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Boundary Value Problems for the Fitzhugh–Nagumo Equations*

MARIA ELENA SCHONBEK

Northwestern University, Department of Mathematics, Evanston, Illinois 60201

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This paper is concerned with the study of the FitzHugh–Nagumo equations. These equations arise in mathematical biology as a model of the transmission of electrical impulses through a nerve axon; they are a simplified version of the Hodgkin–Huxley equations. The FitzHugh–Nagumo equations consist of a non-linear diffusion equation coupled to an ordinary differential equation. $v_t = v_{xx} + f(v) - u$, $u_t = \sigma v - \gamma u$. We study these equations with either Dirichlet or Neumann boundary conditions, proving local and global existence, and uniqueness of the solutions. Furthermore, we obtain L_∞ estimates for the solutions in terms of the L_1 norm of the boundary data, when the boundary data vanish after a finite time and the initial data are zero. These estimates allow us to prove exponential decay of the solutions.

INTRODUCTION

We consider two initial boundary value problems for the system of partial differential equations:

$$\begin{aligned} v_t &= v_{xx} + f(v) - u, \qquad x \geqslant 0, \quad t \geqslant 0; \\ u_t &= \sigma v - \gamma u, \qquad x \geqslant 0, \quad t \geqslant 0; \qquad \gamma > 0, \quad \sigma \geqslant 0 \end{aligned} \tag{1.1}$$

where, qualitatively, the graph of f is as pictured in Fig. 1.

The system (1.1) is an ordinary differential equation coupled to a non-linear diffusion equation. The boundary conditions at $x = 0$ are only given on v. These equations arise as models of the conduction of electrical impulses in a nerve axon. The first such model appeared in 1952 in the work of Hodgkin and Huxley [6]. The form we are using was proposed afterwards by FitzHugh-Nagumo. (For a discussion of this model see [5].)

This paper can be divided into three parts. In the first part we treat the system (1.1) with Dirichlet boundary conditions at $x = 0$. We prove global existence and uniqueness of the solution. To pass from locally to globally defined

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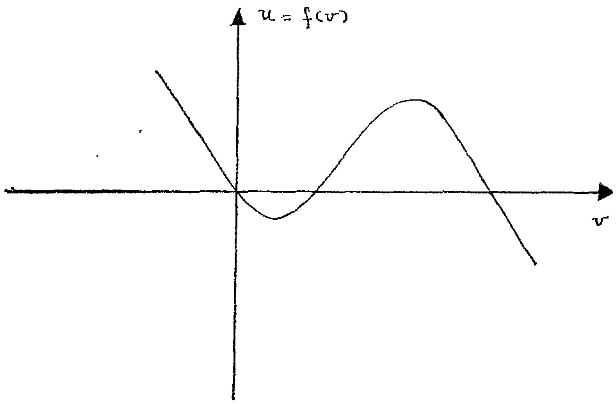

FIGURE 1

solutions we use the invariant regions found by Conley and Smoller, and the method of contracting rectangles developed by Rauch and Smoller in [7].

The main sections of part I are 5, 6, and 7; they deal with the threshold problem for the Fitz Hugh-Nagumo equations (1.1). Numerical and biological evidence seems to indicate that a strong stimulus of short duration, or a weak stimulus of long duration, is sub-threshold. We show that the L_1 norm of the stimulus is one of the critical parameters. We prove that if the initial data are zero and the boundary data $v(t, 0) = h(t)$ have finite sup norm and vanish outside of some interval $[0, T]$, then our solution is bounded, for all $t \geqslant 0$, by a constant (depending on $\| h \|_\infty$ and T) times the total stimulus, $\int_0^T | h(t)| \, dt$. Furthermore, we show that if the total stimulus is sufficiently small, the solution has exponential decay. This proves a conjecture of S. P. Hastings [5]. More precisely we estimate each of the coordinates of $U = (v, u)$. We consider the first coordinate v as the solution to an inhomogeneous heat equation with $f(v) - u$ as the known inhomogeneous term. We employ the integral representation for such solutions, this gives rise to two terms, one due to the boundary data and the other due to the inhomogeneous part $f(v) - u$. The main step is to analyze the contribution from the inhomogeneous part. We do this by establishing a Gronwall type inequality.

In order to analyze the second coordinate u, we solve the ordinary differential equation explicitly and use methods similar to those used to analyze the first coordinate.

Additional information is obtained by energy estimates. Under conditions similar to those in the preceding paragraph, standard multiplier methods are used to show that if $v(t, x)$ is less than the first positive zero of f, for all $t \geqslant T$, $x \geqslant 0$, then the solution $U = (v, u)$ decays exponentially in $L_2 \cap L_\infty$.

Experiments have not made clear what the correct boundary conditions for (1.1) are. Rauch and Smoller studied the Dirichlet problem in [7], but no results were known for the Neumann problem. Part II is devoted to the study of the Fitz Hugh-Nagumo equations (1.1) with Neumann boundary conditions at $x = 0$. First we show the existence of a solution for small t. To construct global solutions we obtain an a priori estimate by comparing the solution with the solution $\Phi(t, x) = (\phi(t, x), \alpha(t, x))$ of the "linear Fitz Hugh-Nagumo equations" $(f(v) = 0)$ with the same initial-boundary data. The difference of the two solutions is a function which satisfies equations similar to (1.1), where $f(v)$ is replaced by $g(v) = f(v + \phi)$. Now the initial and boundary conditions are zero, which enables us to use an argument involving contracting rectangles to obtain the desired estimate.

In the last sections of II, we discuss the threshold problem for the Fitz Hugh-Nagumo equations with Neumann data. Under hypotheses analogous to those for the Dirichlet problem (replacing $v(t, 0)$ by $v_x(t, 0)$) we get, (by similar techniques), the same threshold results. In particular, $\int_0^T |v_x(t, 0)|\, dt$ is the critical parameter. Part III is a short note on the threshold results for the Hodgkin and Huxley equations with zero initial data and compactly supported Dirichlet boundary data. The methods we use are essentially the same as for the Fitz Hugh-Nagumo equations.

