s)} = (XAY)_{I(t)J(s)},\end{aligned} \tag{4.42}$$

completing the proof that $X'AY' = XAY$. ■

We note that the proof of Theorem 1 can be established from strict convexity properties of the objective of Program II, but the arguments used will implicitly coincide with those used in the proof established here.

Proof of Theorem 5. If x' and y' satisfy (3.27) where $\alpha_1, \ldots, \alpha_h$ satisfy (3.26), then

$$\begin{aligned}(x')^T A y' &= \sum_{t=1}^{h} (x'_{I(t)})^T A_{I(t)J(t)} y'_{J(t)} = \sum_{t=1}^{h} (x^*_{I(t)})^T A_{I(t)J(t)} y^*_{J(t)} \\ &= (x^*)^T A y^*.\end{aligned}$$

Also,

$$\prod_{i=1}^{m}(x_i')^{r_i}=\prod_{t=1}^{h}\left[\alpha_t^{\sum_{i\in I(t)}r_i}\left(\prod_{i\in I(t)}(x_i^*)^{r_i}\right)\right]$$

$$=\left(\prod_{t=1}^{h}\alpha_t^{\sum_{i\in I(t)}r_i}\right)\left(\prod_{i=1}^{m}(x_i^*)^{r_i}\right)=1\cdot 1=1.$$

Also, by the equivalence of parts (e) and (f) in Theorem 2, $\sum_{i\in I(t)}r_i=\sum_{j\in J(t)}c_j$ for $t=1,\dots,h$. Hence,

$$\prod_{j=1}^{n}(y_j')^{c_j}=\prod_{t=1}^{h}\left[\alpha_t^{-\sum_{j\in J(t)}c_j}\left(\prod_{j\in J(t)}(y_j^*)^{c_j}\right)\right]$$

$$=\left(\prod_{t=1}^{h}\alpha_t^{-\sum_{i\in I(t)}r_i}\right)\left(\sum_{j=1}^{n}(y_j^*)^{c_j}\right)=\left(\prod_{t=1}^{h}\alpha_t^{\sum_{i\in I(t)}r_i}\right)^{-1}\cdot 1=1.$$

So (x',y') is feasible for Program I and has the same objective value as (x^*,y^*), and therefore (x',y') is optimal for Program I.

Next assume that (x',y') is optimal for Program I. It then follows from part (a) of Theorem 1 and from Theorem 4 that there exist positive numbers $\alpha_1,\dots,\alpha_h$ such that with $\lambda^*=(x^*)^TAy^*/e^Tr$ and $\lambda'=(x')^TAy'/e^Tr$ we have that

$$(\lambda')^{-1}x'_{I(t)}=\alpha_t(\lambda^*)^{-1}x^*_{I(t)}\text{ and }y'_{J(t)}=\alpha_t^{-1}y^*_{I(t)},\qquad t=1,\dots,h.\quad(4.43)$$

As (x^*,y^*) and (x',y') are both optimal for Program I, $(x')^TAy'=(x^*)^TAy^*$ and therefore $\lambda'=\lambda^*$. Thus, (4.43) reduces to (3.27). Further, feasibility of (x^*,y^*) and (x',y') for Program I immediately implies (3.26).

Finally, if A is chainable, then $h=1$ and (3.26) reduces to $\alpha_1=1$. Thus, if (x^*,y^*) and (x',y') are optimal for Program I, (3.27) shows that $x'=x^*$ and $y'=y^*$. ■

APPENDIX

The purpose of this appendix is to summarize some results from the theory of linear inequalities that are used in this paper.

Theorems of the alternative characterize solvability of linear systems in terms of alternative systems. The following three theorem give such results.

