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Interaction Equations for Short and Long Dispersive Waves

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We show the time-local well-posedness for a system of nonlinear dispersive equations for the water wave interaction

$$\begin{cases} i\partial_t u + \partial_x^2 u = uv + |u|^2 u, & t \in (-T, T), \quad x \in \mathbf{R}, \\ \partial_t v + P(D_x) v = \partial_x |u|^2, & \\ u(0, x) = u_0(x), & v(0, x) = v_0(x). \end{cases}$$

It is shown that for any initial data $(u_0, v_0) \in H^s(\mathbf{R}) \times H^{s-1/2}(\mathbf{R})$ $(s \geqslant 0)$, the solution for the above equation uniquely exists in a subset of $C((-T, T); H^s) \times C((-T, T); H^{s-1/2})$ and depends continuously on the data. By virtue of a special structure of the nonlinear coupling, the solution is stable under a singular limiting process. © 1998 Academic Press

1. INTRODUCTION

An interaction phenomenon between long waves and short waves under a weakly coupled nonlinearity has been studied in various physical situations. After the proper rescaling arguments, the harmonics of the wave is subject to a typical form. The short wave term $S(x, t)$: $\mathbf{R} \times \mathbf{R} \to \mathbf{C}$ is described by a Schrödinger type equation and the long wave $L(x, t)$: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is described by some sort of wave equation occasionally accompanied by a dispersive term. In the most general case, the phenomenon is described by the nonlinear coupled system

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$$
\begin{aligned}
&i\,\partial_t S + ic_S\,\partial_x S + \partial_x^2 S = \alpha SL + \gamma\,|S|^2 S, \qquad t,\, x \in \mathbf{R},\\
&\partial_t L + c_L\,\partial_x L + \nu P(D_x)\,L + \lambda\,\partial_x L^2 = \beta\,\partial_x\,|S|^2,
\end{aligned}
\tag{1.1}
$$

where α, β, γ, ν, c_S, c_L, and λ are real constants and $P(D_x)$ is a linear differential operator with constant coefficients. This equation appears in various settings of physics and fluid mechanics. For example,

(i) the internal gravity wave packet [17] ($\beta<0$, $c_S=c_L=\gamma=\lambda=\nu=0$),

(ii) the capillary–gravity interaction wave [12] ($\beta<0$, $c_S=c_L=\gamma=\lambda=\nu=0$) and [24] ($P(D_x)=\partial_x^3$, $c_S=c_L=\alpha=0$, and $\lambda=\nu=1$),

(iii) the sonic–Langmuir wave interaction in plasma physics [22, 41] ($c_L=-1$, $c_S=\lambda=\nu=\gamma=0$),

(iv) the general theory of water wave interaction in a nonlinear medium [5] ($\nu=0$, $\lambda=0$ or 1),

(v) the motion of two fluids under capillary–gravity waves in a deep water flow [13] ($P(D_x)=D_x\,\partial_x$ and $\nu=1$, $c_S=c_L=\gamma=\lambda=0$, $\alpha, \beta>0$), and

(vi) the motion of two fluids under a shallow water flow [13] ($\lambda=\nu=\gamma=c_S=c_L=0$, $\alpha, \beta>0$).

The purpose of this paper is to consider the well-posedness of the Cauchy problem for the interaction equation (1.1). We refer to the word "local well-posedness" in the sense of Hadamard, that is, the solution uniquely exists in a certain time interval (unique existence), the solution has the same regularity as the initial data in a certain time interval (persistence), and the solution varies continuously depending upon the initial data (continuous dependence). Global well-posedness requires that the same properties hold for all time $t>0$.

The most typical case in the theory of wave interaction is described by the following system,

$$
\begin{cases}
i\,\partial_t u + \partial_x^2 u = \alpha v u + \gamma\,|u|^2 u, & t,\, x \in \mathbf{R},\\
\partial_t v + c\,\partial_x v = \beta\,\partial_x(|u|^2), & \\
u(x,0) = u_0(x), & v(x,0) = v_0(x),
\end{cases}
\tag{1.2}
$$

where $c=\pm 1$ and α, β, and γ are real constants. In fact when $\nu=0$ in (1.1), the two systems (1.1) and (1.2) are equivalent by a proper gauge transformation and scaling of the variables (see [39]).

The solvability of the system (1.2) is considered under various settings. Yajima andOikawa [41] applied the inverse scattering method and found the N-soliton solution of (1.2) when $c=1$, $\beta=-1$, and $\gamma=0$. See also Ma [30]. Laurençot [29] considered the orbital stability for a weak solution in $H^1(\mathbf{R})$ around the stationary standing waves. M. Tsutsumi and Hatano [38, 39] considered (1.2) using the method of nonlinear evolution equations and showed local well-posedness where the initial data $(u_0, v_0) \in H^{m+1/2}(\mathbf{R}) \times H^m(\mathbf{R})$ with $m = 0$ for $\gamma = 0$ and with $m = 1, 2, ...$ for $\gamma \neq 0$. They also obtained global well-posedness in similar spaces [39]. Bekiranov *et al.* [1] generalized the result of M. Tsutsumi and Hatano [39] for initial data $(u_0, v_0) \in H^s(\mathbf{R}) \times L^{1/s}(\mathbf{R})$ for $0 < s < 1/2$ when $\gamma = 0$.

Our first purpose is to show that system (1.2) is locally well-posed in $L^2 \times H^{-1/2}$. Hence we improve the results found in [1]. In general, for a nonlinear evolution equation it is more difficult to show well-posedness for initial data in a larger class. The single nonlinear Schrödinger equation,

$$\begin{cases} i\,\partial_t u + \partial_x^2 u = \gamma\,|u|^2 u \\ u(x, 0) = u_0(x), \end{cases}$$

is known to be complete integrable ($\gamma = 1$) and well-posed in the space $L^2(\mathbf{R})$ (Zakharov and Shabat [43], Y. Tsutsumi [40], Kato [23], and Cazenave and Weissler [11]). Hence it is natural to expect that the system (1.2) is well-posed for $u_0 \in L^2(\mathbf{R})$.

From a mathematical point of view, investigating the well-posedness of a dispersive system such as (1.1) is related to the smoothing effects induced by the linear parts of each of the equations that make up the system and the relation of them with the nonlinear terms, in both its degree of nonlinearity and its particular structure. Recently the local well-posedness for single dispersive equations with quadratic nonlinearities has been extensively studied in Sobolev spaces with negative indices. For example, the one-dimensional nonlinear Schrödinger equation with appropriate quadratic nonlinearity is known to be well-posed up to $H^{-1/2+\varepsilon}(\mathbf{R})$, and the KdV equation is well-posed up to $H^{-3/4+\varepsilon}(\mathbf{R})$ ([25, 27, 35]).

In general, a coupled system like (1.2) is more difficult to handle in the same spaces as in the space each of the equations is solved. Due to the asymmetry of the characteristics of each linear part in system (1.2), however, we are still able to show well-posedness for the initial data $(u_0, v_0) \in L^2(\mathbf{R}) \times H^{-1/2}(\mathbf{R})$.

For the system that describes the capillary–gravity interaction [24], (cited in (ii) in the beginning of this Introduction),

$$\begin{cases} i\,\partial_t u + \partial_x^2 u = \gamma\,|u|^2\,u, \\ \partial_t v + \partial_x^3 v + \partial_x v^2 = \partial_x\,|u|^2, \\ u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \end{cases}$$

similar results have been obtained (see M. Tsutsumi [37] and Bekiranov *et al.* [2]). Since well-posedness has been established in a weak space, this interaction model is valid for more singular waves than continuous data.

Next we treat two related systems, (v) and (vi), that describe the interaction of two fluid interfaces under the setting of deep and shallow flows. Funakoshi and Oikawa [13] formulated the interaction waves for the boundary of two fluids with different densities. If the flow is considered in the deep flow setting, the interaction equation is described by the following system,

$$\begin{cases} i\,\partial_t u + \partial_x^2 u = \alpha v u, & t,\, x \in \mathbf{R}, \\ \partial_t v + \nu\,\partial_x\, D_x v = \beta\,\partial_x(|u|^2), & \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), \end{cases} \tag{1.3}$$

where $D_x = H\,\partial_x = |\partial_x|$ and H denotes the Hilbert transform defined by

$$Hu = \mathscr{F}^{-1}\,\frac{-i\xi}{|\xi|}\,\hat{u},$$

while under the shallow flow setting, the governing system is as follows:

$$\begin{cases} i\,\partial_t u + \partial_x^2 u = \alpha u v, & t,\, x \in \mathbf{R}, \\ \partial_t v = \beta\,\partial_x(|u|^2), & \\ u(x,0) = u_0(x), & v(x,0) = v_0(x). \end{cases} \tag{1.4}$$

Benney [5, 6], Grimshaw [17], Djordjevic and Redekopp [12], and Yajima and Satsuma [42] studied these systems numerically. We show that the solution of the system (1.3) is locally well-posed in the space $H^s(\mathbf{R}) \times H^{s-1/2}(\mathbf{R})$ $(s \geqslant 0)$. The presence of a stronger dispersive property due to the term $\nu\,\partial_x\, D_x u$ in the second equation in (1.3) suggests that this system has a better smoothing effect than system (1.4). Nevertheless, local well-posedness in $L^2(\mathbf{R}) \times H^{s-1/2}(\mathbf{R}^2)$ is derived for system (1.4). In other words, the smoothing effect in (1.3) due to the dispersive term $\nu\,\partial_x\, D_x u$ does not play a significant role in the solvability of (1.3). For the case, $\nu = 1, -1$, it seems that for the characteristics of the linear parts in (1.3) there is a type of cancellation and the system does not have sufficient smoothing to guarantee well-posedness in weaker spaces. For the case $|\nu| < 1$, however, the special structure of the nonlinear terms give us some advantage for well-posedness. Thus we have different results depending of the value of ν. As a byproduct of well-posedness, we also show that by a (singular) limiting procedure, as $\nu \to 0$, the solution u_ν to (1.3) strongly converges to the unique solution for (1.4). This limiting procedure implicitly approximates the situation where the depth of the second fluid becomes more shallow and closer to the depth of the first fluid's surface.

To state our result, we introduce the integral equation associated to (1.2) (or (1.4) with $\gamma = 0$),

$$u(t) = U(t)\, u_0 - i \int_0^t U(t-t')\{\alpha v(t')\, u(t') + \gamma\, |u(t')|^2\, u(t')\}\, dt',$$

$$v(t) = W_c(t)\, v_0 + \beta \int_0^t W_c(t-t')\, \partial_x\, |u(t')|^2\, dt',$$

where $U(t) = e^{it\,\partial_x^2}$ and $W_c(t) = e^{-ict\,\partial_x}$ denote the unitary operators for the linear Schrödinger and first order wave equations.

For the particular case (1.4), we understand that $c = 0$ for the group $W_c(t)$, hence the second integral equation has the following simple form

$$v(t) = v_0 + \beta \int_0^t \partial_x |u(t')|^2\, dt'.$$

Also, we shall let $\psi(t)$ denote a smooth cut off function such that $\psi = 1$ for $|x| \leqslant 1$ and $\psi = 0$ for $|x| > 2$. For $T > 0$, $\psi_T = \psi(t/T)$.

We state our first result for the systems (1.2) and (1.4).