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I. The Dirichlet Problem

§2. *Global Theory and Local Solvability*

§2a. Consider the following non linear systems of equations in two independent variables (t, x).

$$U_t = AU_{xx} + F(U), \qquad x \geqslant 0, \quad t \geqslant 0, \tag{2.1}$$

where:

$U = (u_1, ..., u_n)$ is a real n-vector;

F is a smooth R^n valued function with $F(0) = 0$;

A is a non negative diagonal matrix.

$A = \operatorname{diag}\{a_1, ..., a_n\}$,

$$a_i > 0, \qquad 1 \leqslant i \leqslant p,$$
$$a_i = 0, \qquad p < i < n.$$

Let $K_i(t, x) = \operatorname{diag}\{k_1(t, x), ..., k_n(t, x)\}$,

$$k_i(t, x) = (4a_i\pi t)^{-1/2} \exp(-x^2/4a_i t) \qquad \text{if} \quad 1 \leqslant i \leqslant p.$$

where

$$k_i(t, x) = \delta(x), \qquad \text{for} \quad p < i \leqslant n.$$

In what follows we will use the notation

BC for the space of bounded continuous functions on $\bar{R}_+$.

C_0 for the space of continuous function on $\bar{R}_+$ which tend to zero as x tends to infinity.

B for any of the following Banach spaces (with the obvious norm, $\| \|_B$)

$BC^k = \{w: (d/dx)^j w \text{ is in } BC \text{ for } 0 \leqslant j \leqslant k\}\ k \geqslant 0$

$BC^0 \cap L_p \quad p \geqslant 1$

$W_p{}^k = \{w \in L_p: (d/dx)^j w \in L_p \text{ for } 0 \leqslant j \leqslant k\}\ k \geqslant 1$

$BC_0{}^k = \{w \in BC^k, \lim (d/dj)^j w = 0 \text{ as } x \to \infty \text{ for } 0 \leqslant j \leqslant k\}$

(see [7] for a further discussion.)

For $h = (h_1, ..., h_p) \in BC$ and $g = (g_1, ..., g_n) \subset B$, let

$$\tilde{K}_i(t, z, x) = K_i(t, z - x) - K_i(t, -z - x), \qquad 1 \leqslant i \leqslant n.$$

$$H_i(t, x) = -2 \int_0^t h_i(s) \frac{\partial \tilde{K}_i}{\partial z}(t - s, z, x) \Big|_{z=0} ds, \qquad 1 \leqslant i \leqslant p.$$

$$H_i(t, x) = 0, \qquad p < i \leqslant n.$$

$$S_i(t, x) = \int_0^\infty g_i(z) \tilde{K}_i(t, z, x)\, dz, \qquad 1 \leqslant i \leqslant p.$$

$$S_i(t, x) = g_i(x), \qquad p < i \leqslant n.$$

$$S_i(0, x) = g_i(x), \qquad 1 \leqslant i \leqslant n.$$

We recall that $H_i(t, x)$, $1 \leqslant i \leqslant p$, is the solution of the heat equation with Dirichlet boundary condition h_i and zero initial data; $S_i(t, x)$, $1 \leqslant i \leqslant p$, the solution to the heat equation with initial data g_i and boundary data zero. We also note that $S_i(t, x)$ is the restriction to $x > 0$ of the solution to the Cauchy problem for the heat equation on $-\infty < x < \infty$ with initial data of the form $g_i(x)$, for $x > 0$ and $-g_i(-x)$, for $x < 0$.

In the following we will also let

$$\| U(t)\|_\infty = \max_i \| u_i(t)\|_\infty = \max_i \sup_x | u_i(t, x)|$$

$$\| U(t)\|_p = \max_i \| u_i(t)\|_p = \max_i \left(\int_0^\infty | u_i(t, x)|^p\, dx \right)^{1/p}$$

$$\tilde{K} = (\tilde{K}_1, ..., \tilde{K}_n),$$

$$S = (S_1, ..., S_n),$$

$$H = (H_1, ..., H_n).$$

§2b. One can show that a function $U \in C([0, T] \mid B)$ satisfies the system (2.1), in the sense of distributions, with initial and boundary data:

$$\begin{aligned} u_i(0, x) &= g_i(x), \qquad 1 \leqslant i \leqslant n, \quad x \geqslant 0; \\ u_i(t, 0) &= h_i(t), \qquad 1 \leqslant i \leqslant p, \quad t \geqslant 0; \end{aligned} \tag{2.2}$$

if and only if

$$U(t, x) = R(t, x) + \int_0^t \int_0^\infty \tilde{K}(t - s, z, x)\, F(U(s, z))\, dz\, ds, \tag{2.3}$$

$$\begin{aligned} &R_t - AR_{xx} = 0 \\ &R_i(0, x) = g_i(x), \qquad 1 \leqslant i \leqslant n; \\ &R_i(t, 0) = h_i(t), \qquad 1 \leqslant i \leqslant p. \end{aligned}$$

Observations: $R(t, x)$ is given by the explicit formula

$$R(t, x) = H(t, x) + S(t, x).$$

The proof of the only if part is a familiar application of Green's identity with $\tilde{K}$ as one of the entries. A similar argument is given in [1, p. 104], and the proof of the other implication is standard.

The Banach contraction theorem allows us to solve the integral equation (2.3) for a short time interval, the length of which depends only on F, and on the sup norms of the initial and boundary data. More precisely, the following theorem is true.

THEOREM (2.1). *For any $h \in BC$, $g_i \in B$, $1 \leqslant i \leqslant p$, and $g_j \in C^\infty \cap B$, $p + 1 \leqslant j \leqslant n$, with $g(0) = h(0)$, there exists a constant $t_0 > 0$, which depends only on F, $\| g \|_\infty$, and $\| h \|_\infty$, such that the Dirichlet problem for equation* (2.1) *with initial data $U(0, x) = g(x)$ and boundary data $u_i(t, 0) = h_i(t)$, $1 \leqslant i \leqslant p$, has a unique solution U in $C([0, t_0] \mid B_0)$ and $\| U \|_{C([0,t_0] \| B_0)} \leqslant 2(2 \| h \|_\infty + \| g \|_B)$.*

Proof. We refer the reader to J. Rauch and J. Smoller [7], where a similar proof is used to show local existence and uniqueness of weak solutions of the pure initial value problem.

To show that the solutions are smooth, the following regularity theorem can be used.

