Minimal free resolution of the associated graded ring of monomial curves of generalized arithmetic sequences

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\textbf{A B S T R A C T}

Let $A(C)$ be the coordinate ring of a monomial curve $C \subseteq \mathbb{A}^n$ corresponding to the numerical semigroup $S$ minimally generated by a sequence $a_0, \ldots, a_n$. In the literature, little is known about the Betti numbers of the corresponding associated graded ring $gr_m(A)$ with respect to the maximal ideal $m$ of $A = A(C)$. In this paper we characterize the numerical invariants of a minimal free resolution of $gr_m(A)$ in the case $a_0, \ldots, a_n$ is a generalized arithmetic sequence.

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1. Introduction

Let $A = K[[\tau^{a_0}, \ldots, \tau^{a_n}]]/I$ be the coordinate ring of a monomial curve $C \subseteq \mathbb{A}^n$. Assume that the numerical semigroup $S = (a_0, \ldots, a_n)$ is minimally generated by $a_0 < \cdots < a_n$ and denote by $m$ the maximal ideal of $A$.

In the recent years the associated graded ring $G = gr_m(A) = \bigoplus_{i=0}^\infty (m^i/m^{i+1})$ of the maximal ideal $m$ of $A$ has been extensively studied because it encodes several geometric information of $A$. Note that $G$ is a standard graded $K$-algebra, but it is well known that good properties of $A$, e.g., being Cohen–Macaulay, Gorenstein, complete intersection, or level, in general can not be carried from $A$ to the associated graded ring $G$. These investigations often require explicit computations of a standard basis of $I$, that is a system of generators of $I$ (not minimal, in general) whose initial forms with respect to the $m$-adic filtration generate the ideal $I^*$ of $P = K[x_0, \ldots, x_n]$ such that $G \cong P/I^*$. The Hilbert function of $A$ gives partial information about it. We recall that the Hilbert function of $A$ is the numerical function $HF_A(t) = \dim_K (m^t/m^{t+1})$, which coincides by its definition with the Hilbert function of the associated graded ring $G$. Despite the fact that the Hilbert function of a standard graded $K$-algebra is well understood in the Cohen–Macaulay case, very little is known in the local case.

Starting from the fact that $G$ is Cohen–Macaulay if and only if the image of $\tau^{a_0}$ in $G$ is a regular element, several authors, as Robbiano–Valla\textsuperscript{[18]}, Molinelli–Patil–Tamone\textsuperscript{[13,11,12,23]}, Garcia\textsuperscript{[7]}, Arslan–Mete\textsuperscript{[1]}, and Shibuta\textsuperscript{[22]}, tried to find some classes of monomial curves such that their associated graded rings are Cohen–Macaulay or at least have non-decreasing Hilbert function.

More precise information can be deduced from the minimal free resolution of $A$ and from the minimal graded free resolution of $G$. This is a difficult problem and the main goal of this paper is to give information in the case of monomial curves corresponding to a generalized arithmetic sequence. A sequence $a_0, \ldots, a_n$ is called generalized arithmetic sequence if, for each $1 \leq i \leq n$, $a_i = ha_0 + id$ where $h$ and $d$ are given integers and $\gcd(a_0, d) = 1$. Generalized arithmetic sequences were introduced by Lewin\textsuperscript{[10]} and studied by several mathematicians, among others:\textsuperscript{[6,9,14]}

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Molinelli and Tamone in [11] studied the case of an arithmetic sequence $a_0, \ldots, a_n$, i.e. a sequence determined by the equality $a_i = a_0 + id$, where $\gcd(a_0, d) = 1$. Arithmetic sequences can be obtained by generalized arithmetic sequences by $h = 1$. In this case Molinelli and Tamone proved the Cohen–Macaulayness of $G$ and they computed its Hilbert function. Later the technical paper of Sengupta [21] was devoted to determine the minimal free resolution of the local ring $A$ of a monomial curve $C \subseteq \mathbb{A}^4$ corresponding to an arithmetic sequence. Even in this case the problem was still open for the resolution of $G$.

In this paper we study the numerical invariants of the minimal free resolution of the associated graded ring $G$ (as a $P = K[x_0, \ldots, x_n]$-module) in the case of a generalized arithmetic sequence (see Theorem 4.1). We prove that $G$ is Cohen–Macaulay and we characterize the Hilbert function of $A$ in terms of the integers $t$ and $r$ uniquely determined by the equation $a_0 = (t - 1)n + r$ for $1 \leq r \leq n$ and $t \geq 2$. (Corollaries 3.2 and 3.5.)

Generalized arithmetic sequences are particular cases of almost arithmetic sequence already considered in the literature. A sequence $a_0, \ldots, a_n$ is *almost arithmetic sequence* if $a_1, \ldots, a_n$ is an arithmetic sequence and $a_0$ an arbitrary positive integer. In this case, the associated graded ring was studied in several papers by Molinelli, Patil and Tamone (see [13,15,16]) where a very technical machinery has been introduced in order to give some conditions to control the Cohen–Macaulayness property of $G$ and to compute the Hilbert function of $A$ and Cohen–Macaulay type of $G$. Even if a priori these criteria could be used in order to prove that $G$ is Cohen–Macaulay in our class, the easy and new method presented in this paper (see for example Lemma 3.1 and Corollary 3.2) will be useful to produce a larger class of monomial curve with Cohen–Macaulay associated graded ring and to give information on a free resolution of $G$.

By induction steps, we prove that to find the Betti numbers of $G$, it is enough to deal with the case of an arithmetic sequence. The method of finding Betti numbers presented here has an intrinsic interest because it allows us to fit $G$ in to a short exact sequence (Corollary 2.4) involving the defining ideal of the rational normal curve in the standard embedding and, by using the mapping cone procedure [4], to construct a free resolution of $G$. Luckily, in case of our special ideals, the differential maps of this free resolution have some good properties that help us to find Betti numbers explicitly in terms of $t$ and $r$ (see Theorem 4.1).

Several authors studied the local ring $A$ when $a_0, \ldots, a_n$ is a generalized arithmetic sequence. In Corollary 4.11 we compare their results with ours and we conclude that:

- $A$ is complete intersection if and only if $G$ is complete intersection.
- The minimal number of generators of $I$ and the minimal number of generators of $I^c$ coincide.
- The Cohen–Macaulay type of $A$ and the Cohen–Macaulay type of $G$ coincide. In particular $A$ is Gorenstein if and only if $G$ is Gorenstein.

By using a result due to Robbiano (see [17] and also [8]) it is possible to relate a $K[[x_0, \ldots, x_n]]$-free resolution for $A$ with a minimal $K[x_0, \ldots, x_n]$-free resolution of $G$. For the Betti numbers of $A$ and $G$ one has

$$\beta_i(A) \leq \beta_i(G).$$

The resolution of $A$ is minimal if and only if $\beta_i(A) = \beta_i(G)$ for each $i \geq 0$. In this case $A$ is called of homogeneous type. Examples of local rings of homogeneous type can be found in [19,8]. Corollary 4.12 shows that $A = K[[x^{a_0}, \ldots, x^{a_n}]]$ is of homogeneous type provided that $a_0, \ldots, a_n$ is a generalized arithmetic sequence with $n \leq 3$ or $a_0 \leq 2n$.