THEOREM 1.1. *Let $s \geqslant 0$ and $b \in (1/2, 1)$. Then for any $(u_0, v_0) \in H^s(\mathbf{R}) \times H^{s-1/2}(\mathbf{R})$, there exists $T > 0$ and a unique solution $(u(t), v(t))$ of the initial value problem* (1.2) (*and of* (1.4)) *such that*

(1) *The solution (u, v) satisfies*

$$u \in C([0, T); H^s(\mathbf{R})),$$
$$\psi_T U(-t)\, u \in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})),$$
$$v \in C([0, T); H^{s-1/2}(\mathbf{R})),$$
$$\psi_T W_c(-t)\, v \in H_t^b(\mathbf{R}; H_x^{s-1/2}(\mathbf{R})).$$

(2) *The nonlinearities are well-defined and satisfy*

$$u \in L^6((0, T); L_x^6),$$
$$\psi_T W_c(-t)(\partial_x\, |u|^2) \in H_t^b(\mathbf{R}; H_x^{s-1/2}(\mathbf{R})),$$
$$\psi_T U(-t)(uv) \in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})).$$

(3) *The solution of the Schrödinger part u preserves its L^2 norm, i.e., for $0 < t < T$,*

$$\|u(t)\|_2 = \|u_0\|_2.$$

(4) *Moreover, the map $(u_0, v_0) \to (u(t), v(t))$ is Lipschitz continuous from $L^2(\mathbf{R}) \times H^{-1/2}(\mathbf{R})$ to $C([0, T); L^2(\mathbf{R})) \times C([0, T); H^{-1/2}(\mathbf{R}))$.*

Remark 1. The local well-posedness for a solution of (1.2) in $H^s \times L^{1/s}(\mathbf{R})$ ($s>0$ with $\gamma=0$) was obtained by Bekiranov *et al.* [1] using commutator estimates. The above theorem improves the previous result. Combining the above theorem with the argument in [1], it is possible to show that a global result holds for the case $(u_0, v_0) \in H^s(\mathbf{R}) \times H^{s-1/2}(\mathbf{R})$ $(s \geqslant 1/2)$. This result is a generalization of the result by M. Tsutsumi and Hatano [38, 39], where they considered the solution in $H^{m+1/2}(R) \times H^m(\mathbf{R})$ $m=0, 1, 2\ldots$ (with $\gamma=0$ when $m=0$).

Remark 2. The regularity difference of 1/2 is a natural consequence of the structure of the nonlinear terms. One can scale the solution, (u, v), of the system (1.2) or of (1.4) as

$$u_\lambda(x, t) = \lambda^{3/2} u(\lambda x, \lambda^2 t),$$
$$v_\lambda(x, t) = \lambda^2 v(\lambda x, \lambda^2 t),$$

where we ignore the terms $c\, \partial_x v$ and $\gamma |u|^2 u$. That is (u_λ, v_λ) solves (1.2) or (1.4) with initial data $u_{\lambda 0} = \lambda^{3/2} u_0(\lambda x)$ and $v_{\lambda 0} = \lambda^2 v_0(\lambda x)$. Now taking the homogeneous derivative of order s in L^2 for u_λ and $s-1/2$ in L^2 for v_λ yields the following

$$\|D_x^s u_\lambda\|_2^2 = \lambda^{2+2s} \|D_x^s u\|_2^2,$$
$$\|D_x^{s-1/2} v_\lambda\|_2^2 = \lambda^{2+2s} \|D_x^{s-1/2} v\|_2^2.$$

Thus the difference in regularity of 1/2 is needed to keep each norm equivalent under scaling. This coupled with the best result for the cubic nonlinear Schrödinger equation (i.e., well-posedness in L^2 [40]) tells us that Theorem 1.1 is in some sense the best possible.

Remark 3. Related to another kind of nonlinear Schrödinger equation, it should be commented that the weak solution for the derivative nonlinear Schrödinger equation uniquely exists in $H^1(\mathbf{R})$ (see Hayashi [18], Hayashi and Ozawa [20], and Ozawa [33]). These results are obtained by reducing the single equation into a system of nonlinear Schrödinger equations. In our case, the presence of a stronger smoothing effect, which is mainly due to the special structure of the nonlinear interactions, enable us to handle well-posedness in a weaker space.

Remark 4. After completing this work, we are noticed that the similar result is obtained by Ginibre *et al.* [15]. They consider Zakharov system including higher dimensional cases as well as the first order case (1.2) in one space dimension.

We next state the analogous result for the deep flow version of two phase flow of interaction equation (1.3). Note that the integral equation is expressed by

$$u(t) = U(t)\, u_0 - i \int_0^t U(t-t')\{\alpha v(t')\, u(t')\}\, dt',$$

$$v(t) = V_\nu(t)\, v_0 + \beta \int_0^s V_\nu(t-t')\, \partial_x |u(t')|^2\, dt'.$$

where $V_\nu(t) = e^{-\nu t\, \partial_x D_x}$ is the linear Benjamin–Ono propagator.

THEOREM 1.2. *Let $s \geqslant 0$ and $b \in (1/2, 3/4)$. Then for any $(u_0, v_0) \in H^s(\mathbf{R}) \times H^{s-1/2}(\mathbf{R})$ and $|\nu| < 1$, there exists $T > 0$ such that the initial value problem* (1.3) *admits a unique solution $(u(t), v(t))$. Further*:

(1) *The solution satisfies*

$$\begin{aligned} u &\in C([0, T); H^s(\mathbf{R})), \\ \psi_T U(-t)\, u &\in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})), \\ v &\in C([0, T); H^{s-1/2}(\mathbf{R})), \\ \psi_T V_\nu(-t)\, v &\in H_t^b(\mathbf{R}; H_x^{s-1/2}(\mathbf{R})). \end{aligned}$$

(2) *The nonlinearities are well-defined and satisfy*

$$\psi_T U(-t)(uv) \in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})),$$
$$\psi_T V_\nu(-t)(\partial_x |u|^2) \in H_t^b(\mathbf{R}; H_x^{s-1/2}(\mathbf{R})).$$

(3) *For any $s \geqslant 0$, the solution of the Schrödinger part u preserves its L^2 norm, i.e., for any $0 < t < T$,*

$$\|u(t)\|_2 = \|u_0\|_2.$$

(4) *We have the momentum, and the energy conservation laws*:

$$\operatorname{Im} \int u(t)\, \partial_x \bar{u}(t)\, dx - \|v(t)\|_2^2 = \operatorname{Im} \int u_0\, \partial_x \bar{u}_0\, dx - \|v_0\|_2^2 \qquad \text{for} \quad s \geqslant \tfrac{1}{2},$$

$$\|\partial_x u(t)\|_2^2 + \alpha \int v(t) |u(t)|^2\, dx - \tfrac{1}{2} \|D_x^{1/2} v(t)\|_2^2$$

$$= \|\partial_x u_0\|_2^2 + \alpha \int v_0 |u_0|^2\, dx - \tfrac{1}{2} \|D_x^{1/2} v_0\|_2^2 \qquad \text{for} \quad s \geqslant 1.$$

(4) *Moreover, for $T > 0$ the map $(u_0, v_0) \to (u(t), v(t))$ is Lipschitz continuous from $H^s(\mathbf{R}) \times H^{s-1/2}(\mathbf{R})$ to $C([0, T); H^s(\mathbf{R})) \times C([0, T); H^{s-1/2}(\mathbf{R}))$.*

Remark 5. It should be noted that a single dispersive wave in deep fluid flow is generally described by the equation

$$\begin{cases} \partial_t v + \nu\, \partial_x D_x v + \partial_x v^2 = 0 \\ v(x, 0) = v_0(x), \end{cases}$$

which was studied by Benjamin [4] and Ono [31]. Existence and well-posedness results for this single equation are found in [21, 16, 34, 25] (see also [19]). Since (1.3) has no self-interacting nonlinear term, $\partial_x v^2$, the existence result is better than the results obtained in the references above. It should be also noted that the method for the weak solvability of nonlinear Schrödinger equations and the KdV equation found in [7, 8, 26, 27] does not seem to be applicable to this single equation. The smoothing effect for the linear part of the Benjamin–Ono equation is weaker than that of the linear part of the KdV equation although the nonlinear terms are similar. The method of proof which is used for both Theorems 1.1 and 1.2 is based on the analogous argument introduced by Bourgain ([7–9]), in the spatially periodic case and extensively improved by Kenig *et al.* [25, 27, 28] for the Schrödinger and KdV equations. The key fact is that we use appropriate space-time weight norm in the phase space to see the smoothing effect of two dispersive linear equations and smoothing effects of the quadratic nonlinearities which is seen as terms of a convolution of weight potentials. The expression of quadratic nonlinearities by convolution yields some benefit to avoid a loss in regularity. It should be emphasized that the special structure of the nonlinear coupling presents a better smoothing effect. Especially the nonlinear term, $\partial_x |u|^2$, in the second equation has a preferable property of smoothing. In general, $|u|^2$ has a better smoothing property than u or $\bar{u}$ alone. We use this property to obtain well-posedness independent of the dispersive property of the second equation.

Remark 6. For the case $\nu = 1$ or -1, the characteristics between the Schrödinger part and the Benjamin–Ono part in the system (1.3) do not have as strong a cancellation as in the case $|\nu| < 1$; thus we do not see the same smoothing effects. This may be due to some sort of resonance. We expect that if in the regular case $s > 0$, the well-posedness holds for $|\nu| = 1$.

Remark 7. Following the argument in [28] (Theorem 1.4 (ii)) we shall show that the result in Theorem 1.3 is the best possible given by our method (it fails for $s < 0$ except for the limiting case, $s = 0$, which remains open). In [27], it was established that the initial value problem

$$\begin{cases} i\, \partial_t u + \partial_x^2 u = \gamma\, |u|^2 \\ u(x, 0) = u_0(x) \end{cases}$$

is locally well-posed for $s > -1/4$ and that the crucial bilinear estimate

$$\| U(-t)\, |u|^2 \|_{H_t^{b-1}(\mathbf{R};\, H_x^s)} \leqslant C \, \| U(-t) u \|^2_{H_t^b(\mathbf{R};\, H_x^s)}$$

fails for $s < -1/4$. The same example used in [28] to show this failure proves that

$$\| U(-t)\, uv \|_{H_t^{b-1}(\mathbf{R};\, H_x^s)} \leqslant C \, \| U(-t) u \|_{H_t^b(\mathbf{R};\, H_x^s)} \, \| V_1(-t) u \|_{H_t^b(\mathbf{R};\, H_x^s)}$$

also fails for $s < 0$.

As an application of both Theorems 1.1 and 1.2, we state a result of the limiting problem $\nu \to 0$ for the system (1.3).

THEOREM 1.3. *Let* $(u_\nu(t), v_\nu(t))$ *be a solution of* (1.3) *obtained in Theorem* 1.2 *and* $(u(t), v(t))$ *be of* (1.4) *in Theorem* 1.1 *with the same initial data* $(u_0, v_0) \in L^2(\mathbf{R}) \times H^{-1/2}(\mathbf{R})$. *Then for any* $T > 0$ *and* $b \in (1/2, 3/4)$, *we have*

$$\| u_\nu - u \|_{C([0,\, T);\, L^2)} + \| \psi_T (u_\nu - u) \|_{H_t^b(\mathbf{R};\, L_x^2)} \to 0,$$

$$\| v_\nu - v \|_{C([0,\, T);\, H^{-1/2})} \to 0$$

as $\nu \to 0$.

Remark 8. It is not difficult to obtain a similar result for the regular cases once we have a uniform estimate in ν for the local solutions. According to the weak well-posedness results contained in Theorems 1.1 and 1.2, we can approximate the weak solution by regularized solutions.

In what follows we use the following notations. Let $\mathscr{F}_x$ and $\mathscr{F}_t$ be the Fourier transform in the x and t variables. $\langle x \rangle = (1 + |x|)$ is a weight function both used for the time and space phase variables. We use the conventional notation for a differential operator $\langle D_x \rangle = \mathscr{F}_x^{-1} \langle \xi \rangle \mathscr{F}_x$. The norm for $L^2(\mathbf{R})$ and the Sobolev space $H^s(\mathbf{R})$ in the space variable are expressed by $\| \cdot \|_2$ and $\| v \|_{H^s} = \| \langle D_x \rangle^s v \|_2$, respectively. We denote $L_t^p(L_x^q)$ as the Banach spaces $L^p(\mathbf{R};\, L^q(\mathbf{R}))$ in the variables (t, x) and $1 < p, q \leqslant \infty$. $H^b(\mathbf{R};\, H^s) = H_t^b(\mathbf{R};\, H_x^s(\mathbf{R}))$ is the Hilbert space defined by

$$H_t^b(\mathbf{R};\, H_x^s(\mathbf{R})) = \{ u(t, x) \in \mathscr{S}(\mathbf{R}^2);\, \langle D_t \rangle^b u \in L_t^2(\mathbf{R};\, H_x^s(\mathbf{R})) \}.$$

Let $\hat{\hat{f}}$ be the Fourier transform of f in both the x and t variables, that is

$$\hat{\hat{f}}(\tau, \xi) = (2\pi)^{-1} \iint_{\mathbf{R}^2} e^{-it\tau - ix\xi} f(t, x)\, dt\, dx.$$

$\langle\!\langle f, g \rangle\!\rangle$ denotes a dual coupling and $f * g$ is the convolution of f and g in space and time.