THEOREM A.1. *Let $B \in R^{k \times n}$, $C \in R^{p \times n}$, $b \in R^k$, and $c \in R^p$. Then the system*

$$Bx \leqslant b, \quad \text{and} \quad Cx \ll c \tag{A.1}$$

has no solution if and only if for some $y \in R^k$, and $z \in R^p$

$$y^T B + z^T C = 0, \tag{A.2}$$

$$y \geqslant 0, \qquad z \geqslant 0, \tag{A.3}$$

$$y^T b + z^T c \leqslant 0, \tag{A.4}$$

and

$$\text{either} \quad y^T b + z^T c \neq 0 \quad \text{or} \quad z \neq 0. \tag{A.5}$$

Proof. See Schrijver (1986, Corollary 7.1k, p. 94). ∎

THEOREM A.2. *Let $A \in R^{m \times n}$ and $a \in R^m$. Then the system*

$$Ax = a, \qquad x \gg 0 \tag{A.6}$$

has no solution if and only if for some $\lambda \in R^m$

$$\lambda^T A \geqslant 0, \tag{A.7}$$

$$\lambda^T a \leqslant 0, \tag{A.8}$$

and

$$\text{either} \quad \lambda^T a < 0 \quad \text{or} \quad \lambda^T A \neq 0. \tag{A.9}$$

Proof. Apply Theorem A.1 to the case where $B = \begin{pmatrix} A \\ -A \end{pmatrix}$, $b = \begin{pmatrix} a \\ -a \end{pmatrix}$, $C = -I$ and $c = 0$. ∎

THEOREM A.3. *Let $A \in R^{m \times n}$ and $a \in R^m$. Then the system*

$$Ax = a, \qquad x \geqslant 0 \tag{A.10}$$

has no solution if and only if for some $\lambda \in R^m$

$$\lambda^T A \geqslant 0 \quad \text{and} \quad \lambda^T a < 0. \tag{A.11}$$

Proof. See Schrijver (1986, Corollary 7.1d, p. 89). ∎

We next characterize solvability of a linear system which has nonnegativity constraints via an approximate system with strict positivity constraints.

THEOREM A.4. *Let $A \in R^{m \times n}$ and $b \in R^m$. Then the system*

$$Ax = b, \qquad x \geqslant 0 \tag{A.12}$$

has a solution if and only if for every $\varepsilon > 0$ the system

$$\|Ax - b\|_\infty \leqslant \varepsilon, \qquad x \gg 0 \tag{A.13}$$

has a solution.

Proof. Suppose that (A.12) has a solution x^*. Let $e = (1, \dots, 1)^T \in R^n$. For each $\varepsilon > 0$, let $x^*(\varepsilon) = x^* + \varepsilon(\|Ae\|_\infty + 1)^{-1} e$. Then $x^*(\varepsilon) \gg 0$ and, as $Ax^* = b$, we have that $\|Ax^*(\varepsilon) - b\|_\infty = \varepsilon(\|Ae\|_\infty + 1)^{-1} \|Ae\|_\infty < \varepsilon$.

Next assume that (A.12) does not have a solution whereas (A.13) has a solution for each $\varepsilon > 0$, say $x(\varepsilon)$. As (A.12) has no solution, we have from the theorem of the alternative stated in Theorem A.3 that for some $\lambda \in R^m$,

$$\lambda^T A \geqslant 0, \qquad \lambda^T b < 0.$$

In particular, we conclude that for each $\varepsilon > 0$

$$\left(\sum_{i=1}^{m} |\lambda_i| \right) \varepsilon \geqslant \lambda^T [Ax(\varepsilon) - b] = (\lambda^T A) x(\varepsilon) - \lambda^T b \geqslant -\lambda^T b > 0.$$

As $\varepsilon = (-\lambda^T b)/(\sum_{i=1}^m |\lambda_i| + 1)$ does not satisfy the above inequality, we have a contradiction that proves the second implication. ∎

NOTE

The results of this paper were presented at the third Haifa Matrix Theory Conference in January 1987, and the paper appeared as Technical Report 88-28 of the Center for the Mathematical Sciences of the University of Wisconsin in April 1988. Recently a more general scaling problem was introduced by Bapat and Raghavan (1989). Several authors have used optimization techniques to study this and other generalized scaling problems. See the papers by Franklin and Lorenz (1989), Rothblum (1989), and M. H. Schneider (1989). The papers mentioned are published in this issue.

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