## 2. Preliminaries

Let $A = K[[x^{a_0}, \ldots, x^{a_n}]] = K[[x_0, \ldots, x_n]]/I$, be a one dimensional local domain corresponding to a monomial curve in the affine space. Suppose that $0 < a_0 < \cdots < a_n$, $n \geq 2$, with $\gcd(a_0, \ldots, a_n) = 1$ and the numerical semigroup $S = \langle a_0, \ldots, a_n \rangle$ is minimally generated by $a_0, \ldots, a_n$. Let $m = (x^{a_0}, \ldots, x^{a_n})A$ be the maximal ideal of $A$.

The multiplicity of $A$, $e(A)$, is $a_0$ and the embedding dimension of $A$, $\mu(A)$, is $n + 1$ and $n + 1 < a_0$ [24]. Let $x \in S$, as in [13], define the maximal degree of $x$ in $S$ to be $\maxdeg(x) = \max\{\sum_{i=0}^{n} c_i \mid x = \sum_{i=0}^{n} c_i a_i, c_i \geq 0\}$. We also denote the maximal degree as $\maxdeg(x)$ when the semigroup is clear from the context. The highest power of the maximal ideal containing $x^t$ is $m^{\maxdeg(x)}$. Let $S_0^k = \{c_0, \ldots, c_n \mid \sum_{i=0}^{n} c_i = k\}$. Following the notations used in [20], let $S^k = \{x \in S \mid \maxdeg(x) = k\}$. Then $S^k \subseteq S_0^k$ and $[x^t \mid x \in S^k]$ is a minimal generating set for $m^k$ with at most $a_0$ elements. If we write $x \in S$ as $\mathbb{N}$-linear combination of $a_0, \ldots, a_n$, we say $x$ is maximally expressed when the sum of the coefficients is equal to the maximal degree of $x$ in $S$.

**Definition 2.1.** Let $\text{gr}(S)$ be the $K$-vector space with basis $\{x^t \mid x \in S^t\}$. Define $\text{gr}(S) = \bigoplus_{t \geq 0} \text{gr}(S)_t$ with multiplication

$$x^a \cdot x^b := \begin{cases} x^{a+b} & \text{if } \maxdeg(a + b) = \maxdeg(a) + \maxdeg(b), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that with this multiplication $\text{gr}(S)$ is a graded ring with $\text{HF}_{\text{gr}(S)}(i) = |S^i|$, where HF is the Hilbert function. In fact $\text{gr}(S)$ is the associated graded ring $\text{gr}_m(A)$. One can easily see that $\text{gr}(S)$ is Cohen–Macaulay if and only if $x^{a_0}$ is a regular element, and $x^{a_0}$ is a zero divisor in $\text{gr}(S)$ if and only if there exists an $x = \sum_{i=0}^{n} c_i a_i$, maximally expressed in $S$ such that $a_0 + \sum_{i=0}^{n} c_i a_i$ is not maximally expressed in $S$, i.e., there exist non-negative integers $b_0, \ldots, b_n$ such that $a_0 + \sum_{i=0}^{n} c_i a_i = \sum_{i=0}^{n} b_i a_i$ and $1 + \sum_{i=0}^{n} c_i < \sum_{i=0}^{n} b_i$ [20].
Consider the following surjective homomorphism of graded rings:
\[
\phi_S : K[x_0, \ldots, x_n] \to \text{gr}(S)
\]
\[
x_0^{c_0} \cdots x_n^{c_n} \mapsto \begin{cases} 
\sum_{i=0}^{n} c_i x_i & \text{if } \maxdeg \left( \sum_{i=0}^{n} c_i x_i \right) = \sum_{i=0}^{n} c_i, \\
0 & \text{otherwise.}
\end{cases}
\]

It is clear that \(K[x_0, \ldots, x_n]/\ker(\phi_S) \cong \text{gr}(S)\). Since \(\text{gr}(S)\) is the associated graded ring of a monomial curve \(E\), \(\ker(\phi_S)\) is the initial ideal \(I^*\). It is easy to see that \(\ker(\phi_S)\) has a homogeneous generating set consisting of a finite number of monomials and binomials.

From now on in this section we assume that the numerical semigroup \(S\) is generated by \(a_0 < \cdots < a_n\) such that \(\{a_0, \ldots, a_n\}\) is an arithmetic sequence, i.e.,
\[
a_i = a_0 + id
\]
with \(\gcd(a_0, d) = 1\).

Next lemma helps us to characterize \(I^*\).

**Lemma 2.2.** Let \(I^*\) be the initial ideal corresponding to \(S\).

(i) The homogeneous binomial \(x_0^{c_0} \cdots x_n^{c_n} - x_0^{d_0} \cdots x_n^{d_n}\) is in \(I^*\) if and only if neither \(\sum_{i=0}^{n} c_i a_i\) nor \(\sum_{i=0}^{n} d_i a_i\) are maximally expressed, or \(\sum_{i=0}^{n} i c_i = \sum_{i=0}^{n} i d_i\).

(ii) If \(x_0^{c_0} \cdots x_n^{c_n} \in I^*\) then \(x_0^{c_1} \cdots x_n^{c_n} \in I^*\) provided that \(\sum_{i=0}^{n} c_i = \sum_{i=0}^{n} c_i'\) and \(\sum_{i=0}^{n} c_i a_i \leq \sum_{i=0}^{n} c_i' a_i\).

(iii) \(|S_0| = 1 + kn\) for any natural number \(k\).

**Proof.** (i) By definition of \(\phi_S\), \(x_0^{c_0} \cdots x_n^{c_n} - x_0^{d_0} \cdots x_n^{d_n} \in I^*\) if and only if neither \(\sum_{i=0}^{n} c_i a_i\) nor \(\sum_{i=0}^{n} d_i a_i\) are maximally expressed in \(S\) or \(\sum_{i=0}^{n} c_i a_i = \sum_{i=0}^{n} d_i a_i\) and consequently \(\sum_{i=0}^{n} i c_i = \sum_{i=0}^{n} i d_i\).

(ii) It is enough to show that for each \(0 \leq l < n\) such that \(c_l > 0\),
\[
x_0^{c_0} \cdots x_l^{c_l-1} x_{l+1}^{c_{l+1}+1} \cdots x_n^{c_n} \in I^*.
\]
Since \(x_0^{c_0} \cdots x_n^{c_n} \in I^*, x = \sum_{i=0}^{n} c_i a_i\) is not a maximal expression of \(x\). Thus, there exist non-negative integers \(d_0, \ldots, d_n\) such that \(\sum_{i=0}^{n} d_i a_i = \sum_{i=0}^{n} d_i a_i\) and \(\sum_{i=0}^{n} d_i > \sum_{i=0}^{n} c_i\). It is clear that there exists \(0 \leq j < n\) such that \(d_j > 0\). Let \(d_j' = d_j - 1\) for \(i \neq j, j+1, d_{j+1}' = d_{j+1} - 1\). It is obvious that \(\sum_{i=0}^{n} d_j a_i = \sum_{i=0}^{n} c_i a_i + (c_{j+1} + c_{j+1}) a_{j+1} + \sum_{i=j+2}^{n} c_i a_i\), and \(\sum_{i=0}^{n} d_i' = \sum_{i=0}^{n} d_i > \sum_{i=0}^{n} c_i\). Therefore \(x_0^{c_0} \cdots x_l^{c_l-1} x_{l+1}^{c_{l+1}+1} \cdots x_n^{c_n} \in I^*\).