Let $U(t)=e^{it\partial_x^2}$, $V_\nu(t)=e^{-\nu t D_x \partial_x}$ and $W_c(t)=e^{-ct\partial_x}$ be the unitary operators associated with the linear Schrödinger, the linear Benjamin–Ono and the first order wave equation respectively. Finally we introduce the function spaces for constructing the local solutions which are originally used by Bourgain ([7]) (see also [25, 26, 2]). For $s\in\mathbf{R}$ and $-1<b<1$, we let $X^{s,b}$, $Y_\nu^{s,b}$, $Z_c^{s,b}$ be the Hilbert spaces with the norms

$$\begin{aligned}
\|f\|_{X^{s,b}} &= \left(\iint \langle \tau+\xi^2\rangle^{2b} \langle \xi\rangle^{2s} |\hat{\hat{f}}(\tau,\xi)|^2\, d\tau\, d\xi\right)^{1/2} \\
&= \|U(-t)\, f(t)\|_{H_t^b(\mathbf{R};\, H_x^s(\mathbf{R}))}, \\
\|g\|_{Y_\nu^{s,b}} &= \left(\iint \langle \tau+\nu\xi\, |\xi|\rangle^{2b} \langle \xi\rangle^{2s} |\hat{\hat{g}}(\tau,\xi)|^2\, d\tau\, d\xi\right)^{1/2} \\
&= \|V_\nu(-t)\, g(t)\|_{H_t^b(\mathbf{R};\, H_x^s(\mathbf{R}))}, \\
\|h\|_{Z_c^{s,b}} &= \left(\iint \langle \tau+c\xi\rangle^{2b} \langle \xi\rangle^{2s} |\hat{\hat{h}}|\, (\tau,\xi)|^2\, d\tau\, d\xi\right)^{1/2} \\
&= \|W_c(-t)\, h(t)\|_{H_t^b(\mathbf{R};\, H_x^s(\mathbf{R}))}.
\end{aligned}$$

We will avoid using the subscripts ν and c except when this might cause confusion, that is $V(t)$ will denote $V_\nu(t)$, $W(t)$ will denote $W_c(t)$ and $\|\cdot\|_{Y^{s,b}} = \|\cdot\|_{Y_\nu^{s,b}}$, $\|\cdot\|_{Z^{s,b}} = \|\cdot\|_{Z_c^{s,b}}$. $\psi=\psi(t)$ always denote a fixed smooth cut off function such that $\psi(t)=1$ for $|t|\leqslant 1$ and $\psi(t)=0$ for $|t|>2$. $\psi_\delta(t)=\psi(t/\delta)$ for $\delta>0$. Various constants are denoted by C.

2. PRELIMINARY ESTIMATES

The following lemma concerning the basic estimates on the function spaces we consider are established by Kenig *et al.* [25, 27].

LEMMA 2.1 ([25, 27]). *Let* $s\in\mathbf{R}$, $b\in(1/2,1)$ *and* $\delta\in(0,1)$, *then for* $F\in X^{s,b}$ *we have*

$$\|\psi_\delta F\|_{X^{s,b}} \leqslant C\,\delta^{(1-2b)/2}\,\|F\|_{X^{s,b}}. \tag{2.1}$$

Let $a, b\in(0,1/2)$ *with* $a<b$ *and* $\delta\in(0,1)$, *then for* $F\in X^{s,-a}$ *we have*

$$\|\psi_\delta F\|_{X^{s,-b}} \leqslant C\,\delta^{(b-a)/4(1-a)}\,\|F\|_{X^{s,-a}}. \tag{2.2}$$

Similar estimates hold for $Y^{s,b}$ *and* $Z^{s,b}$ *replacing* $X^{s,b}$.

LEMMA 2.2 ([7, 14, 27]). *Let* $s \in \mathbf{R}$, $b \in (1/2, 1)$ *and* $\delta \in (0, 1)$. *Then for* $F \in H_t^b(\mathbf{R}; H_x^s)$, *we have*

$$\left\| \psi_\delta \int_0^t F(t')\, dt' \right\|_{H_t^b(\mathbf{R};\, H_x^s)} \leqslant C\, \delta^{(1-2b)/2}\, \|F\|_{H_t^{b-1}(\mathbf{R};\, H_x^s)}, \tag{2.3}$$

$$\left\| \psi_\delta \int_0^t F(t')\, dt' \right\|_{L^\infty((0,\,T);\, H_x^s)} \leqslant C\, \delta^{(1-2b)/2}\, \|F\|_{H_t^{b-1}(\mathbf{R};\, H_x^s)}. \tag{2.4}$$

Proof of Lemma 2.2. The second inequality (2.4) is immediately obtained from (2.3) by the Sovolev immedding. The first inequality is derived as follows. By the argument in [7] and [27] we divide the following integral into three parts:

$$\begin{aligned}
\psi_\delta \int_0^t F(t')\, dt' &= \psi_\delta \int_0^t \int e^{ix\xi + it'\tau} \hat{\hat{f}}(t', \xi)\, d\xi\, dt' \\
&= \psi_\delta \iint e^{ix\xi} \left(\int_0^t e^{it'\tau}\, dt' \right) \hat{\hat{F}}(\tau, \xi)\, d\tau\, d\xi \\
&= \psi_\delta \iint e^{ix\xi + it\tau}\, \frac{1 - e^{it\tau}}{i\tau}\, \psi(\tau)\; \hat{\hat{F}}(\tau, \xi)\, d\tau\, d\xi \\
&\quad + \psi_\delta \iint e^{ix\xi + it\tau}\, \frac{1}{i\tau}\, (1 - \psi(\tau))\; \hat{\hat{F}}(\tau, \xi)\, d\tau\, d\xi \\
&\quad - \psi_\delta \iint e^{ix\xi}\, \frac{1 - \psi(\tau)}{i\tau}\; \hat{\hat{F}}(\tau, \xi)\, d\tau\, d\xi \\
&\equiv I + II + III.
\end{aligned}$$

Noting that $\mathscr{F}_t \psi_\delta = \delta \hat{\psi}(\delta\tau)$, we have from Lemma 2.1 (2.1),

$$\begin{aligned}
&\|I\|_{H_t^b(\mathbf{R};\, H_x^s)} \\
&= \left\| \langle D_t \rangle^b\, \psi_\delta \iint e^{ix\xi + it\tau} \sum_{k=1}^\infty \frac{(-it)^k}{ik!}\, \tau^{k-1} \psi(\tau)\; \hat{\hat{F}}(\tau, \xi)\, d\tau\, d\xi \right\|_{L_t^2(\mathbf{R};\, H_x^s)} \\
&\leqslant C \sum_{k=1}^\infty \left\| \langle D_t \rangle^b \left(\frac{(-it)^k\, \psi_\delta}{k!} \iint e^{ix\xi + it\tau} \tau^{k-1} \psi(\tau) \langle \xi \rangle^s\, \hat{\hat{F}}(\tau, \xi)\, d\tau\, d\xi \right) \right\|_{L_t^2(\mathbf{R};\, H_x^s)} \\
&\leqslant C \int_{k=1}^\infty \frac{\delta^k}{k!} \left(\|\psi_\delta\|_\infty + \|D_t^b \psi_\delta\|_{L_t^2} \right) \|\langle \tau \rangle^b\, |\tau|^{k-1}\, \psi(\tau) \langle \xi \rangle^s\, \hat{\hat{F}}(\tau, \xi)\|_{L_\tau^2(\mathbf{R};\, L_\xi^2)} \\
&\leqslant C e^\delta (1 + \delta^{(1-2b)/2})\, \|\psi(\tau) \langle \xi \rangle^s\, \hat{\hat{F}}(\tau, \xi)\|_{L_\tau^2(L_\xi^2)} \\
&\leqslant C\, \delta^{(1-2b)/2}\, \|F\|_{H_t^{-\infty}(\mathbf{R};\, H_x^s)},
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\|II\|_{H_t^b(\mathbf{R};\, H_x^s)} &\leqslant C\,\delta^{(1-2b)/2} \left\| \psi_\delta \iint e^{ix\xi+it\tau}\, \frac{1-\psi(\tau)}{i\tau}\, \langle\tau\rangle^b\, \hat{\hat{F}}(\tau,\xi)\, d\tau\, d\xi \right\|_{L_t^2(\mathbf{R};\, H_x^s)} \\ &\leqslant C\,\delta^{(1-2b)/2}\, \|\langle\tau\rangle^{b-1} \langle\xi\rangle^s\, \hat{\hat{F}}(\tau,\xi)\|_{L_t^2(\mathbf{R};\, L_x^2)} \\ &\leqslant C\,\delta^{(1-2b)/2}\, \|F\|_{H_t^{b-1}(\mathbf{R};\, H_x^s)}. \end{aligned} \tag{2.6}$$

For the third term we use the homogeneous estimate (2.3),

$$\begin{aligned}\|III\|_{H_t^b(\mathbf{R};\, H_x^s)} &= \left\| \psi_\delta \iint e^{ix\xi}\, \frac{1-\psi(\tau)}{i\tau}\, \hat{\hat{F}}(\tau,\xi)\, d\tau\, d\xi \right\|_{H_t^b(\mathbf{R};\, H_x^s)} \\ &= \|\psi_\delta\|_{H_t^b(\mathbf{R})} \left\| \int e^{ix\xi} \left(\int \frac{(1-\psi(\tau))\, \hat{\hat{F}}(\tau,\xi)}{i\tau}\, d\tau \right) d\xi \right\|_{H^x} \\ &\leqslant C\,\delta^{(1-2b)/2} \left\| \langle\xi\rangle^s \left(\int t\, \frac{1-\psi(t)}{\tau}\, \hat{\hat{F}}(\tau,\xi)\, d\tau \right) \right\|_{L_\xi^2} \\ &\leqslant C\,\delta^{(1-2b)/2} \left(\int \left(\int \langle\tau\rangle^{-2b}\, d\tau \right) \right. \\ &\qquad \times \left. \left(\int \langle\tau\rangle^{2(b-1)} \langle\xi\rangle^{2s}\, |\hat{\hat{F}}(\tau,\xi)|^2\, d\tau \right) d\xi \right)^{1/2} \\ &\leqslant C\,\delta^{(1-2b)/2}\, \|F\|_{H_t^{b-1}(\mathbf{R};\, H_x^s)}. \end{aligned} \tag{2.7}$$

Combining (2.5)–(2.7), we have the desired estimate. ▮

Next we show the estimates needed for the linear Schrödinger equation and the linear part of the KdV equation which are due to Kenig *et al.* Recall that $U(t)=e^{it\,\partial_x^2}$, $V(t)=e^{-vt\,D_x\,\partial_x}$, and $W(t)=e^{-ct\,\partial_x}$ denote the linear Schrödinger, linear Benjamin-Ono, and linear wave unitary groups, respectively.

PROPOSITION 2.3 ([7, 25, 27]). *Let $s\in\mathbf{R}$, $1/2<b<1$ and $\delta\in(0,1)$. Then we have*

$$\|\psi_\delta\, U(t)\, u_0\|_{X^{s,b}} \leqslant C\,\delta^{(1-2b)/2}\, \|u_0\|_{H^s}, \tag{2.8}$$

$$\left\| \psi_\delta \int_0^t U(t-t')\, F(t')\, dt' \right\|_{X^{s,b}} \leqslant C\,\delta^{(1-2b)/2} \|F\|_{X^{s,b-1}}. \tag{2.9}$$

Similar estimates hold for $V(t)$ and $W(t)$ replacing $U(t)$ and $Y^{s,b}$ and $W^{s,b}$ replacing $X^{s,b}$, respectively.