THEOREM (2.2). *Suppose $U = (u_i, \ldots, u_n) \in C([0, t_0) \mid C_0)$ is a solution of* (2.1), *with initial and boundary conditions* (2.2)

If (1) *$g_i \in C^0(\bar{R}_+)$, $i = 1, \ldots, p$,* (2) *$g_i \in C^\infty(\bar{R}_+)$, $i = p + 1, \ldots, n$ and* (3) *$h_i \in BC(\bar{R}_+)$, $i = 1, \ldots, p$ are satisfied.*

Then $U \in C^\infty(\Omega)$, where $\Omega = (0, \infty) \times R_+$.

Proof. Follows by repeatedly differentiating the integral equation which describes the solution. For details we refer the reader to [4].

§3. *Contracting Rectangles (a general reference for this section is* [7])

DEFINITION (3.1). A bounded convex set $R \subset R^n$ is contracting for the vector field $F(U)$ if for every point $U \in \partial R$ and every outward unit normal n at $U, F(U) \cdot \eta < 0$.

In the proofs of theorems of global existence, stability and asymptotic behavior of solutions of (1.1), an essential part is played by rectangles which are contracting for the vector field $F = (f(v) - u, \sigma v - \gamma u)$, where f is as described in the introduction.

Below we state three technical lemmas which will be needed in the sequel. The proofs can be found in [7].

LEMMA (3.1). *For the linear vector field* $F_L(U) = (-\beta v - u, \sigma v - \gamma u)$, $\gamma, \beta > 0$, $\sigma \geqslant 0$, *there is contracting rectangle containing* 0 *if and only if* $\beta > \sigma/\gamma$.

Assume that $-f'(0) > \sigma/\gamma$ and $\{v: f(v) = -(\sigma/\gamma)v,\ v \neq 0\}$ is non-void. Let $v_c = \min\{|v|: f(v) = -(\sigma/\gamma)v, v \neq 0\}$. Let R_c be the rectangle symmetric in the u and v axes with upper right hand corner at the point $(v_c, (\sigma/\gamma)v_c)$. Then we have:

LEMMA (3.2). *Suppose that* $F(u) = (f(v) - u, \sigma v - \gamma u)$, $-f'(0) > \sigma/\gamma$ *and* R_c *is the rectangle described above. For any compact set* Q *in the interior of* R_c, *there is a rectangle* R *and a constant* $k > 0$ *such that* $Q \subset R \subset R_c$ *and* $F(U) \cdot n < -\kappa\tau$ *for all* $\tau \in [0, 1]$, $U \in \partial(\tau R)$ *and outward unit normals* n *at* U.

Let f satisfy the growth condition,

$$\liminf \left| \frac{f(v)}{v} \right| > \frac{\sigma}{\gamma}, \qquad \text{as } |v| \to \infty. \tag{3.1}$$

Let $v^c = \max\{|v|: f(v) = -(\sigma/\gamma)v\}$, and R^c be the rectangle symmetric in the u and v axes with upper right hand corner at $(v^c, (\sigma/\gamma)v^c)$.

LEMMA (3.3). *Suppose* f *is a smooth function which satisfies* (3.1), $F(U) = (f(v) - u; \sigma v - \gamma u)$. *Then for any compact set* Q *in the exterior of* R^c, *there is a rectangle* R *such that* $R^c \subset R$, Q *is in the exterior of* R *and* τR *is contracting for* f, *for* $1 \leqslant \tau < \infty$.

DEFINITION (3.2). For $U \in R^n$ and R a rectangle in R^n with $0 \in R$, the norm $|\ |_R$ on R^n is defined by

$$|U|_R = \inf\{t \geqslant 0: U \in tR\},$$

Define the norm ν_R on BC as:

$$\nu_R(W) = \sup_{x \in R} | W(x)|_R .$$

Recall the definition of Dini derivatives.

$$\bar{D}\psi(t) = \limsup_{h \to 0} \frac{\psi(t+h) - \psi(t)}{h} .$$

For the Dirichlet problem we have the following result.

LEMMA (3.4). *Let $F(U)$ be a vector field on R^n and let R be a rectangle with $0 \in \mathrm{int}(R)$. Suppose that $U \in C(T - \delta, T + \delta) \mid C_0)$ is a smooth solution of* (1.1) *for $| T - t | < \delta$ and that $\nu_R(U(T)) = s$. Let $\nu_{sR} U(T, 0) < 1$.*

If there is an $\eta > 0$ such that, for any $W \in \partial(sR)$, and $n(W)$ normal to $\partial(sR)$ at W, we have $F(W) \cdot n(W) < -\eta$, then

$$\bar{D}\nu_R(U(T)) \leqslant \frac{-2\eta}{L} \nu_R(U(T)),$$

where L is the length of the shortest side of R.

Proof. We refer the reader to the Basic Lemma of [7] which has a similar proof. Our additional condition, $\nu_{sR}(U(T, 0)) < 1$, is needed to insure that $U(T, 0) \notin \partial sR$. □

We conclude this section with several definitions and remarks which will be needed in the sequel.

DEFINITION (3.3). We say that a convex region R is an invariant region for the Cauchy problem with initial value $U(0, x) = U_0(x)$ if $U_0(x) \in R$ for all $-\infty < x < \infty$ implies $U(t, x) \in R$ for all $t \geqslant 0$ and $-\infty < x < \infty$.

DEFINITION (3.4). We say that a convex region R is an invariant region for the Dirichlet problem with initial value $U(0, x) = U_0(x)$ and boundary value $U(t, 0) = h(t)$ if $U_0(x) \in R$ for all $x \geqslant 0$ and $h(t) \in R$ for all $t \geqslant 0$, imply $U(t, x) \in R$ for all $x \geqslant 0$ and $t \geqslant 0$.

Observation 1. Lemmas (3.2) and (3.3) give us two families of contracting rectangles for the solution to the Cauchy problem for the Fitz Hugh-Nagumo equations, namely:

i. Small rectangles $\{\tau R\}$, with $R \subset R_c$ and $\tau \in (0, 1]$.
ii. Large rectangles $\{\lambda Q\}$, with $R^c \subset Q$ and $\lambda \in [1, \infty)$.