(iii) It is straightforward by using induction on \(k\). \(\square\)

We recall that \(G\) is Cohen–Macaulay (Proposition 1.1 in [11]) and, if \(a_0 = (t - 1)n + r, t \geq 2\) and \(1 \leq r \leq n\), then the \(h\)-polynomial of \(G\) is
\[
1 + nz + \cdots + nz^{l-1} + (r - 1)z^l
\]
(see Corollary 1.10 in [11]). We denote \(I^*\) by \(I^*(t,r)\). For example if \(S\) is generated by \(a_0 < \cdots < a_5\) and \(a_0 = 15\) the corresponding ideal is \(I^*(3,5)\).

A crucial fact will relate \(I^*\) with the ideal \(J\) of \(P = K[x_0, \ldots, x_n]\) generated by \(2 \times 2\) minors of the matrix
\[
M = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}.
\]

It is well known that \(J\) is the ideal of a rational normal curve in the standard embedding. By Lemma 2.2 one can easily see that \(J \leq I^*(t,r)\), for each \(t, r\). Using this easy fact and the Hilbert function of \(G\), in the next lemma we find a nice relation between \(J, I^*(t,r)\). As a corollary we have a minimal system of generators for each \(I^*(t,r)\), and some short exact sequences which will be very useful for our main purpose that is computing Betti numbers. This is a new method to find a minimal system of generators for \(I^*\) avoiding standard basis. We denote the minimal number of generators of an ideal \(I\) in \(R = k[x_0, \ldots, x_n]\) or \(P = k[x_0, \ldots, x_n]\) by \(\mu(I)\).

**Lemma 2.3.** (i) \(I^*(t,n) = J + \langle x_n^2 \rangle\).

(ii) For each \(1 \leq i \leq n - 1, I^*(t,n-i) = I^*(t,n-i+1) + \langle x_{n-i} x_{n-i-1} \rangle\).

(iii) For each \(1 \leq i \leq n - 1, (I^*(t,n-i+1) : x_{n-i} x_{n-i-1}^2) = \langle x_1, \ldots, x_n \rangle\).
Proof. We denote by $HSP_{I_{(t,r)}^*}$ the Hilbert series of $G$, when $a_0 = (t - 1) \times n + r$.

(i) Suppose that $S$ is a numerical semigroup, generated by an arithmetic sequence, corresponding to $I_{(t,n)}^r$. We know that $J \subseteq I_{(t,n)}^r$. The equality $HSP_{I_{(t,n)}^r}(z) = \frac{1+nz+\cdots+nz^{t-1}+(n-2)z^t}{1-z}$ implies that $HF_{P/I_{(t,n)}^r}(j) = 1 + jn$ for $1 \leq j \leq t - 1$. By Lemma 2.2, $S' = S_0$ for any $1 \leq j \leq t - 1$. Therefore, there is no monomial of degree less than $t$ in $I_{(t,n)}^r$. But $HF_{P/I_{(t,n)}^r}(t) = tn$ and hence $|S'| = |S_0| - 1$ which implies that $x_{n}^r \in I_{(t,n)}^r$.

The short exact sequence $0 \to [P/J](t) \to P/J \to P/(J + \langle x_{n}^r \rangle) \to 0$ and Hilbert series of $P/J$, which is $HSP_{I_{(t,n)}^r}(z) = \frac{1+nz+\cdots+nz^{t-1}+(n-2)z^t}{1-z}$.

(ii), (iii) Suppose that $i = 1$. The Hilbert series of $I_{(t,n-1)}^r$ is

$$HSP_{I_{(t,n-1)}^r}(z) = \frac{1 + nz + \cdots + nz^{t-1} + (n - 2)z^t}{1 - z}.$$

As previous, suppose $S$ is a numerical semigroup corresponding to $I_{(t,n-1)}^r$. For each $1 \leq j \leq t - 1, S' = S_0$ and $|S'| = |S_0| - 2$. Thus, there exist no monomial of degree less than $t$ in $I_{(t,n-1)}^r$ and Lemma 2.2 implies $x_{n-1}^r, x_{n-1}^r \in I_{(t,n-1)}^r$. Therefore $I_{(t,n)}^r + (x_{n-1}^r x_{n}^r - 1) \subseteq I_{(t,n-1)}^r$.

We prove $I_{(t,n)}^r : x_{n-1}^r x_{n}^r - 1 = \langle x_1, \ldots, x_n \rangle$. For each $1 \leq j \leq n, x_j x_{n-1}^r x_{n}^r - x_{j-1} x_{n}^r = x_{j} x_{n-1}^r - x_{j-1} x_{n}^r \in I_{(t,n)}^r$,

hence, $x_j x_{n-1}^r x_{n}^r - 1 \in I_{(t,n)}^r$. So, $\langle x_1, \ldots, x_n \rangle \subseteq I_{(t,n)}^r : x_{n-1}^r x_{n}^r - 1$. Note that the Cohen–Macaulayness of $gr_{m}(A)$ implies that for each $1 \geq 1, x_j x_{n-1}^r x_{n}^r - 1 \not\in I_{(t,n)}^r$ (see the paragraph just before Definition 2.1). Therefore, $(I_{(t,n)}^r : x_{n-1}^r x_{n}^r - 1) = \langle x_1, \ldots, x_n \rangle$.

Consider the short exact sequence

$$0 \to [P/(x_1, \ldots, x_n)](t) \to P/I_{(t,n)}^r \to P/(I_{(t,n)}^r + (x_{n-1}^r x_{n}^r - 1)) \to 0,$$

where $\phi$ is multiplication by $x_{n-1}^r x_{n}^r - 1$. Additive property of Hilbert function shows that $HF_{P/(I_{(t,n)}^r + (x_{n-1}^r x_{n}^r - 1))}(i) = HF_{P/I_{(t,n)}^r}(i) - HF_{P/(x_1, \ldots, x_n)}(i - t) = HF_{P/I_{(t,n-1)}^r}(i)$ for each $i \geq 0$. Thus, the equality $I_{(t,n-1)}^r = I_{(t,n)}^r + (x_{n-1}^r x_{n}^r - 1)$ holds.

By induction and repeating the above method, one can prove (ii), (iii) for any $2 \leq i \leq n - 1$. □

Corollary 2.4. Let $A = K[[\tau^{a_0}, \ldots, \tau^{a_n}]]$ be a local ring corresponding to a semigroup $S$ minimally generated by arithmetic sequence $a_0, \ldots, a_n$, $a_0 = (t - 1)n + r, t \geq 2$ and $1 \leq r \leq n$. Then

(i) For each $m > t, S'^m = (m - t)a_0 + S' = \{(m - t)a_0 + a | a \in S'\}$.

(ii) If a monomial $x_{0}^{s_0} \cdots x_{n}^{s_n}$ belongs to $I_{(t,r)}^*$, then $\sum_{1}^{n} s_i \geq t$.

(iii) $I_{(t,r)}^* = J + \langle x_{0}^{r}, x_{0}^{s_0} \cdots x_{n}^{s_n}, x_{n}^{r} \rangle$, where $J$ is the ideal of $P$ generated by $2 \times 2$ minors of the matrix $M = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$.

(iv) $\mu(I_{(t,r)}^*) = \binom{n}{2} + n - r + 1$.