Proof of Proposition 2.3. Since $U(t)\,\psi_\delta = \psi_\delta\, U(t)$, we see

$$\|\psi_\delta u(t)\, U_0\|_{X^{s,b}} = \|U(-t)\,\psi_\delta\, U(t)\, u_0\|_{H_t^b(\mathbf{R};\, H_x^s)} = \|\psi_\delta\|_{H_t^b} \|u_0\|_{H_x^s}$$

and the first inequality (2.8) follows from $\|\psi_\delta\|_{H_t^b} = C\,\delta^{(1-2b)/2}$. The second inequality (2.9) is a direct consequence of Lemma 2.2. ∎

The following corollary follows immediately from Proposition 2.3.

COROLLARY 2.4. *Let $b \in (1/2, 1)$ and $\delta \in (0, 1)$. Then we have*

$$\left\| \psi_\delta \int_0^t U(t-t')\, F(t')\, dt \right\|_{L^\infty((0,T);\, H_x^s)} \leqslant C\,\delta^{(1-2b)/2}\, \|F\|_{X^{s,\, b-1}}. \tag{2.10}$$

Similar estimates hold for $F \in Y^{s,b}$ and $F \in Z^{s,b}$ with replacing $U(t)$ by $V(t)$ and $W(t)$, respectively.

The following estimate due to Strichartz [36] is well known and used often in the various areas of the study of nonlinear Schrödinger equations.

PROPOSITION 2.4 ([36]). *Let $u_0 \in L^2(\mathbf{R})$ then*

$$\|U(t)\, u_0\|_{L_t^6(L_x^6)} \leqslant C\, \|u_0\|_2.$$

Finally we give some elementary estimates needed for the nonlinear estimates in Section 3.

LEMMA 2.5 ([27]). (1) *For $p, q > 0$ and $r = \min(p, q, p+q-1)$ with $p+q>1$, there exists $C>0$ such that*

$$\int \frac{dx}{\langle x-a\rangle^p \langle x-b\rangle^q} \leqslant \frac{C}{\langle a-b\rangle^r}. \tag{2.11}$$

(2) *For $p>1$ and $q>1/2$*

$$\int \frac{dx}{\langle ax-b\rangle^p} \leqslant \frac{C}{|a|}, \tag{2.12}$$

$$\int \frac{dx}{\langle a_0 + a_1 x + a_2 x^2\rangle^q} \leqslant C. \tag{2.13}$$

3. NONLINEAR ESTIMATES

In this section we give four estimates for the nonlinear terms appearing in systems (1.2) and (1.3). First we treat the cubic nonlinear term.

LEMMA 3.1. *For* $s \geqslant 0$, $a \leqslant 0$ *and* $b \in (1/2, 1)$ *we have that*

$$\| |u|^2 u \|_{X^{s,a}} \leqslant C \|u\|^3_{X^{s,b}}. \tag{3.1}$$

Proof of Lemma 3.1. It is sufficient to show (3.1) for $u \in \mathcal{S}(\mathbf{R}^2)$. Since for any $\xi, \eta, \zeta \in \mathbf{R}$ and $s \geqslant 0$, $\langle \xi \rangle^s \leqslant \langle \xi - \eta \rangle^s \langle \eta - \zeta \rangle^s \langle \zeta \rangle^s$, we see that for $a \leqslant 0$ and $1/2 < b < 1$,

$$\begin{aligned} \| |u|^2 u \|_{X^{s,a}} &= \| \langle \tau + \xi^2 \rangle^a \langle \xi \rangle^s \, (\widehat{|u|^2 u}) \|_{L^2_\tau(L^2_\xi)} \\ &\leqslant \sup_{\xi, \tau} \langle \tau + \xi^2 \rangle^a \, \| |w|^2 w \|_{L^2_t(L^2_x)} \\ &\leqslant \|w\|^3_{L^6_t(L^6_x)}, \end{aligned}$$

where $\hat{w}(\tau, \xi) = \langle \xi \rangle^s \, \hat{u}(\tau, \xi)$.

Let $f(\tau, \xi) = \langle \tau + \xi^2 \rangle^b \, \hat{w}(\tau, \xi)$, then

$$\|w\|_{L^6_t(L^6_x)} = \left\| \iint e^{it\tau + ix\xi} \frac{f(\tau, \xi)}{\langle \tau + \xi^2 \rangle^b} \, d\xi \, d\tau \right\|_{L^6_t(L^6_x)}.$$

Using the change of variables $\omega = \tau + \xi^2$ and rewriting the expression, it follows that

$$\begin{aligned} &\left\| \int \left\{ e^{it\omega} \langle \omega \rangle^{-b} \left(\int e^{ix\xi - it\xi^2} f(\omega - \xi^2, \xi) \, d\xi \right) \right\} d\omega \right\|_{L^6_t(L^6_x)} \\ &\quad = \left\| \int e^{it\omega} \langle \omega \rangle^{-b} \{ U(t) \, g_\omega \} \, d\omega \right\|_{L^6_t(L^6_x)}, \end{aligned}$$

where we put $\mathcal{F}_x g_\omega(\xi) = f(\omega - \xi^2, \xi)$.

Next we use Minkowski's inequality and the Strichartz estimates on the Schrödinger equation found in Proposition 2.5 to get the following inequalities

$$\begin{aligned} \int \| e^{it\omega} \langle \omega \rangle^{-b} \, U(t) \, g_\omega \|_{L^6_t(L^6_x)} \, d\omega &\leqslant \int \langle \omega \rangle^{-b} \, \| U(t) \, g_\omega \|_{L^6_t(L^6_x)} \, d\omega \\ &\leqslant C \int \langle \omega \rangle^{-b} \, \| g_\omega \|_{L^2_x} \, d\omega \\ &\leqslant C \, \| \langle \omega \rangle^{-b} \|_{L^2_\omega} \, \| g_\omega(x) \|_{L^2_\omega(L^2_x)} \\ &\leqslant C \, \| g_\omega(x) \|_{L^2_\omega(L^2_x)}. \end{aligned} \tag{3.2}$$

Finally by Plancherel's identity, we have from (3.2),

$$\|g_\omega(x)\|_{L^2_\omega(L^2_x)}=\left(\iint |f(\omega-\xi^2,\xi)|^2\,d\xi\,d\omega\right)^{1/2}=\|f\|_{L^2_\tau(L^2_\xi)}$$

and this completes the proof. ∎

The following lemmas are proved by a method originally due to Bourgain and considerably improved by Kenig *et al.*, and contain the remaining nonlinear estimates needed for the construction of local solutions.

The estimate in Lemma 3.2 shows that the nonlinear interaction $\partial_x|u|^2$ has the best smoothing property of the four nonlinear terms by our method and it plays a key role in our results.

LEMMA 3.2. *Let $s\geqslant 0$ and $b>1/2$. Suppose that $u\in X^{s,b}$. Then there is a constant, $C>0$ only depending on b such that*

$$\|\partial_x|u|^2\|_{L^2_t(H^{s-1/2}_x)}\leqslant C\,\|u\|^2_{X^{s,b}}. \tag{3.3}$$

The following corollary is an immediate consequence of Lemma 3.2.

COROLLARY 3.3. *For $u\in X^{s,b}$ with $1/2<b$, $s\geqslant 0$ and for any $a\leqslant 0$, there is a constant $C>0$ only depending on b such that*

$$\|\partial_x|u|^2\|_{Y^{s-(1/2),a}}\leqslant C\,\|u\|^2_{X^{s,b}}, \tag{3.4}$$

$$\|\partial_x|u|^2\|_{Z^{s-(1/2),a}}\leqslant C\,\|u\|^2_{X^{s,b}}. \tag{3.5}$$

Proof of Corollary 3.3. Since

$$\begin{aligned}\|\partial_x|u|^2\|_{Y^{s-(1/2),a}}&=\|\langle\tau+v\xi\,|\xi|\rangle^a\langle\xi\rangle^{s-1/2}\,\widehat{\partial_x|u|^2}\|_{L^2_\tau(L^2_\xi)}\\&\leqslant\|\partial_x|u|^2\|_{L^2_t(H^{s-1/2}_x)},\end{aligned}$$

(3.4) follows immediately from (3.3). Similarly (3.5) follows from (3.3). ∎

Proof of Lemma 3.2. By letting $f(\tau,\xi)=\langle\tau+\xi^2\rangle^b\langle\xi\rangle^s\hat{u}$ and $f^*(\tau,\xi)=\langle\tau-\xi^2\rangle^b\langle\xi\rangle^s\hat{\bar{u}}$, we have

$$\begin{aligned}&\|\partial_x|u|^2\|_{L^2_t(H^{s-1/2}_x)}\\&=\|i\xi\langle\xi\rangle^{s-1/2}\,\widehat{|u|^2}\|_{L^2_\tau(L^2_\xi)}\\&=\left\|i\xi\langle\xi\rangle^{s-1/2}\left(\frac{f}{\langle\xi\rangle^s\langle\tau+\xi^2\rangle^b}*\frac{f^*}{\langle\xi\rangle^s\langle\tau-\xi^2\rangle^b}\right)\right\|_{L^2_\tau(L^2_\xi)}\end{aligned}$$

$$= \left\| i\xi \langle \xi \rangle^{s-1/2} \left(\iint \frac{f(\sigma, \eta)}{\langle \eta \rangle^{s} \langle \sigma + \eta^2 \rangle^{b}} \right. \right.$$

$$\left. \left. \times \frac{f^*(\tau - \sigma, \xi - \eta)}{\langle \xi - \eta \rangle^{s} \langle \tau - \sigma - (\xi - \eta)^2 \rangle^{b}} \, d\sigma \, d\eta \right) \right\|_{L^2_\tau(L^2_\xi)}$$

$$\leqslant \left\| |\xi|^{s+1/2} \left(\iint \frac{1}{\langle \eta \rangle^{2s} \langle \sigma + \eta^2 \rangle^{2b} \langle \xi - \eta \rangle^{2s} \langle \tau - \sigma - (\xi - \eta)^2 \rangle^{2b}} \, d\sigma \, d\eta \right)^{1/2} \right.$$

$$\left. \times (|f|^2 * |f^*|^2)^{1/2} \right\|_{L^2_\tau(L^2_\xi)}$$

$$\leqslant \| |f|^2 * |f^*|^2 \|^{1/2}_{L^1_\tau(L^1_\xi)}$$

$$\times \left\| |\xi|^{1/2} \left(\iint \frac{1}{\langle \sigma + \eta^2 \rangle^{2b} \langle \tau - \sigma - (\xi - \eta)^2 \rangle^{2b}} \, d\sigma \, d\eta \right)^{1/2} \right\|_{L^\infty_\tau L^\infty_\xi}$$

$$\leqslant \|f\|_{L^2_\tau(L^2_\xi)} \, \|f^*\|_{L^2_\tau(L^2_\xi)}$$

$$\times \left\| |\xi|^{1/2} \left(\iint \frac{1}{\langle \sigma + \eta^2 \rangle^{2b} \langle \tau - \sigma - (\xi - \eta)^2 \rangle^{2b}} \, d\sigma \, d\eta \right)^{1/2} \right\|_{L^\infty_\tau L^\infty_\xi},$$

where we have used $|\xi| \leqslant \langle \xi - \eta \rangle \langle \eta \rangle$.