By the Basic Lemma of [7] these rectangles are invariant regions for the solution to the Cauchy problem.

Observation 2. By Lemma (3.4) the two families of rectangles $\{\tau R\}$, $\{\lambda Q\}$, are also invariant regions for the solution to the Dirichlet problem.

§4. *Global Existence*

Consider the following system of equations

$$\begin{aligned} v_t &= v_{xx} + f(v) - u, & x \geqslant 0, t \geqslant 0, \\ u_t &= \epsilon u_{xx} + \sigma v - \gamma u, & x \geqslant 0, t \geqslant 0, \epsilon \geqslant 0. \end{aligned} \tag{4.1}$$

THEOREM (4.1). *Let $g \in B \cap C_0$ and $h \in BC$, with $g(0) = h(0)$, and let f be as in Lemma* (3.3). *Then the system of equations* (4.1) *has a unique solution in $C([0, \infty) \mid (B \cap C_0)$ with initial and boundary conditions as follows:*

1. *If $\epsilon \neq 0$.*

$$\begin{aligned} U(0, x) &= g(x) = (g_1(x), g_2(x)), \\ U(t, 0) &= h(t) = (h_1(t), h_2(t)). \end{aligned} \tag{4.2}$$

2. *If $\epsilon = 0$ the second equation in* (4.2) *is replaced by*

$$v(t, 0) = h(t) \qquad \textit{(here } h(t) \textit{ is a real valued function, and we impose no boundary condition on } u\textit{)}.$$

Proof. By Lemma (3.3) we can choose a sufficiently large rectangle R such that R is contracting for the vector field

$$\begin{aligned} &F(U) = (f(v) - u, \sigma v - \gamma u) \text{ and} \\ &\nu_R(g(x)) < 1 \text{ for all } x \geqslant 0, \text{ and} \\ &\nu_R(h(t)) < 1 \text{ for all } t \geqslant 0. \end{aligned}$$

By Theorem (2.1) we get a solution $U \in C([0, t_0] \mid B)$ of (4.1) with initial and boundary conditions (4.2).

Claim. $\nu_R[U(t)] < 1$ for $0 < t < t_0$.
If this is not true, let $\bar{t} = \inf\{t \in (0, t_0) \mid \nu_R U(t) = 1\}$. Thus $\bar{t} \geqslant 0$, by continuity of $\nu_R U(t)$.

$\nu_R U(\bar{t}) > \nu_R(U(\bar{t}, 0))$ since $\nu_R U(\bar{t}, 0) < 1$ by construction of R. Therefore, by Lemma (3.4),

$$\bar{D}\nu_R U(\bar{t}) < 0.$$

Thus for any $t \in (\bar{t} - \epsilon, \bar{t})$ we have $\nu_R U(t) > 1$, which contradicts the definition of $\bar{t}$.

The estimate $\nu_R U(t) < 1$ for $t \in [0, \text{to}]$ is the sup norm estimate we need to extend U from a local solution to a global solution with $\nu_R U(t) < 1$ for all $t \geqslant 0$.

Uniqueness follows from the uniqueness of the local solutions. □

THEOREM (4.2). (*Sup Norm Estimate for the Dirichlet Problem with Compactly Supported Boundary Data*). *If* $U(t, x) \in C([0, \infty) \mid C_0)$ *is a solution of the Fitz Hugh–Nagumo equations with Dirichlet boundary data* $h(t) \in BC$, *and* $h(t) = 0$ *for* $t \geqslant T$, *we have*

$$\| U(t)\|_\infty \leqslant \text{const} \| U(T)\|_\infty , \qquad \textit{for all} \quad t \geqslant T.$$

Proof. By observation 2, of Section 3, we know that there exists a family of small invariant rectangles $\{\tau R\}$, and a family of large invariant rectangles $\{\lambda \underset{\sim}{Q}\}$, for the solution of the Dirichlet problem, such that for all $x \geqslant 0$, $U(T, x)$ lies in one of these rectangles, which is sufficient to establish the estimate:

$$\| U(t)\|_\infty \leqslant \text{const} \| U(T)\|_\infty \qquad t \geqslant T. \qquad \square$$

§5. *The Treshold Problem*

We discuss a model of a semi-infinite nerve stimulated at $x = 0$. Numerical and biological evidence supports the conjecture that a strong stimulus of short duration, or a weak stimulus of long duration is subthreshold. In sections 5 and 6 we show that the first part of the conjecture is correct. The second part was proven in [7].

THEOREM (5.1). *Let* f *be a smooth function which satisfies*

1. $f(0) = 0$
2. $-f'(0) > \sigma/\gamma$
3. $\liminf | f(v)/v | > \sigma/\gamma$ *as* $| v | \to \infty$.

Suppose $h \in BC$ *satisfies*

4. $h(t) = h(0) = 0$ *for all* $t \geqslant t_0 \geqslant 0$
5. $\| h \|_\infty \leqslant M$,

let $U(t, x) = (v(t, x), u(t, x)) \in C([0, \infty) \mid C_0)$ *be the unique solution of*

$$\begin{aligned} v_t &= v_{xx} + f(v) - u \qquad && x \geqslant 0, t \geqslant 0 \\ u_t &= \sigma v - \gamma u && x \geqslant 0, t \geqslant 0 \end{aligned} \tag{5.1}$$

$$\begin{aligned} v(0, x) &= u(0, x) = 0 \qquad && x \geqslant 0 \\ v(t, 0) &= h(t) && t \geqslant 0. \end{aligned} \tag{5.2}$$

Denote by $F = (F_1 , F_2)$ *the vector field* $(f(v) - u, \ \sigma v - \gamma u)$. *Then for any* $T > t_0 \geqslant 0$ *there exists a constant* $k = k(T, t_0 , M, F)$, *growing at most like* $\max\{1/(T - t_0), \exp T\}$, *such that*

$$\| U(t, \cdot)\|_\infty \leqslant k \| h \|_1 \qquad \textit{for all} \quad t \geqslant T.$$

Proof. By Theorem (4.1) we know that (5.1) has a unique solution $U = (v, u)$ which satisfies

$$| U(t, x)| \leqslant \tilde{k} \qquad \text{for all} \quad x \geqslant 0, t \geqslant 0, \qquad \text{where} \quad \tilde{k} = \tilde{k}(F, M). \tag{5.3}$$

The coordinates of U satisfy

$$v(t, x) = H(t, x) + \int_0^t \int_0^\infty \tilde{K}(t-s, z, x) F_1(U(s, z))\, dz\, ds, \tag{5.4}$$

$$u(t, x) = \int_0^t F_2(U(s, x))\, ds = \sigma \int_0^t v(s, x)\, e^{-\gamma(t-s)}\, ds, \tag{5.5}$$

where $H(t, x) = -1/\pi^{1/2} \int_0^t h(s)/((t-s)^{-3/2})x \exp - [x^2/4(t-s)]\, ds$.