(v) The following short sequences are exact:

$$0 \to [P/J](t) \to P/J \to P/I_{(t,n)}^* \to 0.$$

and for $1 \leq j \leq n - 1,$

$$0 \to [P/(x_1, \ldots, x_n)](t) \to P/I_{(t,n)}^* \to P/I_{(t,n-j)}^* \to 0.$$

3. Generalized arithmetic sequences

The main idea of this section is to relate the numerical properties of a semigroup $S'$ minimally generated by a generalized arithmetic sequence $a_0', \ldots, a_n'$ to the properties of the semigroup $S$ generated by the arithmetic sequence that "gives rise" to $S'$.

Let $\{a_0, \ldots, a_n\}$ be an arithmetic sequence, i.e., $a_i = a_0 + id$, gcd$(a_0, d) = 1$ and $S$ be the semigroup generated by them. Let $h > 1, a_0 = a_0$ and for $1 \leq i \leq n, a_i = ha_0 + id$ and $S'$ be the semigroup minimally generated by $a_0', \ldots, a_n'$. For example if $S = (7, 10, 13, 16, 19)$ and $h = 2$, then $S' = (7, 17, 20, 23, 26)$.

We prove that $gr(S')$ is Cohen–Macaulay and the Hilbert function is the same as Corollary 1.10 of [11]. We recall that a generalized arithmetic sequence is a special case of almost arithmetic sequence which means a sequence $m_0, \ldots, m_k$ where $m_0, \ldots, m_k - 1$ are an arithmetic sequence and $m_k$ is an arbitrary integer. Molinelli, Patil and Tamone in [13] gave some inequalities, using Apery set and positive integers $\lambda, \mu, \omega, \nu, \upsilon, z, u$ (see [13] for definitions and more information) to control Cohen–Macaulayness property in some cases. One might be able to prove Corollary 3.2 with their method. We
avoid this and prove it directly by a very easy comparison between elements of $S'$ and $S$. It is worthy to remark that in [15], the authors computed the Hilbert function for the case of almost arithmetic sequences just for $m_k > m_0$.

Next lemma plays a key role in our method.

**Lemma 3.1.** If $x = \sum_{i=0}^{n}c_i a_i$ is not maximally expressed in $S$, then $x' = \sum_{i=0}^{n}c_i a_i'$ is not maximally expressed in $S'$.

**Proof.** Let $a_0 = (t - 1) \times n + r$ and $I_{(r,t)}^t$ be the initial ideal corresponding to $S$. Since $x = \sum_{i=0}^{n}c_i a_i$ is not maximally expressed in $S$ and $\text{gr}(S)$ is Cohen–Macaulay, $\sum_{i=0}^{n}c_i a_i$ is not maximally expressed in $S$ too. Therefore $x_1^{c_1} \cdots x_n^{c_n} \in I_{(r,t)}^t$. By Corollary 2.4, $\sum_{i=1}^{n}c_i \geq t$ and $\text{maxdeg}_S(\sum_{i=1}^{n}c_i a_i) > t$. Supposing that $\text{maxdeg}_S(\sum_{i=1}^{n}c_i a_i) = m$, there exist non-negative integers $b_0, \ldots, b_n$ in such a way that $\sum_{i=0}^{n}b_i = t$ and $\sum_{i=0}^{n}c_i a_i = (m-t)a_0 + \sum_{i=0}^{n}b_i a_i$. This equality implies that $\sum_{i=1}^{n}c_i a_i' = (m-t+b_0 + \sum_{i=1}^{n}c_i - \sum_{i=1}^{n}b_i)(h-1)a_0 + \sum_{i=1}^{n}b_i a'_i$. Therefore $\text{maxdeg}_S(\sum_{i=1}^{n}c_i a_i') > m - t + \sum_{i=0}^{n}b_i = m > \sum_{i=1}^{n}c_i$ and hence $\sum_{i=0}^{n}c_i a_i'$ is not maximally expressed in $S'$. □

**Corollary 3.2.** The ring $\text{gr}(S')$ is Cohen–Macaulay.

**Proof.** Suppose that $\text{gr}(S')$ is not Cohen–Macaulay. So there exists $x = \sum_{i=0}^{n}c_i a'_i$ maximally expressed in $S'$ such that $a'_0 + \sum_{i=0}^{n}c_i a'_i$ is not maximally expressed. Therefore there exist non-negative integers $b_0, \ldots, b_n$ such that $a'_0 + \sum_{i=0}^{n}c_i a'_i = \sum_{i=0}^{n}b_i a'_i$, and $\sum_{i=0}^{n}b_i = 1 + \sum_{i=0}^{n}c_i$. Since $\sum_{i=0}^{n}c_i a'_i$ is maximally expressed in $S'$ we have $b_0 = 0$. Lemma 3.1 implies that $\sum_{i=0}^{n}c_i a_i$ is maximally expressed in $S$ so $a_0 + \sum_{i=0}^{n}c_i a_i$ is also maximally expressed in $S$ (gr(S) is Cohen–Macaulay). Despite this, $a_0 + \sum_{i=0}^{n}c_i a_i = (\sum_{i=0}^{n}b_i - \sum_{i=0}^{n}c_i)(h-1)a_0 + \sum_{i=0}^{n}b_i a_i$ and thus $\text{maxdeg}_S(a_0 + \sum_{i=0}^{n}c_i a_i) \geq \sum_{i=1}^{n}b_i > 1 + \sum_{i=0}^{n}c_i$ which is a contradiction. So gr(S') is Cohen–Macaulay. □

In the following we show that the Hilbert function of $\text{gr}(S)$ is the same as the Hilbert function of $\text{gr}(S')$. For this we need one more lemma.

**Lemma 3.3.** If $a_i + ma_n$ is maximally expressed in $S$ for some $1 \leq s \leq n$, then $a'_i + ma'_n$ is maximally expressed in $S'$.

**Proof.** Assume that $a'_i + ma'_n$ is not maximally expressed in $S'$. Then there exist positive integers $b_0, \ldots, b_n$ such that

$$\sum_{i=0}^{n}b_i > 1 + m \quad \text{and} \quad a'_i + ma'_n = \sum_{i=0}^{n}b_i a'_i. \quad (1)$$

Therefore $a_i + ma_n + (1 + m)(h-1)a_0 = \sum_{i=0}^{n}b_i a_i + \sum_{i=0}^{n}b_i(h-1)a_0. \text{If} \sum_{i=0}^{n}b_i \geq 1 + m \text{then} a_i + ma_n = \sum_{i=0}^{n}b_i a_i + (\sum_{i=0}^{n}b_i - 1)(h-1)a_0. \text{Thus} a_i + ma_n \text{is not maximally expressed, which is a contradiction. Hence we may assume that} \sum_{i=0}^{n}b_i \leq m. \text{This inequality and (1) imply that} b_0 > h-1 \text{and therefore} a_i + ma_n = (b_0 - h + 1)a_0 + \sum_{i=0}^{n}b_i a'_i. \text{One can easily check that} b_0 - h + 1 + \sum_{i=0}^{n}b_i \geq 1 + m. \text{Also} a_i + ma_n + m(h-1)a_0 = (b_0 - h + 1)a_0 + \sum_{i=0}^{n}b_i a_i + \sum_{i=0}^{n}b_i(h-1)a_0. \text{Now, if} \sum_{i=0}^{n}b_i = m, \text{there will be a contradiction again because} a_i + ma_n \text{is maximal expression and there is no expression} \sum_{i=0}^{n}c_i a_i \text{for} a_i + ma_n \text{in S with the property that} \sum_{i=0}^{n}c_i = 1 + m \text{and} c_0 \neq 0. \text{So we may assume that} \sum_{i=0}^{n}b_i \leq m - 1. \text{This inequality and (1) imply that} b_0 > 2(h-1) \text{and hence} a_i + a_n + (m-1)a'_n = (b_0 - 2h + 2)a_0 + \sum_{i=0}^{n}b_i a'_i. \text{It is easy to check that} b_0 - 2h + 2 + \sum_{i=0}^{n}b_i \geq 1 + m. \text{This inequality and equation} a_i + ma_n + (m-1)(h-1)a_0 = (b_0 - 2h + 2)a_0 + \sum_{i=0}^{n}b_i a_i + \sum_{i=0}^{n}b_i(h-1)a_0 \text{indicate that} \sum_{i=0}^{n}b_i \leq m - 2. \text{Continuing this process, finally reaches a contradiction. Therefore} a'_i + ma'_n \text{is not maximally expressed in S'.} \square$