By (2.11) and (2.12) in Lemma 2.5 we have

$$|\xi|^{1/2} \left(\iint \frac{1}{\langle \sigma + \eta^2 \rangle^{2b} \langle \tau - \sigma - (\xi - \eta)^2 \rangle^{2b}} \, d\sigma \, d\eta \right)^{1/2}$$

$$\leqslant C \, |\xi|^{1/2} \left(\int \frac{1}{\langle \tau - \xi^2 + 2\eta\xi \rangle^{2b}} \, d\eta \right)^{1/2} \leqslant C,$$

which shows (3.4). ∎

Lemma 3.4. *Let* $s \geqslant 0$, $a \leqslant -1/4$, $1/2 < b < 3/4$, *and* $|v| < 1$. *Suppose that* $u \in X^{s,b}$ *and* $v \in Y_v^{s-1/2,b}$, *then there is a* $C > 0$ *only depending on* b, a *and* $(1 - |v|)^{-1}$ *such that*

$$\|uv\|_{X^{s,a}} \leqslant C \, \|u\|_{X^{s,b}} \, \|v\|_{Y_v^{s-1/2,b}}. \tag{3.6}$$

If $v \in Z_c^{s-1/2,b}$, *then there is a* $C > 0$ *only depending on* b, *and* a *such that*

$$\|uv\|_{X^{s,a}} \leqslant C \, \|u\|_{X^{s,b}} \, \|v\|_{Z_c^{s-1/2,b}}. \tag{3.7}$$

Proof of Lemma 3.4. First we note that by duality we have that

$$\|uv\|_{X^{s,a}} = \|\langle\xi\rangle^s \langle\tau+\xi^2\rangle^a \widehat{uv}\|_{L^2_\tau(L^2_\xi)}$$

$$= \sup_{\|\phi\|_{L^2_\tau(L^2_\xi)}\leqslant 1} \left| \left\langle\!\!\left\langle \frac{\langle\xi\rangle^s}{\langle\tau+\xi^2\rangle^{|a|}} \hat{u} * \hat{\bar{v}}, \phi \right\rangle\!\!\right\rangle \right|.$$

Now setting $f(\tau,\xi) = \langle\tau+\xi^2\rangle^b \langle\xi\rangle^s \hat{u}$ and $g(\tau,\xi) = \langle\tau+v\xi\,|\xi|\rangle^b \langle\xi\rangle^{s-1/2} \hat{\bar{v}}$,

$$\left\langle\!\!\left\langle \frac{\langle\xi\rangle^s}{\langle\tau+\xi^2\rangle^{|a|}} \hat{u} * \hat{\bar{v}}, \bar{\phi} \right\rangle\!\!\right\rangle$$

$$= \left\langle\!\!\left\langle \frac{\langle\xi\rangle^s}{\langle\tau+\xi^2\rangle^{|a|}} \left(\frac{f}{\langle\xi\rangle^s \langle\tau+\xi^2\rangle^b} * \frac{g}{\langle\xi\rangle^{s-1/2}\langle\tau-\xi^3\rangle^b} \right), \bar{\phi} \right\rangle\!\!\right\rangle$$

$$= \iint \frac{\langle\xi\rangle^s}{\langle\tau+\xi^2\rangle^{|a|}}$$

$$\times \left(\iint \frac{g(\sigma,\eta)\, f(\tau-\sigma,\xi-\eta)}{\langle\eta\rangle^{s-1/2} \langle\sigma+v\eta\,|\eta|\rangle^b \langle\xi-\eta\rangle^s \langle\tau-\sigma+(\xi-\eta)^2\rangle^b}\, d\sigma\, d\eta \right)$$

$$\times \bar{\phi}(\tau,\xi)\, d\tau\, d\xi$$

$$= \iiiint_{R_1} + \iiiint_{R_2}$$

$$+ \iiiint_{R_3} \frac{\langle\xi\rangle^s g(\sigma,\eta)\, f(\tau-\sigma,\xi-\eta)\, \bar{\phi}(\tau,\xi)}{\left(\begin{array}{c} \langle\eta\rangle^{s-1/2} \langle\xi-\eta\rangle^s \langle\tau+\xi^2\rangle^{|a|} \langle\sigma+v\eta\,|\eta|\rangle^b \\ \times \langle\tau-\sigma+(\xi-\eta)^2\rangle^b \end{array} \right)}\, d\sigma\, d\eta\, d\tau\, d\xi$$

$$= I_1 + I_2 + I_3, \tag{3.8}$$

where the regions R_1, R_2 and R_3 make up $\mathbf{R}^4$ and are defined as follows.

First we split $\mathbf{R}^4$ into three regions, A, B, and C:

$$A = \{(\tau,\sigma,\xi,\eta) \in \mathbf{R}^4 : |\eta| \leqslant 2\}, \tag{3.9}$$

$$B = \left\{ (\tau,\sigma,\xi,\eta) \in \mathbf{R}^4 : |2(1+v\,\mathrm{sgn}(\eta))\,\eta - 2\xi| \geqslant \frac{1-|v|}{2}|\eta|,\ |\eta| > 2 \right\}, \tag{3.10}$$

$$C = \left\{ (\tau,\sigma,\xi,\eta) \in \mathbf{R}^4 : |(1+v\,\mathrm{sgn}(\eta))\,\eta - 2\xi| \geqslant \frac{1-|v|}{2}|\eta|,\ |\eta| > 2 \right\}. \tag{3.11}$$

For $|v|<1$ we have that the set

$$\Big\{(\tau,\sigma,\xi,\eta)\in\mathbf{R}^4\colon |\eta|>2,\ |2(1+v\,\mathrm{sgn}(\eta))\,\eta-2\xi|<\frac{1-|v|}{2}\,|\eta|$$
$$\text{and } |(1+v\,\mathrm{sgn}(\eta))\,\eta-2\xi|<\frac{1-|v|}{2}|\eta|\Big\}$$

is empty, so that we have $\mathbf{R}^4=A\cup B\cup C$.

Noting that for points in C,

$$|\tau+\xi^2|+|\sigma+v\eta\,|\eta||+|\tau-\sigma+(\xi-\eta)^2|\geqslant|v\eta\,|\eta|+\eta^2-2\xi\eta|>\frac{1-|v|}{2}\,|\eta|^2, \tag{3.12}$$

we separate C into three parts,

$$C_1=\{(\tau,\sigma,\xi,\eta)\in C\colon |\sigma+v\eta\,|\eta||,\ |\tau-\sigma+(\xi-\eta)^2|\leqslant|\tau+\xi^2|\},$$
$$C_2=\{(\tau,\sigma,\xi,\eta)\in C\colon |\tau+\xi^2|,\ |\tau-\sigma+(\xi-\eta)^2|\leqslant|\sigma+v\eta\,|\eta||\},$$
$$C_3=\{(\tau,\sigma,\xi,\eta)\in C\colon |\tau+\xi^2|,\ |\sigma+v\eta\,|\eta||\leqslant|\tau-\sigma+(\xi-\eta)^2|\},$$

so that one of the following $|\tau+\xi^2|$, $|\sigma+v\eta\,|\eta||$, or $|\tau-\sigma+(\xi-\eta)^2|$ is larger than $(1-|v|)/6\,|\eta|^2$.

We can now define the three sets that we separate $\mathbf{R}^4$ into:

$$R_1=A\cup B\cup C_1,$$
$$R_2=C_2,$$
$$R_3=C_3.$$

Now to estimate I_1 we integrate with respect to τ and ξ first, then we use Hölder's and the Cauchy–Schwartz inequalities to obtain the following:

$$\begin{aligned}|I_1|&\leqslant\|\bar\phi(\tau,\xi)\|_{L^2_\tau(L^2_\xi)}\Bigg\|\frac{\langle\xi\rangle^s}{\langle\tau+\xi^2\rangle^{|a|}}\\&\quad\times\iint\frac{f(\tau-\sigma,\xi-\eta)\,g(\sigma,\eta)\,\chi_{R_1}(\tau,\sigma,\xi,\eta)\,d\sigma\,d\eta}{\langle\eta\rangle^{s-1/2}\langle\xi-\eta\rangle^s\langle\sigma+v\eta\,|\eta|\rangle^b\langle\tau-\sigma+(\xi-\eta)^2\rangle^b}\Bigg\|_{L^2_\tau(L^2_\xi)}\\&\leqslant\|f\|_{L^2_\tau(L^2_\xi)}\|g\|_{L^2_\tau(L^2_\xi)}\|\phi\|_{L^2_\tau(L^2_\xi)}\Bigg\|\frac{\langle\xi\rangle^s}{\langle\tau+\xi^2\rangle^{|a|}}\\&\quad\times\left(\iint\frac{\langle\eta\rangle\,\chi_{R_1}}{\left(\begin{aligned}&\langle\eta\rangle^{2s}\langle\xi-\eta\rangle^{2s}\langle\sigma+v\eta\,|\eta|\rangle^{2b}\\&\times\langle\tau-\sigma+(\xi-\eta)^2\rangle^{2b}\end{aligned}\right)}\,d\sigma\,d\eta\right)^{1/2}\Bigg\|_{L^\infty_\tau(L^\infty_\xi)}.\end{aligned} \tag{3.13}$$

For the second term, we integrate over σ and η first and follow the same steps as above to obtain

$$|I_2| \leqslant \|f\|_{L^2_\sigma(L^2_\eta)} \|g\|_{L^2_\sigma(L^2_\eta)} \|\phi\|_{L^2_\sigma(L^2_\eta)} \left\| \frac{\langle \eta \rangle^{1/2}}{\langle \eta \rangle^s \langle \sigma + v\eta \, |\eta| \rangle^b} \right. \\ \left. \times \left(\iint \frac{\langle \xi \rangle^{2s} \chi_{R_2}}{\langle \xi - \eta \rangle^{2s} \langle \tau + \xi^2 \rangle^{2\,|a|} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \, d\tau \, d\xi \right)^{1/2} \right\|_{L^\infty_\sigma(L^\infty_\eta)}. \tag{3.14}$$

Using the change of variables $\sigma - \tau = \rho$ and $\eta - \xi = \zeta$ the third region, R_3, is transformed into the set $\tilde{R}_3$ defined as

$$\tilde{R}_3 = \left\{ (\sigma, \rho, \eta, \zeta) \in \mathbf{R}^4 : 2 < |\eta|, \frac{1 - |v|}{2} \, |\eta|^2 \leqslant \left| v\eta \, |\eta| - \eta^2 + 2\eta\zeta \right| \leqslant 3 \, |\rho - \zeta^2| \right\}$$

and it follows that the last term can be estimated as before as

$$|I_3| \leqslant \left| \iint \frac{f_-(\rho, \zeta)}{\langle \zeta \rangle^s \langle \rho - \zeta^2 \rangle^b} \right. \\ \left. \times \left(\iint \frac{\langle \eta - \zeta \rangle^s g(\sigma, \eta) \, \bar{\phi}(\sigma - \rho, \eta - \xi) \, \chi_{\tilde{R}_3}(\rho, \sigma, \eta, \zeta)}{\langle \eta \rangle^{s - 1/2} \langle \sigma + v\eta \, |\eta| \rangle^b \langle \sigma - \rho + (\eta - \xi)^2 \rangle^{|a|}} \, d\sigma \, d\eta \right) d\rho \, d\zeta \right| \\ \leqslant \|f\|_{L^2_\rho(L^2_\xi)} \|g\|_{L^2_\rho(L^2_\xi)} \|\phi\|_{L^2_\rho(L^2_\zeta)} \left\| \frac{1}{\langle \zeta \rangle^s \langle \rho - \zeta^2 \rangle^b} \right. \\ \left. \times \left(\iint \frac{\langle \eta - \zeta \rangle^{2s} \chi_{R_3}}{\langle \eta \rangle^{2s - 1} \langle \sigma + v\eta \, |\eta| \rangle^{2b} \langle \sigma - \rho + (\eta - \zeta)^2 \rangle^{2\,|a|}} \, d\sigma \, d\eta \right)^{1/2} \right\|_{L^\infty_\rho(L^\infty_\xi)}, \tag{3.15}$$

where $f_-(\rho, \zeta) = f(-\rho, -\zeta)$ and $\phi_-(\rho, \zeta) = \phi(-\rho, -\zeta)$. We note that $\overline{f_-} = \langle \zeta \rangle^s \langle \rho - \zeta^2 \rangle^b \, \hat{\hat{u}}$ and $\|\overline{f_-}\|_{L^2_\rho(L^2_\zeta)} = \|f\|_{L^2_\rho(L^2_\xi)} = \|u\|_{X^s_b}$ and $\|\phi\|_{L^2_\rho(L^2_\zeta)} \leqslant 1$.