Since F is smooth and U verifies (5.3), there exists a constant $\bar{k} = \bar{k}(M, F)$ such that

$$| FU(t, x)| \leqslant \bar{k}\, | U(t, x)|, \qquad \text{for all} \quad x \geqslant 0. \tag{5.6}$$

Thus, by (5.4), (5.5) and (5.6), we have, for all $x \geqslant 0$, $t \geqslant 0$,

$$\begin{aligned} | U(t, x)| &\leqslant | v(t, x)| + | u(t, x)| \\ &\leqslant | H(t, x)| + \bar{k} \int_0^t \int_0^\infty \tilde{K}(t-s, z, x\, | U(s, z)|\, dz\, ds \\ &\quad + \bar{\bar{k}} \int_0^t | U(s, x)|\, ds. \end{aligned} \tag{5.7}$$

We observe that by Theorem (4.2), we only need to show

$$\| U(T)\|_\infty \leqslant \text{const}(T, M, F, t_0)\, \| h \|_1 \tag{5.8}$$

or equivalently

$$\begin{aligned} &\text{a.} \quad \| v(T)\|_\infty \leqslant \text{const}(T, M, F, t_0)\, \| h \|_1 \\ &\text{b.} \quad \| u(T)\|_\infty \leqslant \text{const}(T, M, F, t_0)\, \| h \|_1 . \end{aligned} \tag{5.9}$$

To prove (5.9).a, we estimate the first two terms on the right hand side of (5.7) for $t = T$. Bound for the first term:

$$\| H(T, x)\| \leqslant \frac{\text{const}}{(T-t_0)} \int_0^{t_0} | h(s)|\, ds \leqslant \text{const}(T, t_0)\, \| h \|_1 \tag{5.10}$$

this follows from

$$z^\alpha \exp(-z^2) \leqslant \text{const} \exp(-z^2/2) \leqslant \text{const}, \qquad \text{for any} \quad z, \alpha \geqslant 0. \tag{5.11}$$

To estimate the second term we observe that

$$\int_0^T \int_0^\infty \tilde{K}(T-s, z, x)\, | U(s, z)|\, dz\, ds \leqslant \text{const} \int_0^T \int_0^\infty \frac{| U(s, z)|}{(T-s)^{1/2}}\, dz\, ds. \tag{5.12}$$

Therefore we only need to show

$$\int_0^T \int_0^\infty \frac{|U(s, z)|}{(T-s)^{1/2}} \, dz \, ds \leqslant \text{const}(T, M, P, t_0) \| h \|_1 . \tag{5.13}$$

For this we require the preliminary result

$$\int_0^T \int_0^\infty | U(s, x)| \, dx \, ds \leqslant \text{const}(T, F, M) \| h \|_1 , \tag{5.14}$$

which is obtained by means of a Gronwall inequality.

Integral representations (5.4), (5.5), yield for $0 \leqslant t \leqslant T$

$$\begin{aligned} \int_0^t \int_0^\infty | U(s, x)| \, dx \, ds &\leqslant \frac{1}{\pi^{1/2}} \int_0^t \int_0^\infty \int_0^s \frac{|h(q)|}{(s-q)^{3/2}} x \exp\left(\frac{-x^2}{4(s-q)}\right) dq \, dx \, ds \\ &\quad + k \int_0^t \int_0^\infty \int_0^s \int_0^\infty | \tilde{K}(s-q, z, x)(U(q, z))| \, dz \, dq dx ds \\ &\quad + k \int_0^t \int_0^\infty \int_0^s | U(q, x)| \, dq \, dx \, ds \\ &= I(t) + II(t) + III(t). \end{aligned} \tag{5.15}$$

Changing the order of integration in $H(T)$ and integrating over x first we get

$$II(t) + III(t) \leqslant \text{const} \int_0^t \int_0^\infty \int_0^s | U(q, x)| \, dq \, dx \, ds. \tag{5.16}$$

We observe that

$$\begin{aligned} I(t) &\leqslant \text{const} \int_0^t | h(q)| \int_q^t \frac{1}{(s-q)^{1/2}} \int_0^\infty \frac{-2x}{4(s-q)} \exp - \left(\frac{x^2}{4(s-q)}\right) dx \, ds \, dq \\ &\leqslant \text{const}(T^{1/2}) \| h \|_1 . \end{aligned} \tag{5.17}$$

Now in view of (5.16) and (5.17) we apply Gronwall's inequality to (5.15), establishing the estimate (5.14).

Let us return to (5.13).

By (5.4), (5.5) and the same argument used to obtain (5.6) we have

$$\begin{aligned} &\int_0^T \int_0^\infty \frac{| U(s, x)|}{(T-s)^{1/2}} \, dx \, ds \\ &\quad \leqslant \text{const} \int_0^T \int_0^\infty \int_0^s \frac{| h(q)|}{(T-s)^{1/2}} \frac{x}{(s-q)^{3/2}} \exp - \left(\frac{x^2}{4(s-q)}\right) dq \, dx \, ds \\ &\quad\quad + \text{const} \int_0^T \int_0^\infty \int_0^s \frac{| U(q, x)|}{(T-s)^{1/2}} \, dq \, dx \, ds = IV(T) + V(T). \end{aligned} \tag{5.18}$$