**Theorem 3.4.** The Hilbert function of $\text{gr}(S')$ is the same as the Hilbert function of $\text{gr}(S)$.

**Proof.** It is enough to prove that for each positive integer $j$, $|S'| = |S^j|$. Note that if $x \in S'$, then $x = j a_0$ or there exist unique integers $0 \leq m \leq j - 1$ and $1 \leq s \leq n$ such that $x = (j - m - 1)a_0 + a_i + ma_n$. Define $\alpha : S' \rightarrow S^j$ in such a way that if $x = j a_0$ then $\alpha(x) = x$ and if $x = (j - m - 1)a_0 + a_i + ma_n$ then $\alpha(x) = (j - m - 1)a'_0 + a'_i + ma'_n$. By Lemma 3.3 and the Cohen–Macaulayness of gr(S) this map is well defined. On the other hand, if we fix one maximal expression $x' = \sum_{i=0}^{n}c_i a'_i$ for each element $x'$ of $S'$ and define the map $\beta : S^j \rightarrow S'$ as $\beta(x') = \sum_{i=0}^{n}c_i a_i$, then this map is also well defined, by Lemma 3.1. One can easily check that both $\alpha$ and $\beta$ are one to one. Therefore $|S'| = |S^j|$. □

**Corollary 3.5.** If $S'$ is a numerical semigroup minimally generated by the generalized arithmetic sequence $a_0, a_0 + d, \ldots, a_0 + nd$, where $h > 1$ and $a_0 = (t - 1)n + r, 1 \leq r \leq n$, then,

(i) The $h$-polynomial of the associated graded ring corresponding to $S'$ is $1 + nz + \cdots + nz^{t-1} + (r-1)z^t$.

(ii) The initial ideal corresponding to $\text{gr}(S')$ is

$$I_{(r,t)}^t \leftarrow \langle x_i, x_{i-1}x_{i+1}^{t-1}, \ldots, x_{i}x_{i-1}^{t-1}, x_{i+1}^{t-1}, x_1, x_2, \ldots, x_i x_{i-1} \rangle + \langle x_{i-1}x_{i-2} - x_{i-2}x_{i-1} | 2 \leq i_1 < i_2 \leq n \rangle.$$

(iii) $\mu(I_{(r,t)}^t) = \left(\frac{n}{2}\right) + n - r + 1$. 

Proof. (i) It is clear by Theorem 3.4 and Corollary 1.10 in [11].
(ii), (iii) Lemma 3.1 and Corollary 2.4 imply that \((x_n, x_{n−1}x_n^{i−1}, \ldots, x_kx_n^{i−1}) \subseteq I_n^{e∗} \). Also for each \(1 \leq i \leq n − 1, \)
\[ a_i + d_i = h(a_i) + d_{i+1}. \]
Therefore, \((x_n^2, x_1x_2, \ldots, x_ix_{n−1}) \subseteq I_n^{e∗} \) and hence
\[ L = (x_n^2, x_{n−1}x_n^{i−1}, \ldots, x_2x_n^{i−1}, x_1x_2, \ldots, x_1x_{n−1}) + (x_i−1x_i−2 − x_i−2x_i−1 | 2 \leq i_1 < i_2 \leq n) \subseteq I_n^{e∗}. \]
In addition, \(x_0 = P/I_n^{e∗} \)-regular and \( (P/I_n^{e∗})/x_0(P/I_n^{e∗}) \equiv P/(x_0) + L \) as a \( P \)-module. Therefore, for each \( i \geq 0, \)
\[ H_{P/(i)} = H_{P/I_n^{e∗}}(i) = H_{P/I_n^{e∗}}(i). \]
Thus, \( L = I_n^{e∗}. \) \( \square \)

Example 3.6. Let \( S = \{7, 17, 20, 23, 26\} \), then \( \text{gr}(S) \) is Cohen–Macaulay. 7 = 1 × 4 + 3. So, \( t = 2 \) and \( r = 3. \)
\[ I_n^{e∗}_{(2,3)} = (x_n^2, x_2x_3, x_1x_2, x_1x_3) + (x_{2i−1}x_{2i−2} − x_{2i−2}x_{2i−1} | 2 \leq i_1 < i_2 \leq 4). \]
The \( h \)-polynomial of \( G \) is \( 1 + 4z + 2z^2. \)

4. The Betti numbers

Through this section we compute the Betti numbers of an associated graded ring \( G \) of a numerical semigroup \( S \) corresponding to a generalized arithmetic sequence. Our method is based on Corollary 2.4 and the mapping cone technique. Although the result of a mapping cone is not a minimal free resolution in general, it is possible to take advantage from it to get information about Betti numbers.

As a result we have the Betti numbers of \( G \) for the case of generalized arithmetic sequence. We prove that \( G \) is level and give a necessary and sufficient condition for Gorenstein property of \( G \). And finally, we compare properties of the local ring \( A \) with graded ring \( G \).

The following theorem is the main result of this paper.

Theorem 4.1. Let \( a_0, \ldots, a_n \) be a generalized arithmetic sequence minimally generating \( S \). Let \( a_0 = (t − 1)n + r, t \geq 2, \)
\( 1 \leq r \leq n \). The non-zero Betti numbers of \( G = \text{gr}(S) \) are:
\[ \beta_0, \beta_1, \beta_2 = \begin{pmatrix} n \end{pmatrix}, \beta_3, \beta_4 = \begin{pmatrix} n \end{pmatrix}. \]

To prove the Theorem 4.1, we need some remarks and theorems.

Let \( a_0, \ldots, a_n \) be an arithmetic sequence minimally generating \( S \), \( t \) and \( r \) as before and, \( I_n^{e∗} \), the ideal defining \( G \) in \( P \). In the following we find a free resolution \( (\mathcal{F}, \delta) \) for \( G \) which is not minimal in most cases but the differential maps are very special and we can read the Betti numbers from this resolution directly. For our purpose we need to recall that the minimal free resolution of \( P/(x_1, \ldots, x_n) \) is the Koszul complex associated to \( x_1, \ldots, x_n \) and the minimal free resolution of \( P/J \) is \( J \) is given in Remark 4.3, is the Eagon–Northcott complex of the matrix \( M \) ([5], Corollary 6.2).

We describe the Koszul complex and the Eagon–Northcott complex by means of exterior algebras on symbols \( e_1, \ldots, e_n \).

In the following will be useful to fix the notations and the properties of the well known Eagon–Northcott complex.