Thus, reviewing the estimates (3.8), (3.13)–(3.15) and noting that $\langle \xi \rangle \leqslant \langle \eta \rangle \langle \xi - \eta \rangle$ (or equivalently $\langle \eta - \zeta \rangle \leqslant \langle \zeta \rangle \langle \eta \rangle$), we will establish estimate (3.6) once the following lemma is shown.

LEMMA 3.5. *All the following expressions are bounded by a constant* $C = C((1 - |v|)^{-1})$:

$$\left\| \frac{1}{\langle \tau+\xi^2\rangle^{|a|}} \left(\iint \frac{\langle \eta\rangle \, \chi_{R_1}}{\langle \sigma+v\eta\,|\eta|\rangle^{2b}\,\langle \tau-\sigma+(\xi-\eta)^2\rangle^{2b}}\, d\sigma\, d\eta \right)^{1/2} \right\|_{L^\infty_\tau(L^\infty_\xi)},$$

$$\left\| \frac{\langle \eta\rangle^{1/2}}{\langle \sigma+v\eta\,|\eta|\rangle^{b}} \left(\iint \frac{\chi_{R_2}}{\langle \tau+\xi^2\rangle^{2\,|a|}\,\langle \tau-\sigma+(\xi-\eta)^2\rangle^{2b}}\, d\tau\, d\xi \right)^{1/2} \right\|_{L^\infty_\sigma(L^\infty_\eta)},$$

$$\left\| \frac{1}{\langle \rho-\zeta^2\rangle^{b}} \left(\iint \frac{\langle \eta\rangle \, \chi_{\tilde{R}_3}}{\langle \sigma+v\eta\,|\eta|\rangle^{2b}\,\langle \sigma-\rho+(\eta-\zeta)^2\rangle^{2\,|a|}}\, d\sigma\, d\eta \right)^{1/2} \right\|_{L^\infty_\rho(L^\infty_\xi)}.$$

Proof of Lemma 3.5. According to Lemma 2.5 (2.11), it suffices to get bounds for

$$\frac{1}{\langle \tau+\xi^2\rangle^{|a|}} \left(\int \frac{\langle \eta\rangle\, d\eta}{\langle \tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta\rangle^{2b}} \right)^{1/2} \qquad \text{on } R_1, \tag{3.16}$$

$$\frac{\langle \eta\rangle^{1/2}}{\langle \sigma+v\eta\,|\eta|\rangle^{b}} \left(\int \frac{d\xi}{\langle \sigma-\eta^2+2\eta\xi\rangle^{2\kappa}} \right)^{1/2} \qquad \text{on } R_2, \tag{3.17}$$

$$\frac{1}{\langle \rho-\zeta^2\rangle^{b}} \left(\int \frac{\langle \eta\rangle\, d\eta}{\langle \rho-\zeta^2+v\eta\,|\eta|-\eta^2+2\zeta\eta\rangle^{2\kappa}} \right)^{1/2} \qquad \text{on } \tilde{R}_3, \tag{3.18}$$

where $\kappa=\min(b, |a|)$.

We start with (3.16) in region $R_1=A\cup B\cup C_1$. In region A, we have $|\eta|<2$ and it is easy to see that

$$\frac{1}{\langle \tau+\xi^2\rangle^{|a|}} \left(\int \frac{d\eta}{\langle \tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta\rangle^{2b}} \right)^{1/2} \leqslant C. \tag{3.19}$$

Next we estimate (3.16) in region B. By the change of variables $\eta'=\tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta$ and the condition $(1-|v|)/2\,|\eta|\leqslant|2v\,\mathrm{sgn}(\eta)\,\eta+2\eta-2\xi|$ on B yields

$$\begin{aligned}
&\frac{1}{\langle \tau+\xi^2\rangle^{|a|}} \left(\int \frac{\langle \eta\rangle\, d\eta}{\langle \tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta\rangle^{2b}} \right)^{1/2}\\
&\qquad\leqslant C\left(\int \frac{\langle \eta\rangle\, d\eta'}{\langle \eta'\rangle^{2b}\,|2v\,\mathrm{sgn}(\eta)\,\eta+2\eta-2\xi|} \right)^{1/2}\\
&\qquad\leqslant C(1-|v|)^{-1/2}\left(\int \langle \eta'\rangle^{-2b}\, d\eta' \right)^{1/2} \leqslant C(1-|v|)^{-1}.
\end{aligned}\tag{3.20}$$

Note that $2b>1$ is need to obtain the last line in (3.20) above and that $|v|<1$. Similarly on region C_1, we have that for $|v|<1$, $|\eta|\leqslant C(1-|v|)^{-1}\langle \tau+\xi^2\rangle^{2\,|a|}$ for $a\leqslant -1/4$, and it follows from Lemma 2.5 (2.13) that

$$\frac{1}{\langle\tau+\xi^2\rangle^{|a|}}\left(\int\frac{\langle\eta\rangle\,d\eta}{\langle\tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta\rangle^{2b}}\right)^{1/2}$$
$$\leqslant\left(\int\frac{|\eta|\,d\eta}{\langle\tau+\xi^2\rangle^{2\,|a|}\,\langle\tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta\rangle^{2b}}\right)^{1/2}$$
$$\leqslant C(1-|v|)^{-1}\left(\int\frac{d\eta}{\langle\tau+\xi^2+v\eta\,|\eta|+\eta^2-2\xi\eta\rangle^{2b}}\right)^{1/2}\leqslant C(1-|v|)^{-1}. \tag{3.21}$$

Thus gathering (3.19)–(3.21), we have shown estimate (3.16).

Next we show (3.17). Since $\frac{1}{2}-\kappa-b<0$, the change of variables $\eta'=\sigma-\eta^2+2\xi\eta$, Lemma 2.5 (2.12), and the restriction on the region (3.12) yields the following

$$\frac{|\eta|^{1/2}}{\langle\sigma+v\eta\,|\eta|\rangle^{b}}\left(\int\frac{d\xi}{\langle\sigma-\eta^2+2\xi\eta\rangle^{2\kappa}}\right)^{1/2}$$
$$\leqslant\frac{C}{\langle\sigma+v\eta\,|\eta|\rangle^{b}}\left(\int_{|\eta'|\leqslant 2\,|\sigma+v\eta\,|\eta||}\langle\eta'\rangle^{-2\kappa}\,d\eta'\right)^{1/2}$$
$$\leqslant C\frac{\langle\sigma+v\eta\,|\eta|\rangle^{1/2-\kappa}}{\langle\sigma+v\eta\,|\eta|\rangle^{b}}\leqslant C.$$

Last, in the region $\tilde{R}_3$ we note that $|\eta|/\langle\rho-\zeta^2\rangle^{2b}<C(1-|v|)^{-1}$ and from Lemma 2.5 (2.13), we have that

$$\frac{1}{\langle\rho-\zeta^2\rangle^{b}}\left(\int\frac{\langle\eta\rangle\,d\eta}{\langle\rho-\zeta^2+v\eta\,|\eta|-\eta^2+2\zeta\eta\rangle^{2\kappa}}\right)^{1/2}$$
$$\leqslant C(1-|v|)^{-1}\left(\int\frac{d\eta}{\langle\rho-\zeta^2+v\eta\,|\eta|-\eta^2+2\zeta\eta\rangle^{2\kappa}}\right)^{1/2}\leqslant C(1-|v|)^{-1}.$$

Here we have used $\kappa>\frac{1}{4}$ and $b\geqslant\frac{1}{4}$. Now (3.16)–(3.18) are shown to be bounded and proof of Lemma 3.5 and hence, proof of (3.6) is completed.

For the proof of (3.7), we choose

$$A=\{(\tau,\sigma,\xi,\eta)\in\mathbf{R}^4\colon |\eta|\leqslant 2\},$$
$$B=\{(\tau,\sigma,\xi,\eta)\in\mathbf{R}^4\colon |2\eta-2\xi+c|>\tfrac{1}{2}|\eta|,\ |\eta|>2\},$$
$$C=\{(\tau,\sigma,\xi,\eta)\in\mathbf{R}^4\colon |\eta-2\xi+c|>\tfrac{1}{2}|\eta|,\ |\eta|>2\}$$

and define R_1, R_2, and R_3 in a similar manner as before. The proof of estimate (3.7) is then similar to the proof of estimate (3.6) and is therefore omitted. Note that this case, we have no restriction on the parameter c. This completes the proof of Lemma 3.4. ∎

LEMMA 3.6. *Let $u, \tilde{u} \in X^{s,b}$ and $v, \tilde{v} \in Y_\nu^{s-1/2,b}$ with $s \geq 0$ and $1/2 < b < 3/4$. Then there are constants $C_1 > 0$ and $C_\nu > 0$ such that for any $a \leqslant -1/4$,*

$$\|\partial_x |u|^2 - \partial_x |\tilde{u}|^2\|_{Y_\nu^{s-1/2,a}} \leqslant C_1(\|u\|_{X^{s,b}} + \|\tilde{u}\|_{X^{s,b}})\|u-\tilde{u}\|_{X^{s,b}}, \tag{3.22}$$

$$\|uv - \tilde{u}\tilde{v}\|_{X^{s,a}} \leqslant C_\nu(\|u-\tilde{u}\|_{X^{s,b}}\|v\|_{Y_\nu^{s-1/2,b}} + \|\tilde{u}\|_{X^{s,b}}\|v-\tilde{v}\|_{Y_\nu^{s-1/2,b}}), \tag{3.23}$$

where the constant C_ν depends on b and $(1-|\nu|)^{-1}$.

Proof of Lemma 3.6. Lemma 3.6 is an immediate consequence of Lemma 3.1, Corollary 3.3, and Lemma 3.4. ∎

Remark 8. As we mentioned in the introduction, the crucial estimate (3.6) in the proof of Lemma 3.4 is sharp, that is it fails for $|\nu| = 1$ and $s < 0$ (the case $s = 0$ remains open). To see this we follow [28] Theorem 1.4(ii) to construct a counterexample:

We take

$$f_N(\xi, \tau) = \psi(\xi - N)\,\psi(\tau + \xi^2)$$

and

$$g_N(\xi, \tau) = \psi(\xi + N)\,\psi(\tau - \xi^2).$$

Thus

$$f_N * g_N(\xi, \tau) \sim c\chi_R(\xi, \tau),$$

where R is a rectangle of dimension $N \times N^{-1}$ centered at the origin with its longest side pointing in the $(1, -2N)$ direction. Inserting this information in (3.24) we get (taking $b = 1/2$, see [28])

$$\frac{1}{N^{1/2}} \times \frac{N^{1/2-s}}{N^s} \leqslant C,$$

which shows the need for $s \geqslant 0$.

The next lemma is a simple application of the nonlinear estimates above and we state it without proof.

LEMMA 3.7. *Let $u, \tilde{u} \in X^{s,b}$ and $v, \tilde{v} \in Z^{s-1/2,b}$ with $s \geqslant 0$ and $1/2 < b < 3/4$. Then there is a constant $C > 0$ such that for any $a \leqslant -1/4$,*

$$\| |u|^2 u - |\tilde{u}|^2 \tilde{u}\|_{X^{s,a}} \leqslant C(\|u\|^2_{X^{s,b}} + \|\tilde{u}\|^2_{X^{s,b}})\,\|u-\tilde{u}\|_{X^{s,b}},$$

$$\|uv - \tilde{u}\tilde{v}\|_{X^{s,a}} \leqslant C\,\|u-\tilde{u}\|_{X^{s,b}}\,\|v\|_{Z^{s-(1/2),b}} + \|\tilde{u}\|_{X^{s,b}}\,\|v-\tilde{v}\|_{Z^{s-(1/2),b}}, \tag{3.32}$$

$$\|\partial_x |u|^2 - \partial_x |\tilde{u}|^2\|_{Z^{s-(1/2),a}} \leqslant C(\|u\|_{X^{s,b}} + \|\tilde{u}\|_{X^{s,b}})\,\|u-\tilde{u}\|_{X^{s,b}}.$$

4. LOCAL WELL POSEDNESS—PROOF OF THEOREM 1.1 AND 1.2

In this section we show the proof of well posedness of (1.2) and (1.3). We basically follow the arguments given in [27, 2]. Note that in our case, we cannot employ the scaling argument in [27] because of the presence of the pure power term in the Schrödinger part. Recall that ψ is the smooth cut-off function as defined in the Section 1, where we denote $\psi_\delta = \psi(t/\delta)$.