By Fubini's theorem,

$$\begin{aligned}
&IV(T)\\
&= \text{const} \int_0^T | h(q)| \int_q^T \frac{1}{(T-s)^{1/2}} \frac{1}{(s-q)^{1/2}} \int_0^\infty \frac{x}{(s-q)} \exp\left(\frac{-x^2}{4(s-q)}\right) dx\, ds\, dq\\
&\leqslant \text{const} \int_0^T | h(q)| \int_q^T \frac{1}{(T-s)^{1/2}} \frac{1}{(s-q)^{1/2}} ds\, dq\\
&\leqslant \text{const} \int_0^T | h(q)|\, dq. \qquad (5.19)
\end{aligned}$$

Observe that by (5.14)

$$V(T) \leqslant \text{const}(T, t_0, M, F) \| h \|_1 \int_0^T \frac{1}{(T-s)^{1/2}} ds. \qquad (5.20)$$

Combining (5.18), (5.19) and (5.20) we get the desired estimate (5.13), which together with (5.12) and (5.10) yields (5.9).a. □

To prove the inequality (5.9).b we observe that by (5.5) and the integral representation of v we have

$$\begin{aligned}
| u(T, x)| &\leqslant \sigma \int_0^T | v(s, x)|\, ds\\
&\leqslant \int_0^T \int^s \frac{| h(q)|}{(s-q)^{3/2}}\, x \exp\left(\frac{-x^2}{4(s-q)}\right) dq\, ds\\
&\quad + k \int_0^T \int_0^s \int_0^\infty \tilde{K}(s-q, z, x) \mid U(s, z)|\, dz\, dq\, ds\\
&\leqslant \text{const} \int_0^T | h(q)| \int_q^T \frac{x}{(s-q)^{3/2}} \exp\left(\frac{-x^2}{4(s-q)}\right) ds\, dq\\
&\quad + k \int_0^T \int_0^s \int_0^\infty \frac{| U(q, z)|}{(s-q)^{1/2}} dz\, dq\, ds. \qquad (5.21)
\end{aligned}$$

Combining (5.13) and (5.21) we get inequality (5.9).b, completing the proof. □

§6. *Stability by Contracting Rectangles*

In this section we study the stability of the zero solution. We show that for solutions with zero initial data, whose boundary value vanishes outside a finite interval, $[0, t_0]$, and for which $\int_0^{t_0} | h(t) |\, dt$ is sufficiently small, we have exponential decay. This proves a conjecture of S. P. Hastings [5]. More precisely we have the result below.

THEOREM (6.1). *If conditions* 1 *to* 5 *of Theorem* (5.1) *hold, there exist constants* c, k, *and* λ, *such that if*

$$\| h \|_1 \leqslant \lambda,$$

then

$$\| U(t, \cdot)\|_\infty \leqslant k \exp(-ct), \qquad t \geqslant 0$$

where k *and* λ *depend on* T, t_0, M *and* F, *and* c *depends only on* F.

Proof. It suffices to show that for $t \geqslant t_0$, $x \geqslant 0$, there exists an invariant rectangle $R \subset R_c$ (see Lemma (3.2)) contracting for the vector field $(f(v) - u, \sigma v - \gamma u)$, with the additional property:

$$\bar{D}\nu_R U(t) \leqslant -c\nu_R U(t). \tag{6.1}$$

To construct R, recall that by Theorem (5.1)

$$\| U(t)\|_\infty \leqslant k \| h \|_1, \qquad t > t_0$$

thus if $\| h \|_1$ is sufficiently small, there is a compact set $Q \subset \operatorname{int} R_c$, such that $U(t, x) \subset Q$. Now by Lemma (3.2) there is a contracting rectangle R for the vector field $F(U) = (f(v) - u, \sigma v - \gamma u)$ and $\nu_R(U(t)) < 1$ for $t > t_0$.

We divide the proof that R has property (6.1) into two cases. Suppose $t > t_0$:

1. If t is such that $\nu_R U(t) > \nu_R U(t, 0)$ we have by Lemma (3.4)

$$\bar{D}\nu_R U(t) \leqslant -\frac{2\eta}{L}\nu_R U(t) \tag{6.2}$$

(L the shortest side in R).

2. If t is such that $\nu_R U(t) = \nu_R U(t, 0)$, let $\nu_R U(t) = s$ and set

$$X = \{x : U(t, x) \in \partial sR\}.$$

Remark:

1. X is not empty, since $x = 0$ is in X.

2. X is compact. We know that $\lim_{x\to\infty} U(t, x) = 0$ so X is bounded, and it is obvious that X is closed.

Let $\theta = \theta_1 \cup \{0\} \supset X$, where θ is a bounded neighborhood of X, and $0 \notin \theta_1$. Since for $t \geqslant t_0$

$$U(t + h, 0) = e^{-\gamma h} U(t, 0)$$

and

$$\bar{D}\nu_R U(t + h, 0) = -\gamma\nu_R U(t, 0) < -\frac{\gamma}{2}\nu_R U(t, 0) = -\frac{\gamma}{2} s.$$

we have for small $| h |$, $h \neq 0$,

$$\nu_R U(t + h, 0) \leqslant s\left(1 - \frac{\gamma h}{2}\right). \tag{6.3}$$

By a result of [7] we know that if:

1. $|h|$ is small and $\theta \in \theta_1$,

$$\nu_R U(t+h, \theta) \leqslant s\left(1 - \frac{2\eta}{L} h\right). \tag{6.4}$$

2. $|h|$ is small and $x \in R_+ \backslash \theta$,

$$\nu_R U(t+h, x) \leqslant s\left(1 - \frac{2\eta}{L} h\right). \tag{6.5}$$

Now by (6.3), (6.4) and (6.5), if $\nu_R U(t) = \nu_R U(t, 0)$ we have for $t > t_0$, $x \geqslant 0$,

$$\nu_R U(t+h, x) \leqslant s(1 - (h/L)\rho),$$

where $\rho = \min(2\eta, L\gamma/2)$. Thus

$$\bar{D}\nu_R U(t) \leqslant \frac{-\rho s}{L}, \tag{6.6}$$

which together with (6.2) implies

$$D\nu_R U(t) \leqslant -\frac{\rho}{L} \nu_R U(t) \qquad \text{for} \quad t > t_0 \quad x \geqslant 0.$$

Therefore there exist positive constants k and c such that

$$\nu_R U(t) \leqslant k \exp(-ct), \qquad t \geqslant 0,$$

and the proof is complete.