Notation 4.2. For \( k = 0, 1, \)
\[ \Delta_k(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{s} (-1)^{i+1} x_{i+k−1}e_1 \wedge \cdots \wedge e_i \wedge \cdots \wedge e_n \quad (1 \leq i_1 < \cdots < i_s \leq n) \]

Remark 4.3 (See [3] for the General Setting). Let \( J \) be the ideal of \( P \) generated by \( 2 \times 2 \) minors of the matrix \( M = (x_0 \ x_1 \cdots \ x_{n−1} \ x_n). \) The Eagon–Northcott complex for \( J \) is given by:
\[ \mathcal{F} : \quad 0 \to F_{n−1} \to \cdots \to F_2 \to F_1 \to P, \]
where for \( 1 < s \leq n − 1, F_s \) is the free \( P \)-module generated by
\[ e_1 \wedge \cdots \wedge e_{i+s−1} \otimes \lambda_0 \lambda_1 e_{i+s} \quad (1 \leq i_1 < \cdots < i_{s−1} \leq n, v_0 + v_1 = s − 1, v_0 \geq 0, v_1 \geq 0), \]
where \( \lambda_0 \) and \( \lambda_1 \) are symbols and \( F_s \) is generated by \( e_i \wedge e_{i+s} \otimes 1 \) \( (1 \leq i_1 < i_2 \leq n) \) or briefly \( e_i \wedge e_{i+s} \).

Note that for each \( 1 \leq s \leq n − 1, F_s \otimes P^{\binom{n}{s−1}}(−s − 1). \) The differential maps are:
\[ d_1(e_1 \wedge e_{i+s}) = x_{i−1}x_{i+s−2} − x_{i+s−2}x_{i−1}, \]
and for \( s \geq 2, \)
\[ d_s(e_1 \wedge \cdots \wedge e_{i+s−1} \otimes \lambda_0 \lambda_1 e_{i+s} = \Delta_0(e_1 \wedge \cdots \wedge e_{i+s−1} \otimes \lambda_0^{i+s−1} e_{i+s} + \Delta_1(e_1 \wedge \cdots \wedge e_{i+s−1} \otimes \lambda_0^{i+s−1} e_{i+s}) \otimes \lambda_0^{n−i−1} e_{i+s−1}, \]
where in the right hand side we consider just the summands with non-negative powers on \( \lambda_0 \) and \( \lambda_1. \)
Now, we are ready to apply mapping cone to short exact sequences in Corollary 2.4v. For \( P/L^*_t \), we use the first sequence,
\[
0 \to [P/\mathcal{J}]/(-t) \xrightarrow{\delta_{\mathcal{J}}^t} P/\mathcal{J} \to P/L^*_t \to 0,
\]
to obtain a free resolution of \( P/L^*_t \), as a mapping cone of complex homomorphism \( \psi(\mathcal{J}) : G \to F \) which is a lifting of \([P/\mathcal{J}]/(-t) \xrightarrow{\delta_{\mathcal{J}}^t} P/\mathcal{J} \) where \( G \) is a shift of \( F \). We denote this resolution by \( (\mathcal{F}(\mathcal{J}), \delta_{\mathcal{J}}) \). One can easily see that \( \psi(\mathcal{J}) \) is also multiplication by \( \delta_{\mathcal{J}}^t \) for \( 0 \leq i \leq n - 1 \), i.e., if \( e_i \) is the basis of \( G \) and \( e_i^1 \wedge \cdots \wedge e_{i+1}^1 \wedge \lambda^0 \lambda^1 \cdots \lambda^t \) \((1 \leq i_1 < \cdots < i_{n+1} \leq n, v_0 + v_1 = s - 1)\) are the bases of \( G \), and \( e_0 \) with \( e_0^1 \wedge \cdots \wedge e_{i+1}^1 \wedge \lambda^0 \lambda^1 \cdots \lambda^t \) \((1 \leq i_1 < \cdots < i_{n+1} \leq n, v_0 + v_1 = s - 1)\) are the corresponding bases of \( F \). Then \( \psi(\mathcal{J})e_i = e_i^t e_0 \) and \( \psi(\mathcal{J})e_0 = e_1^t \wedge \cdots \wedge e_{i+1}^t \wedge \lambda^0 \lambda^1 \cdots \lambda^t \).

Thus, by Proposition 6.15 of [5], \( (\mathcal{F}(\mathcal{J}), \delta_{\mathcal{J}}) \) is the minimal free resolution of \( P/L^*_t \). The following theorem is the precise description of this resolution.

**Theorem 4.4.** The minimal free resolution of \( P/L^*_t \) is:
\[
\mathcal{F}(\mathcal{J}) : 0 \to P^{n-1}(-n-t) \to P^{n-1}(-n) \oplus P^{(n-2)}(-n) \to P^n(-n) \to \cdots \to P^{n-3}(-3) \oplus P^{(2)}(-2) \to P^{(1)}(-1) \oplus P(-1) \to P.
\]

Where for \( 2 \leq i \leq n, \mathcal{F}(\mathcal{J})i = P^{(i+1)}(-i-1) \oplus P^{(i-1)}(-i) \). \( \delta(\mathcal{J}) \) is given as the differentials in the Eagon–Northcott complex on the bases corresponding to \( F \) and
\[
\delta(\mathcal{J})1 = \delta_{\mathcal{J}}^t \colon e_i \mapsto x_i^t e_0, \quad \delta(\mathcal{J})2 = \delta_{\mathcal{J}}2 \colon (e_i^1 \wedge e_{i+1}^1) \mapsto (x_i e_{i+1} - x_{i+1} e_i) e_1 + x_i^t e_1 e_{i+1} - x_{i+1}^t e_1 e_i.
\]

and for \( s \geq 3 \),
\[
\delta(\mathcal{J})s = \delta_{\mathcal{J}}s \colon e_i^1 \wedge \cdots \wedge e_{i+s}^1 \wedge \lambda^0 \lambda^1 \cdots \lambda^t \mapsto \Delta_{\mathcal{J}}(e_i^1 \wedge \cdots \wedge e_{i+s}^1) \wedge \lambda^0 \lambda^1 \cdots \lambda^t - \Delta(\mathcal{J})e_i^1 \wedge \cdots \wedge e_{i+s}^1 \wedge \lambda^0 \lambda^1 \cdots \lambda^t.
\]

Here on the right hand side we consider just the summands with non-negative power on \( \lambda_0 \) and \( \lambda_1 \).

**Example 4.5.** Let \( a_0, \ldots, a_5 \) be an arithmetic sequence and \( d_0 = 5t = (t - 1)5 + 5 \). Then the ideal defining \( G \) in \( P \) is \( I_t \) and the minimal free resolution of it, is given by Theorem 4.4.

To find a free resolution for each \( P/L^*_t \) we apply iterated mapping cone in the second short exact sequences in Corollary 2.4v, which is for \( 1 \leq j \leq n - 1 \),
\[
0 \to [P/\mathcal{J}_1, \ldots, \mathcal{J}_n]/(-t) \xrightarrow{x^t_{\mathcal{J}_1, \ldots, \mathcal{J}_n}} P/\mathcal{J}_{t=1} \to P/\mathcal{J}_{t=n-j} \to 0.
\]

Since in each step we need the Koszul complex and the complex obtained in the previous step, we use the following notation to avoid confusion.

In a step \( j \), \( 1 \leq j \leq n - 1 \), we show the Koszul complex by \( K_j \) and its basis by \( e_1^{(1)} \wedge \cdots \wedge e_s^{(1)} \) \((1 \leq i_1 < \cdots < i_s \leq n)\).