Proof of Theorem 1.1. We consider the following function space where we seek our solution. For $(u_0, v_0) \in H^s(\mathbf{R}) \times H^{s-(1/2)}(\mathbf{R})$ and $b \in (1/2, 3/4)$, let

$$\mathscr{X}_{MN} = \{(u, v) \colon u \in X^{s,b},\, v \in Z_c^{s-(1/2),b},$$

$$\text{such that } \|u\|_{X^{s,b}} \leqslant M,\ \|v\|_{Z_c^{s-1/2,b}} \leqslant N\},$$

where $M = 2C_0 \|u_0\|_{H^s}$ and $N = 2C_0 \|v_0\|_{H^{s-1/2}}$. Then $\mathscr{X}_{MN}$ is a complete metric space with norm

$$|\!|\!|(u, v)|\!|\!|_{\mathscr{X}_{MN}} = \|u\|_{X^b} + \|v\|_{Z^b}.$$

Hereafter we use the abbreviation $X^b = X^{s,b}$ and $Z^b = Z_c^{s-(1/2),b}$. Without loss of generality, we may assume that $1 < M$ and $1 < N$. For $(u, v) \in \mathscr{X}_{MN}$, we define the maps,

$$\Xi[u, v] = \psi(t)\, U(t)\, u_0 - i\psi(t) \int_0^t U(t-t')\, \psi_\delta(t') \{\alpha uv(t') + \gamma\, |u|^2\, u(t')\}\, dt',$$

$$\Psi[u, v] = \psi(t)\, W(t)\, v_0 + \psi(t) \int_0^t W(t-t')\, \psi_\delta(t')\, \beta\, \partial_x\, |u|^2\, (t')\, dt'.$$

Then according to Lemma 2.1, Proposition 2.3, Lemma 3.1, Corollary 3.3 and (3.7) in Lemma 3.4, we have for $b < b' < 3/4$ and $m = (b' - b)/4b'$,

$$\begin{aligned}
\|\Xi[u, v]\|_{X^b} &\leqslant C_0\, \|u_0\|_{H^s} + C\, \|\psi_\delta\{\alpha uv + \gamma\, |u|^2 u\}\|_{X^{b-1}} \\
&\leqslant C_0\, \|u_0\|_{H^s} + C\, \delta^\mu(\|uv\|_{X^{b'-1}} + \||u|^2\, u\|_{X^{b'-1}}) \\
&\leqslant C_0\, \|u_0\|_{H^s} + C\, \delta^\mu(\|u\|_{X^b}\, \|v\|_{Z^b}\, \|u\|^3_{X^b}), \qquad (4.1) \\
\|\Psi[u, v]\|_{Z^b} &\leqslant C_0\, \|v_0\|_{H^{s-1/2}} + C\, \|\psi_\delta \beta\, \partial_x\, |u|^2\|_{X_{b-1}} \\
&\leqslant C_0\, \|v_0\|_{H^{s-1/2}} + C\, \delta^\mu(\|\partial_x\, |u|^2\|_{Z^{b'-1}}) \\
&\leqslant C_0\, \|v_0\|_{H^{s-1/2}} + C\, \delta^\mu(\|u\|^2_{X^b}). \qquad (4.2)
\end{aligned}$$

It follows from (4.1) and (4.2) that

$$\|\Xi[u, v]\|_{X^b} \leqslant \frac{M}{2} + C_1\,\delta^\mu(M^3 + MN),$$

$$\|\Psi[u, v]\|_{Z^b} \leqslant \frac{N}{2} + C_2\,\delta^\mu M^2.$$

If we set

$$\delta^\mu \leqslant \frac{1}{2\max(C_1, C_2)(M^2+N)}$$

then we have that $\|\Xi[u, v]\|_{X^b} \leqslant M$ and $\|\Psi[u, v]\|_{Y^b} \leqslant N$ hence $(\Xi[u, v], \Psi[u, v]) \in \mathscr{X}_{\mathscr{M}\mathscr{N}}$.

Similarly by (3.24) in Lemma 3.7, we have that

$$\begin{aligned}\|\Xi[u, v] - \Xi[\tilde{u}, \tilde{v}]\|_{X^b} &\leqslant C\,\delta^\mu(\|v\|_{Z^b}\,\|u-\tilde{u}\|_{X^b} + \|\tilde{u}\|_{X^b}\,\|v-\tilde{v}\|_{Z^b}\\ &\quad + (\|u\|^2_{X^b} + \|\tilde{u}\|^2_{X^b})\,\|u-\tilde{u}\|_{X^b})\\ &\leqslant C\,\delta^\mu(N\,\|u-\tilde{u}\|_{X^b} + M\,\|v-\tilde{v}\|_{Z^b}) + M^2\,\|u-\tilde{u}\|_{X^b})\\ &\leqslant C_3\,\delta^\mu((M^2+N)\,\|u-\tilde{u}\|_{X^b} + M\,\|v-\tilde{v}\|_{Z^b})\\ &\leqslant \tfrac{1}{4}(\|u-\tilde{u}\|_{X^b} + \|v-\tilde{v}\|_{Z^b})\end{aligned}$$

and

$$\begin{aligned}\|\Psi[u, v] - \Psi[\tilde{u}, \tilde{v}]\|_{Z^b} &\leqslant C\,\delta^\mu(\|u\|_{X^b} + \|\tilde{u}\|_{X^b})\,\|u-\tilde{u}\|_{X^b}\\ &\leqslant C_4\,\delta^\mu M\,\|u-\tilde{u}\|_{X^b}\\ &\leqslant \tfrac{1}{4}(\|u-\tilde{u}\|_{X^b} + \|v-\tilde{v}\|_{Z^b})\end{aligned} \tag{4.3}$$

if $\delta^\mu \leqslant (4\max(C_3, C_4)(M^2+N))^{-1}$.

Therefore the map $\Xi \times \Psi$: $(u, v) \to (\Xi[u, v], \Psi[u, v])$ is a contraction mapping from $\mathscr{X}_{\mathscr{M}\mathscr{N}}$ into itself and we obtain a unique fixed point which solves the equation for $T < \delta$.

Next we show the uniqueness of the solution in the above class. For simplicity, we show the uniqueness of the solution to (1.2) with $\gamma = 0$. The general case follows similar argument. We introduce the following auxiliary norms. For $T > 0$, we let

$$\|u\|_{X_T} = \inf_w \{ \|w\|_{X^{s,b}} : w \in X^{s,b} \text{ such that}$$
$$u(t) = w(t)\ t \in [0, T] \text{ in } H^s\},$$
$$\|v\|_{Y_T} = \inf_w \{ \|w\|_{Y^{s-(1/2),b}} : w \in Y^{s-1/2,b} \text{ such that}$$
$$u(t) = w(t)\ t \in [0, T] \text{ in } H^{s-(1/2)}\}.$$

Obviously, if $\|u_1 - u_2\|_{X_T} = 0$, we have $u_1(t) = u_2(t)$ in H^s for $t \in [0, T]$.

Let (u_1, v_1) be the solution obtained above and (u_2, v_2) be a solution of the integral equation with the same initial data (u_0, v_0). We assume that for some $M, N > 0$,

$$\|u_1\|_{X^{s,b}},\ \|\psi u_2\|_{X^{s,b}} \leqslant M,$$
$$\|v_1\|_{Y^{s-(1/2),b}},\ \|\psi v_2\|_{Y^{s-(1/2),b}} \leqslant N.$$

Without loss of generality, we may assume that $1 < M + N$ and $T < 1$. For some $T^* < T$ which will be fixed later, we have

$$\begin{aligned} \psi u_2(t) &= \psi(t)\, U(t)\, u_0 - i\psi(t) \int_0^t U(t-t')\, \psi_{T^*}(t')\, \psi^2(t')\, u_2(t')\, v_2(t')\, dt', \\ \psi v_2(t) &= \psi(t)\, W(t)\, v_0 + \psi(t) \int_0^t W(t-t')\, \psi_{T^*}(t')\, \beta\, \partial_x |\psi(t')\, u_2(t')|^2\, dt' \end{aligned} \tag{4.4}$$

for $t \in [0, T^*]$.

Consider the difference $u_1 - \psi u_2$ and $v_1 - \psi v_2$. For any $\varepsilon > 0$, there exists $(\omega, \phi) \in X^{s,b} \times Y^{s-(1/2),b}$ such that for $t \in [0, T^*]$,

$$\begin{aligned} \omega(t) &= u_1(t) - \psi(t)\, u_2(t), \\ \phi(t) &= v_1(t) - \psi(t)\, v_2(t) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \|\omega\|_{X^{s,b}} &\leqslant \|u_1 - \psi u_2\|_{X_{T^*}} + \varepsilon, \\ \|\phi\|_{Y^{s-(1/2),b}} &\leqslant \|v_1 - \psi v_2\|_{Y_{T^*}} + \varepsilon. \end{aligned} \tag{4.6}$$

Set $(\tilde{\omega}, \tilde{\phi})$ satisfying

$$\begin{cases} \tilde{\omega}(t) = -i\psi(t) \displaystyle\int_0^t U(t-t')\, \psi_{T^*}\{\omega(t')\, v_1(t') + \psi(t')\, u_2(t')\, \phi(t')\}\, dt', \\ \tilde{\phi}(t) = \psi(t) \displaystyle\int_0^t W(t-t')\, \psi_{T^*}(t')\, \partial_x \{\bar{u}_1(t')\, \omega(t') + \psi(t')\, u_2(t')\, \bar{\omega}(t')\}\, dt'. \end{cases}$$

By (4.5) we have $\tilde{\omega}(t)=\omega(t)=u_1-\psi(t)\,u_2(t)$ and $\tilde{\phi}(t)=\phi(t)=v_1-\psi(t)\,v_2(t)$ for $t\in[0,T^*]$.

Then according to Lemma 2.1, Proposition 2.3, Lemma 3.1, Corollary 3.3 and (3.7) in Lemma 3.4, we have for $b<b'<3/4$ and $\mu=(b'-b)/4b'$,

$$\begin{aligned}\|u_1-\psi u_2\|_{X_{T^*}} &\leqslant \|\tilde{\omega}\|_{X^{s,b}} \leqslant C\,\|\psi_{T^*}\{\omega v_1+\psi u_2\phi\}\|_{X^{s,b'-1}}\\ &\leqslant C_3 T^{*\mu}(\|\omega\|_{X^{s,b}}\,\|v_1\|_{Y^{s-(1/2),b}}+\|\phi\|_{Y^{s-(1/2),b}}\,\|\psi u_2\|_{X^{s,b}})\\ &\leqslant C_3 T^{*\mu}(M+N)(\|\omega\|_{X^{s,b}}+\|\phi\|_{Y^{s-(1/2),b}}),\end{aligned}$$

If $T^{*\mu}\leqslant 1/4C_3(M+N)$, we have

$$\|u_1-\psi u_2\|_{X_{T^*}}\leqslant \tfrac{1}{4}(\|\omega\|_{X^{s,b}}+\|\phi\|_{Y^{s-(1/2),b}}).$$

Similarly, we have

$$\|v_1-\psi v_2\|_{Y_{T^*}}\leqslant \tfrac{1}{4}\|\omega\|_{X^{s,b}}$$

if $T^{*\mu}\leqslant 1/8C_4M$, where C_4 appears in (4.3).