§7. *The Energy Method*

In this section we consider the solution of equations (5.1) with initial and boundary conditions (5.2), where the boundary condition $h(t)$ satisfies properties 4 and 5 of Theorem (5.1), and for all $t \geqslant 0$, $x \geqslant 0$, $v(t, x)$ is smaller than α, where α is the first positive zero of $f(v)$. Under these conditions we prove in Theorem (7.1) using energy estimates that $U(t)$ decays exponentially in $L_2 \cap L_\infty$.

More precisely, we prove the following.

THEOREM (7.1). *Let* $U = (v, u) \in C([0, \infty) \mid L_2 \cap C_0)$ *be the solution of the Fitz Hugh-Nagumo equations* (5.1) *with initial and boundary conditions* (5.2). *Suppose f is smooth and satisfies*

1. $\lim |f(v)|/|v| > \sigma/\gamma$ *as* $|v| \to \infty$.
2. $f'(0) < 0$, $f(0) = 0$.
3. $f(v) > 0$, *if* $v < 0$,

and $v(t, 0) = h(t)$ *satisfies,*

1. $h(t) \in BC$.
2. $h(t) = h(0) = 0$, *for* $t \geqslant t_0$.

Let $\alpha = \inf\{v > 0 \mid f(v) = 0\}$. If $\sup_{t,x} v(t, x) \leqslant \alpha_0 < \alpha$, *then, for any* $t \geqslant 0$, *there exist constants* k *and* c *such that*

$$\| U(t)\|_2 + \| U(t)\|_\infty \leqslant k \exp(-ct) \qquad \textit{for} \quad t \geqslant 0.$$

Proof. First we show that there exist constants k and c such that

$$\| U(t)\|_2 \leqslant k \exp(-ct) \qquad \text{for} \quad t \geqslant t_0 . \tag{7.1}$$

Hypotheses 2 and 3 on f and the fact that $v(t, x) \leqslant \alpha_0 < \alpha$ guarantee that there exists a $\delta > 0$, such that

$$vf(v) \leqslant -\delta v^2 \qquad \text{for} \quad t \geqslant 0, \quad x \geqslant 0. \tag{7.2}$$

Multiplying the first equation of (5.1) by σv and the second by u, and adding, we get, for the slab$(0, T] \times (0, N]$ $(T > 0, N > 0)$,

$$\begin{aligned} 1/2 \frac{d}{dt} (u^2 + \sigma v^2) &= \sigma(v v_x)_x - \sigma v_x{}^2 + \sigma v f(v) - \gamma u^2 \\ &\leqslant [(v v_x)_x - v_x{}^2] - [\sigma\delta v^2 + \gamma u^2]. \end{aligned} \tag{7.3}$$

Note that in the slab $(0, T] \times (0, N]$, $(vv_x)_x$ and $v_x{}^2$ are well defined as a consequence of the interior regularity theorem (2.2). Therefore to prove (7.1), we shall integrate (7.3) over intervals of the form $[1/n, N]$, $n \geqslant 1$ and then pass to the limit as n, N tend to infinity. That this limiting process is justified is a tedious, but straightforward exercise, which we will delete.

Let

$$P_{n,N}(t) = \int_{1/n}^{N} u^2(t, x) + \sigma v^2(t, x)\, ds \qquad n \geqslant 1$$

$$P(t) = \lim_{n\to\infty} \lim_{N\to\infty} P_{n,N}(t) = \int_0^\infty u^2(t, x) + \sigma v^2(t, x)\, dx.$$

We need to show

$$P(t) \leqslant k \exp(-ct), \qquad \text{for} \quad t \geqslant t_0 . \tag{7.4}$$

We break the proof into three steps.

1. $\lim_{n\to\infty} \lim_{N\to\infty} \frac{d}{dt} P_{n,N}(t) = \frac{d}{dt} P(t), \quad t_0 \leqslant t \leqslant T, \quad T$ arbitrary.
2. $\frac{d}{dt} P(t) \leqslant -\text{const}\, P(t)$ for $t_0 \leqslant t \leqslant T$, (some positive const.)
3. $P(t) \leqslant \text{const}(t_0) \exp(-kt)$.

Step 1, which we will delete, is, as mentioned above, a long straightforward calculation. Step 2, follows by integrating (7.3) over the slab $[1/n, N]$ and passing to the limit using Step 1. Note that when we integrate the right hand side of (7.3) the boundary terms vanish as n and N tend to infinity.

Step 3, is immediate from Step 2.

This proves the L^2-decay of the solution.

To complete the proof we show that the sup norm has exponential decay.

Observe that

$$v(T, 0) = h(T) = 0, \qquad \text{for} \quad T \geqslant t_0,$$

$$u(T, 0) = \int_0^{t_0} \sigma h(s)\, e^{-\gamma(T-s)}\, ds.$$

Suppose $T - 1 \geqslant t_0$. Then,

$$v(T) = \int_0^\infty \tilde{K}(1, z, x)\, v(T-1, z)\, dz + \int_{T-1}^T \int_0^\infty \tilde{K}(T-s, z, x)\, F_1 U(s, z)\, dz\, ds. \tag{7.5}$$

Since the solution $U(t, x)$ is bounded (Theorem (4.1)), it follows that

$$\| F_1 U(s)\|_2 \leqslant k \| U(s)\|_2 . \tag{7.6}$$

Using (7.6) and applying the Schwarz inequality to (7.5) we have

$$\| v(T)\|_\infty \leqslant \| K(1)\|_{L_2(R)} \| v(T-1)\|_{L_2(R_+)} + c_1 \| K \|_{L_2([0,1]\times R)} \| U \|_{L_2([T-1,T]\times R_+)} . \tag{7.7}$$

This proves the decay of v in the sup norm since the L_2 norm of K is finite.