And, the result of mapping cone of \( P/\mathcal{J}_{t=1} \) will be shown by \( (\mathcal{F}(\mathcal{J}), \delta(\mathcal{J})) \). For our purpose, we need to compute the graded complex homomorphisms \( \psi(\mathcal{J}) : K_j \to \mathcal{F}(\mathcal{J}_{t=1}) \). Since \( \mathcal{F}(\mathcal{J}_{j-1}) = \mathcal{F}(\mathcal{J}_{j-1}) \oplus K_{j-1} \), we show the basis of \( \mathcal{F}(\mathcal{J}_{j-1}) \) by the same symbols corresponding to the basis of \( \mathcal{F}(\mathcal{J}_{t=1}) \) and \( K_{j-1} \). More precisely:

- In \( \mathcal{F}(\mathcal{J}_1) \): \( e_i^0 \wedge e_j^0 \) \((1 \leq i_1 < i_2 \leq n)\) for the bases of degree \(-2\) and \( e_1, \ldots, e_{j+1} \) for the bases of degree \(-t\).
- In \( \mathcal{F}(\mathcal{J}_2) \): \( e_i^0 \wedge e_j^0 \wedge \lambda_1^0 \) \((k = 0, 1 \leq i_1 < i_2 \leq i_3 \leq n)\) for the bases of degree \(-3\), \( e_1^1 \wedge e_i^1 \) \((1 \leq i_1 < i_2 \leq n)\) for the bases of degree \(-t - 2\) and \( e_1^1 \wedge e_2^1 \) \((1 \leq i_1 \leq n, 2 \leq k \leq j + 1)\) for the bases of degree \(-t - 1\).
- In \( \mathcal{F}(\mathcal{J}_3) \): \( x_i^0 \wedge e_i^0 \wedge \lambda_1^0 \lambda_2^1 \) \((1 \leq i_1 < \cdots < i_4 \leq n, v_0 + v_1 = s - 1, v_0 > 0, v_1 > 0)\) for the bases of degree \(-s - 1\).
- In \( \mathcal{F}(\mathcal{J}_4) \): \( e_i^0 \wedge \cdots \wedge e_{i+s}^0 \wedge \lambda_1^0 \lambda_2^1 \) \((1 \leq i_1 < \cdots < i_5 \leq n, v_0 + v_1 = s - 2, v_0 > 0, v_1 > 0)\) for the basis of degree \(-t - s\).

Via computations by the mapping cone technique, we obtain the following proposition.

**Proposition 4.6.** As a \( P \)-module, \( P/L^*_t \) \((1 \leq j \leq n - 1, \ t \geq 2)\) has a free resolution \((\mathcal{F}(\mathcal{J}_t), \delta(\mathcal{J}_t)) \) of the form:
\[
\mathcal{F}(\mathcal{J}_t) : 0 \to P/(t - n - 1) \to P^{n-1}(-n-t) \oplus P^{(n-2)}(-t-1) \to P^{n-1}(-n) \oplus P^{(n-2)}(-n) \oplus P^{(n-3)}(-n) \oplus P^{(n-4)}(-n) \oplus P^{(n-5)}(-n) \oplus \cdots \to P^3(-2) \oplus P^2(-2) \oplus P^1(-2) \oplus P(1) \oplus P^0 \to P.
\]
where for $2 \leq s \leq n + 1$,

$$\mathcal{F}_{(j)} = p_{0}^{(s-1)}(-t) \oplus p_{(-s)}^{(n)}(-t) \oplus p_{0}^{(n-1)}(-t - s + 1).$$

The differential maps on the common maps with Theorem 4.4 act the same and for $2 \leq k \leq j + 1$:

- $\delta_{(j)}(e_{k}) = x_{n-k+1}x_{n-1}^{-1}$.
- For $1 \leq i \leq n$, $\delta_{(j)}(e_{i}) = -x_{n}^{s}e_{i}^{0} \wedge e_{n-k+2}^{0} + x_{i-1}e_{k-1} - x_{i}e_{k}$.
- For $s \geq 3$:

$$\delta_{(j)}(e_{i}^{k} \wedge \cdots \wedge e_{i-s+1}^{k}) = \sum_{l=1}^{k-1}(-1)^{l}x_{n}^{s-l}e_{i}^{0} \wedge \cdots \wedge e_{i-l}^{0} \wedge e_{n-k+s+1}^{0} \wedge \lambda_{0}^{-1}x_{n}^{l}^{-1} + (-1)^{s+k+1}e_{i}^{1} \wedge \cdots \wedge e_{i-s+2}^{1} \wedge \lambda_{0}^{-2}x_{n}^{s-1}$$

$$+ \wedge \cdots \wedge \Delta_{0}(e_{i}^{k-1} \wedge \cdots \wedge e_{i-s+1}^{k}) - \Delta_{0}(e_{i}^{k-1} \wedge \cdots \wedge e_{i-s+1}^{k})$$

where, in this formula $\Delta_{0}(e_{i}^{k-1} \wedge \cdots \wedge e_{i-s+1}^{k})$ is omitted if $k = 2$ and the summands with negative power on $\lambda_{0}$ or $\lambda_{1}$ are also omitted.

**Proof.** The proof is by induction on $j$. We compute an appropriate lifting $\psi_{(j)}$, for $0 \rightarrow [P / \langle x_{1}, \ldots, x_{n} \rangle](-t) \rightarrow P / I_{(n-j)}^{*}$.

The first step is $j = 1$.

Let $K_{(1)}$ be the Koszul complex generated by $x_{1}, \ldots, x_{n}$ shifted in degree by $-t$ and denote the bases of $K_{(1)}$, $1 \leq r \leq n$ by $e_{r}^{1} \wedge \cdots \wedge e_{r}^{2}$, $1 \leq i \leq i_{1} \leq \cdots \leq i_{s} \leq n$. The differential maps of $K_{(1)}$ are given by $\Delta_{1}(e_{i}^{1} \wedge \cdots \wedge e_{i}^{2})$. Suppose $(\mathcal{F}_{(0)}, \delta_{(0)})$ is the free resolution of $P / I_{(n)}^{*}$ described in Theorem 4.4. Define $\psi_{(1)}$ by multiplication by $x_{n-k}^{-1}x_{n}^{-1}$, $\psi_{(1)}(e_{r}^{1}) = -x_{n}^{-1}e_{r}^{0} \wedge e_{n-k}^{0} \wedge e_{i}^{1} \wedge e_{i}^{2} \wedge e_{n-k-i}^{0}$ and for $s \geq 2$, $\psi_{(1)}(e_{i}^{1}) = -x_{n}^{-1}e_{i}^{0} \wedge e_{n-k}^{0} \wedge e_{n-k-i}^{0} \wedge e_{i}^{1} \wedge e_{i}^{2} \wedge e_{n-k-i}^{0} \wedge e_{i}^{1} \wedge e_{i}^{2}$.

One can easily check that the above maps provide the lifting. So the mapping cone of this complex homomorphism implies the desired resolution for $j = 1$.