Combining the above, it follows that

$$\|u_1-\psi u_2\|_{X_{T^*}}+\|v_1-\psi v_2\|_{Y_{T^*}}\leqslant \tfrac{1}{2}(\|\omega\|_{X^{s,b}}+\|\phi\|_{Y^{s-(1/2),b}}).$$

By (4.6), we conclude

$$\|u_1-\psi u_2\|_{X_{T^*}}+\|v_1-\psi v_2\|_{Y_{T^*}}\leqslant 2\varepsilon.$$

This proves $u_1=u_2$ and $v_1=v_2$ on $[0,T^*]$. Repeating this procedure, we obtain the uniqueness result for any existence interval.

The additional regularity

$$u\in C([0,T);H^s),\qquad v\in C([0,T);H^{s-(1/2)})$$

both follow from (2.5) of Corollary 2.3 and the H^s, $H^{s-(1/2)}$ boundedness of the unitary operators $U(t)$ and $V(t)$ and we complete the proof of local well-posedness. ∎

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1. We only point out that the bound is uniform for $0<\nu<1-\varepsilon$.

Proof of Theorem 1.2. We use the abbreviation $X^b=X^{s,b}$ and $Y^b=Y_\nu^{s-(1/2),b}$. For $(u_0,v_0)\in H^s(\mathbf{R})\times H^{s-(1/2)}(\mathbf{R})$ and $b\in(1/2,3/4)$, let

$$\mathscr{X}_{MN}=\{(u,v): u\in X^b,\ v\in Y^b,\text{ such that }\|u\|_{X^b}\leqslant M,\ \|v\|_{Y^b}\leqslant N\},$$

where $M = 2C_0 \|u_0\|_{H^s}$ and $N = 2C_0 \|v_0\|_{H^{s-1/2}}$ with the norm

$$|||(u, v)|||_{\mathscr{X}_{\mathscr{M}\mathscr{N}}} = \|u\|_{X^b} + \|v\|_{Z^b}.$$

Without loss of generality, we may assume that $1 < M$ and $1 < N$.

Similarly by Lemma 2.1, Proposition 2.3, Lemma 3.1, Corollary 3.3 and Lemma 3.4, we have for $b < b' < 3/4$, $\mu = (b' - b)/4b'$, and $(u, v) \in \mathscr{X}_{MN}$ that the maps

$$\Xi[u, v] = \psi(t)\, U(t)\, u_0 - i\alpha\psi(t) \int_0^t U(t - t')\, \psi_\delta(t')\, uv(t')\, dt',$$

$$\Psi[u, v] = \psi(t)\, V_\nu(t)\, v_0 + \beta\psi(t) \int_0^t V_\nu(t - t')\, \psi_\delta(t')\, \partial_x |u|^2(t')\, dt'.$$

satisfy

$$\|\Xi[u, v]\|_{X^b} \leqslant \frac{M}{2} + C_1(1 - |\nu|)^{-1}\, \delta^\mu MN,$$

$$\|\Psi[u, v]\|_{Z^b} \leqslant \frac{N}{2} + C_2\, \delta^\mu M^2.$$

If we set

$$\delta^\mu \leqslant \frac{1}{2 \max(C_1(1 - |\nu|)^{-1}, C_2)(M^2 + N)}$$

then we have that $\|\Xi[u, v]\|_{X^b} \leqslant M$ and $\|\Psi[u, v]\|_{Y^b} \leqslant N$ hence $(\Xi[u, v], \Psi[u, v]) \in \mathscr{X}_{\mathscr{M}\mathscr{N}}$ and showing that it is a contraction follows as before. ∎

5. THE LIMITING PROBLEM—PROOF OF THEOREM 1.3

In this section we prove Theorem 1.3. To show our theorem, we first consider the regular solutions (u_ν, v_ν) for (1.3) and the regular solutions (u, v) for (1.4) with the same initial data $(u_0, v_0) \in H^{5/2} \times H^2$. By an approximation procedure the conclusion follows from the well-posedness results already established.

Proof of Theorem 1.3. Let (u_ν, v_ν) be a unique solution of (1.3) in $C([0, \infty); H^{5/2}) \times C([0, \infty); H^2)$ and let (u, v) be a solution of (1.4) in the same space both with initial data $(u_0, v_0) \in H^{5/2} \times H^2$. Without loss of generality, we may assume that $0 < \nu < 1/2$. Note that $Y_0^{s-(1/2), b} = H_t^b(H_x^s)$.

We fix the time interval $[0, T]$ so that each of the solutions satisfy the following

$$\|u_\nu\|_{X^{0,b}},\ \|u\|_{X^{0,b}} \leqslant M$$
$$\|v\|_{H_t^b(H_x^{-1/2})} \leqslant N$$
$$\|D_x^{1/2}\,\partial_x v_\nu\|_{L_t^2(L_x^2)} \leqslant \tilde{N}_\nu.$$

We note that (u_ν, v_ν) has enough regularity to satisfy (1.3) in the strong sense, so we have the solution to the integral equation associated with (1.3)

$$\begin{cases} u_\nu(t) = U(t)\,u_0 - i\alpha \int_0^t U(t-t')\,v_\nu(t')\,u_\nu(t')\,dt', \\ v_\nu(t) = v_0 + \beta \int_0^t \partial_x\,|u_\nu(t')|^2\,dt' - \nu \int_0^t D_x\,\partial_x v_\nu(t')\,dt', \end{cases}$$

and we have the same for a solution (u, v) of (1.4),

$$\begin{cases} u(t) = U(t)\,u_0 - i\alpha \int_0^t U(t-t')\,v(t')\,u(t')\,dt', \\ v(t) = v_0 + \beta \int_0^t \partial_x\,|u(t')|^2\,dt'. \end{cases}$$

Then the difference satisfies

$$\begin{cases} u_\nu(t) - u(t) = -i\alpha\psi \int_0^t U(t-t')(v_\nu(t')\,u_\nu(t') - v(t')\,u(t'))\,dt', \\ v_\nu(t) - v(t) = \beta\psi \int_0^t (\partial_x\,|u_\nu(t')|^2 - \partial_x\,|u(t')|^2)\,dt' - \nu\psi \int_0^t \partial_x\,D_x v_\nu(t')\,dt'. \end{cases}$$

Then by taking the norms $\|\cdot\|_{X^{0,b}}$ and $\|\cdot\|_{Y_0^{-1/2,b}} = \|\cdot\|_{H_t^b(H_x^{-1/2})}$, we have by (2.4) in Proposition 2.3 and (3.6) in Lemma 3.4 that

$$\begin{aligned} \|u_\nu - u\|_{X^{0,b}} &\leqslant |\alpha| \left\| \psi \int_0^t U(t-t')(v_\nu(t')\,u_\nu(t') - v(t')\,u(t')))\,dt' \right\|_{X^{0,b}}, \\ &\leqslant C(\|v_\nu u_\nu - v u_\nu\|_{X^{0,b-1}} + \|v u_\nu - v u\|_{X^{0,b-1}}) \\ &\leqslant C\,\delta^\mu(\|v_\nu - v\|_{H_t^b(H_x^{-1/2})}\,\|u_\nu\|_{X^{0,b}} + \|u_\nu - u\|_{X^{0,b}}\,\|v\|_{H_t^b(H_x^{-1/2})}) \end{aligned} \tag{5.1}$$

Using (2.6) in Corollary 2.3 and (3.22) in Lemma 3.6 yields

$$
\begin{aligned}
&\|v_\nu - v\|_{H_t^b(H_x^{-1/2})} \\
&\quad \leqslant |\beta| \left\| \psi \int_0^t (\partial_x |u_\nu(t')|^2 - \partial_x |u_\nu(t')|^2)\, dt' \right\|_{H_t^b(H_x^{-1/2})} \\
&\qquad + |\nu| \left\| \psi \int_0^t D_x \partial_x v_\nu \, dt' \right\|_{H_t^b(H_x^{-1/2})} \\
&\quad \leqslant C \, \|\partial_x |u_\nu|^2 - \partial_x u_\nu \bar{u}\|_{H_t^{b-1}(H_x^{-1/2})} \\
&\qquad + C \, \|\partial_x u_\nu \bar{u} - \partial_x |u|^2\|_{H_t^{b-1}(H_x^{-1/2})} + C\, |\nu| \, \|D_x^{1/2} \partial_x v_\nu\|_{H_t^{b-1}(L_x^2)} \\
&\quad \leqslant C \, \|u_\nu\|_{X^{0,b}} \|u_\nu - u\|_{X^{0,b}} + C \, \|u\|_{X^{0,b}} \|u_\nu - u\|_{X^{0,b}} \\
&\qquad + C\, |\nu| \, \|D_x^{1/2} \partial_x v_\nu\|_{H_t^{b-1}(L_x^2)}.
\end{aligned} \tag{5.2}
$$

Next, by (5.2) and (5.1) we obtain

$$
\|u_\nu - u\|_{X^{0,b}} \leqslant C\,\delta^\mu (M^2 + N)\, \|u_\nu - u\|_{X^{0,b}} + C\,\delta^\mu M \,|\nu|\, \|D_x^{1/2} \partial_x v_\nu\|_{L_t^2(L_x^2)}.
$$

If

$$
\delta^\mu \leqslant \frac{1}{2C(M^2 + N)}
$$

we have

$$
\|u_\nu - u\|_{X^{0,b}} \leqslant \frac{|\nu|}{2} \|D_x^{1/2} \partial_x v_\nu\|_{L_t^2(L_x^2)}. \tag{5.3}
$$

Combining (5.2) and (5.3) we have

$$
\begin{aligned}
\|u_\nu - u\|_{X^{0,b}} &\to 0 \\
\|v_\nu - v\|_{H_t^b(H_x^{-1/2})} &\to 0
\end{aligned} \tag{5.4}
$$

as $\nu \to 0$.

Now we prove the general case. For any initial data $(u_0, v_0) \in L^2 \times H^{-1/2}$ we choose a sequence $(u_0^n, v_0^n) \in H^{5/2} \times H^2$ such that

$$
\begin{aligned}
u_0^n &\to u_0 \qquad \text{in } L^2 \\
v_0^n &\to v_0 \qquad \text{in } H^{-1/2}.
\end{aligned}
$$

Then by combining Theorem 1.1 and 1.2 with the above conclusion (5.4), we have that

$$\begin{aligned}
\|u_\nu - u\|_{X^{0,b}} &\leqslant \|u_\nu - \tilde{u}_\nu\|_{X^{0,b}} + \|\tilde{u}_\nu - \tilde{u}\|_{X^{0,b}} + \|\tilde{u} - u\|_{X^{0,b}} \\
&\leqslant C(T)(\|\tilde{u}_0 - u_0\|_{L^2_x} + \|\tilde{v}_0 - v_0\|_{H_x^{-1/2}}) \\
&\quad + |\nu|\ \|D_x^{1/2}\,\partial_x v_\nu\|_{L^2_t(L^2_x)}, \\
\|v_\nu - v\|_{C([0,T);H^{-1/2})} &\leqslant \|v_\nu - \tilde{v}_\nu\|_{C([0,T);H^{-1/2})} + \|\tilde{v}_\nu - \tilde{v}\|_{C([0,T);H^{-1/2})} \\
&\quad + \|\tilde{v} - v\|_{C([0,T);H^{-1/2})} \\
&\leqslant C(T)(\|\tilde{u}_0 - u_0\|_{L^2_x} + \|\tilde{v}_0 - v_0\|_{H_x^{-1/2}}) \\
&\quad + |\nu|\ (CM+1)\ \|D_x^{1/2}\,\partial_x v_\nu\|_{L^2_t(L^2_x)}.
\end{aligned}$$

By letting $\nu \to 0$ and $n \to \infty$ we have that $u_0^n \to u_0$, $v_0^n \to v_0$, and

$$\begin{aligned}
u_\nu^n &\to u \qquad \text{in} \quad X^{0,b} \cap C([0,T);L^2) \\
v_\nu^n &\to v \qquad \text{in} \quad C([0,T);H^{-1/2}).
\end{aligned}$$

Repeating this procedure, we have the conclusion for the desired time interval. ∎

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