By induction hypothesis suppose for $1 \leq j \leq n - 1$, we have found the free resolution described for $P / I_{(n-j)}^{*}$ in the theorem. We prove the theorem for $j + 1$. We apply (2) for $j + 1$. So we have the following short exact sequence:

$$0 \rightarrow [P / \langle x_{1}, \ldots, x_{n} \rangle](-t) \rightarrow P / I_{(n-j-1)}^{*} \rightarrow P / I_{(n-j)}^{*} \rightarrow 0.$$
Using these properties, we can change the basis of \( \mathcal{F}_{(j)} \) and find a trivial subcomplex \( g_{(j)} \) of the form:

\[
0 \to P^i(-t - n) \to P^i(-t - n) \oplus P^n(n^{-1})(-t - n + 1) \to \cdots \to
\]

\[
P^{nq}(q)(-t + 3) \oplus P^{q}(q)(-t + 2) \to P^{q}(q)(-t + 2) \to 0 \to 0,
\]

(where for \( 3 \leq s \leq n \), \( g_{(j)} = P^{nq}(q)(-t - s + 1) \oplus P^{nq+1}(q)(-t - s) \), embedded in \( \mathcal{F}_{(j)} \) in such a way that \( \mathcal{F}_{(j)}/g_{(j)} \) is again a free complex. Since \( g_{(j)} \) is exact, the long exact sequence on homologies corresponding to the short exact sequence of complexes

\[
0 \to g_{(j)} \to \mathcal{F}_{(j)} \to \mathcal{F}_{(j)}/g_{(j)} \to 0
\]

shows that \( \mathcal{F}_{(j)}/g_{(j)} \) is a free resolution of \( P/I_{r,n-j}^* \). In this way we delete all the basis corresponding to invertible elements and the result is the minimal free resolution.

Using the above discussion, we can prove the Theorem 4.1. For the rest of this section we suppose that \( S \) is a numerical semigroup minimally generated by the generalised arithmetic sequence \( a_0, h a_0 + d, \ldots, h a_0 + nd \), where \( n \geq 2, a_0, d, h \) are positive integers and \( \gcd(a_0,d) = 1 \). Moreover, there exist integers \( t \geq 2 \) and \( 1 \leq r \leq n \) such that \( a_0 = (t - 1)n + r \). Let \( A = K[[t^{a_0}, \ldots, t^{a_0}]] \) and \( G = \text{gr}_m(A) \).

**Proof of Theorem 4.1.** Let \( h = 1 \). If \( r = n \), Theorem 4.4 shows the Betti numbers and if \( r < n \), then one can apply Proposition 4.6 and Remark 4.7.

If \( h > 1 \), considering Corollary 3.5, the defining ideal of \( G \) in \( P, I_{r,t}^* \), is

\[
\langle x_i'^t, x_{i+1}x_{n-1}^{-1}, \ldots, x_{n-1}^{-1}, x_1x_2, \ldots, x_1x_{n-1}^{-1} \rangle + \langle x_{i+1} - x_i, x_{i+1}x_i - x_{i+1} \rangle \mid 2 \leq i_1 < i_2 \leq n \}
\]

the element \( x_0 \) is \( P \)-regular and \( P/I_{r,t}^* \)-regular. Also,

\[
(P/I_{r,t}^*)/x_0(P/I_{r,t}^*) \cong P/(x_0 + I_{r,t}^*). \]

Therefore, the tensor product of the minimal free resolution of \( P/I_{r,t}^* \) with \( P/(x_0) \), provides the minimal free resolution of \( K[x_1, \ldots, x_6]/I_{r,t}^* \) as \( K[x_1, \ldots, x_6] \)-module (so as \( P \)-module). Thus, the Betti numbers of \( P/I_{r,t}^* \) and \( P/I_{r,t}^* \) are the same.

**Example 4.8.** Let \( n = 6 \) and \( a_0 = 22 \). The corresponding initial ideal to a semigroup generated by arithmetic sequence \( a_0 < \cdots < a_6 \), is \( I_{4,4} \), (note that \( 22 = 3 \times 6 + 4 \)). Thus, \( t = 4 \) and \( r = 4 \). By Theorem 4.1, the minimal free resolution of the associated graded ring is given by

\[
0 \to P^3(-10) \to P^3(-6) \oplus P^12(-9) \to P^{24}(-5) \oplus P^{15}(-8) \to P^{45}(-4)
\]

\[
\oplus P^{15}(-6) \to P^{40}(-3) \oplus P^{12}(-5) \to P^{15}(-2) \oplus P^3(-4) \to P.
\]

**Corollary 4.9.** If \( t = 2 \), i.e., \( n + 1 \leq a_0 \leq 2n \), then \( G \) has linear syzygies only for \( n - r + 1 \) steps, and for \( t > 2 \), \( G \) has no linear syzygy.

Now, we prove that the ring \( G \) is level. It is worthy to remark that if we start with local ring of an almost arithmetic sequence with Cohen–Macaulay associated graded ring, \( G \) is not necessarily level. Check for instance \( K[[t^8, t^{11}, t^{14}, t^{17}, t^{21}]] \).

Denote the Cohen–Macaulay type of a graded or local ring \( A \) by \( \text{type}(A) \).

**Corollary 4.10.** The ring \( G \) is level.

**Proof.** It is known that \( \text{type}(G) \) is equal to the dimension of the last syzygy module of \( G \). Therefore, Theorem 4.1 implies that for \( r > 1 \), \( \text{type}(G) = r - 1 \) and for \( r = 1 \), \( \text{type}(G) = n \). Hence the result can be concluded by Corollary 3.5.

Let \( I^* \) be the ideal defining \( G \) and \( I \) be the ideal defining \( A \).

**Corollary 4.11.** 1. \( \mu(I^*) = \mu(I) = \binom{n}{r} + n - r + 1 \).

2. \( \text{type}(A) = \text{type}(G) = \begin{cases} n \quad &\text{if } r = 1, \\ n - 1 \quad &\text{otherwise}. \end{cases} \)

3. \( A \) is Gorenstein if and only if \( G \) is Gorenstein if and only if \( r = 2 \).

4. \( A \) is complete intersection if and only if \( G \) is complete intersection if and only if \( n = r = 2 \).

**Proof.** For \( G \) it is enough to consider Theorem 4.1 and for \( A \) we refer to Theorem 3.3.7 and 3.4.6 of [14] to control 1 and 2. Conditions 3 and 4 are easy consequences of 1 and 2. See also [6,9].
It is clear that from Theorem 4.1, we also have total Betti numbers of graded ring $G$. We show (total) Betti numbers of $A$ and $G$ by $\beta_i(A)$ and $\beta_i(G)$.

Let $R = k[x_0, \ldots, x_n]$. There is a general result due to Robbiano (see [17] and also [8]) which says that from a minimal $P$-free resolution of the associated graded ring $G$, we can build up an $R$-free resolution for the local ring $A$, which is not necessarily minimal. Hence, for the Betti numbers of $A$ and $G$ one has

$$\beta_i(A) \leq \beta_i(G).$$

The resolution of $A$ is minimal if and only if $\beta_i(A) = \beta_i(G)$ for each $i \geq 0$. In this case $A$ is called of homogeneous type. In next corollary we show that there are some classes of homogeneous type local rings corresponded to some generalized arithmetic sequences.

**Corollary 4.12.** Let $A$ be the one dimensional local domain corresponding to a generalized arithmetic sequence. Then, $A$ is of homogeneous type provided that $t = 2$ or $n \leq 3$.

**Proof.** For $t = 2$ see Proposition 1.13 of [19]. For $n = 2, 3$, it is enough to apply the Corollary 4.11. □

We remark that the Betti numbers of $A$, just in the case of arithmetic sequence and $n = 3$ have been computed in [21].

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**References**