To estimate u recall (5.5)

$$u(t, x) = \sigma \int_0^t v(s, x)\, e^{-\gamma(t-s)}\, ds. \tag{7.8}$$

If $T \geqslant t_0 + 1$, by (7.7) we have

$$\| v(T)\|_\infty \leqslant k \exp(-cT),\ k,\ c,\ >0,$$

Therefore (7.7) yields

$$\begin{aligned}\| u(T)\|_\infty &\leqslant \sup_{t\leqslant t_0+1} \| v(t)\|_\infty \int_0^{t_0+1} \sigma e^{-\gamma(T-s)}\, ds \\ &\quad + \int_{t_0+1}^{T} \sigma k e^{-\xi s} e^{-\gamma(T-s)}\, ds \\ &\leqslant \alpha e^{-\gamma T} \frac{e^{\xi(t_0+1)}}{\gamma} + k\sigma e^{-\gamma T} \int_{t_0+1}^{T} e^{(\gamma-\xi)s}\, ds \\ &\leqslant \text{const} \exp(-kt). \end{aligned} \tag{7.9}$$

From (7.7) and (7.9), we get

$$\| U(t)\|_\infty \leqslant \text{const} \exp(-kt) \qquad \text{for} \quad t \geqslant t_0 + 1,$$

and the proof of Theorem (7.1) is complete. □

We next observe that according to Theorem (5.1) the hypothesis $v \leqslant \alpha_0 < \alpha$ for $t \geqslant T$ is satisfied if $\| h \|_1$ is small, hence we have:

COROLLARY (7.1). *Suppose* $U = (v, u)$ *satisfies the Fitz Hugh-Nagamo equations* (5.1) *with initial and boundary data*

$$u(0, x) = 0 = v(0, x)$$

$$v(t, 0) = h(t).$$

Let f be as in Theorem (7.1). *Let*

$$\alpha = \inf\{v > 0: f(v) = 0\}.$$

Let $h(t)$ satisfy

1. $h \in BC$.
2. $h(t) = h(0) = 0$, *for* $t \geqslant t_0$.
3. $\| h \|_1$ *is sufficiently small, depending on* $\| h \|_\infty$.

Then there exist positive constant k and c such that

$$\| U(t)\|_\infty + \| U(t)\|_2 \leqslant k \exp(-ct), \qquad \textit{for} \quad t \geqslant 0.$$

II. THE NEUMANN PROBLEM

§8. *Local Solvability*

We now consider the non-linear system of equations (2.1) with Neumann boundary conditions at $x = 0$.

Notation:

$$\bar{K}_i(t, z, x) = K_i(t, z - x) + K_i(t, -z - x), \qquad 1 \leqslant i \leqslant n.$$

For

$$h = (h_1, ..., h_p) \in BC \qquad \text{and} \qquad g = (g_1, ..., g_n) \in B$$

$$\bar{H}_i(t, x) = -\int_0^t h_i(s)\, \bar{K}_i(t - s, x)\, ds, \qquad 1 \leqslant i \leqslant p.$$

$$\bar{H}_i(t, x) = 0, \qquad p < i \leqslant n.$$

$$\bar{S}_i(t, x) = \int_0^\infty g_i(z)\, \bar{K}_i(t, z, x)\, dz, \qquad 1 < i \leqslant n.$$

Recall that $\bar{H}_i(t, x)$, $1 \leqslant i \leqslant p$, is the solution of the heat equation with Neumann boundary condition $h_i(t)$ and zero initial data. $\bar{S}_i(t, x)$ is the restriction to $x > 0$ of the solution to the Cauchy problem for the heat equation with initial data of the form $g_i(x)$, for $x > 0$, and $g_i(-x)$ for $x < 0$.

We will write,

$$\bar{K} = (\bar{K}_1, \bar{K}_2, \bar{K}_3, ..., \bar{K}_n),$$
$$\bar{S} = (\bar{S}_1, ..., \bar{S}_n),$$
$$\bar{H} = (\bar{H}_1, ..., \bar{H}_n).$$

Let $R(t, x)$ be the unique element of $C([0, \infty) \mid B)$ such that

$$R_j = R_{xx}, \qquad t > 0, \quad x > 0,$$
$$R(0, x) = g(x), \qquad x > 0,$$
$$\frac{d}{dx}(R_1, ..., R_p)(t, 0) = h(t), \qquad t > 0.$$

It is easy to show that $U \in C([0, \infty) \mid B)$ satisfies (2.1) (in the sense of distributions) with initial and boundary data.

$$\begin{aligned} &U(0, x) = g(x) \\ &(u_i)_x(t, 0) = h_i(t) \text{ where } (u_i)_x \text{ is continuous in } x \geqslant 0,\ t \geqslant 0. \end{aligned} \tag{8.1}$$

if and only if

$$U(t, x) = R(t, x) + \int_0^t \int_0^\infty \bar{K}(t - s, z, x)\, F(u(s, z))\, dz\, ds. \tag{8.2}$$

We mention without proof several results on existence and regularity for solutions of (2.1) with data (8.1). (The proofs are standard).

THEOREM (8.1). *For any* $h = (h_1, ..., h_p) \in BC$ *and* $g = (g_1, ..., g_n) \in C_0$ *there is a constant* t_0, $0 \leqslant t_0 \leqslant 1$, *depending only on* F, $\| g \|_\infty$, *such that the Neumann problem* (2.1) *with initial and boundary data* (8.1) *has a unique solution* $U \in C([0, t_0] \mid B)$ *and*

$$\| U \|_{C([0,t_0] \mid C_0)} \leqslant 2 \left(\frac{\| h \|_\infty}{2c\pi^{1/2}} + \| g \|_\infty \right)$$

where $c^2 = \min(a_1, ..., a_p)$.

THEOREM (8.2) (Interior regularity). *Suppose* $U = (u_1, ..., u_n) \in C([0, t_0] \mid C^0(\bar{R}_+))$ *is a solution of* (2.1) *with initial and boundary conditions* (8.1).

If:

1. $g_i \in C^0(\bar{R}_+)$, $\quad i = 1, ..., n$.
2. $g_i \in C^\infty(\bar{R}_+)$, $\quad i = 1, ..., n$,
3. $h_i \in BC(\bar{R}_+)$, $\quad i = 1, ..., p$.

Then:

$$U \in C^\infty(\Omega) \qquad \textit{where} \quad \Omega = (0, \infty) \times R_+ ,$$

THEOREM (8.3). *If the system* (8.1) *is linear, i.e.,* $F(U) = FU$ *for a constant matrix* F, *then for any* $g = (g_1, ..., g_n) \in C_0$, *and* $h = (h_1, ..., h_p) \in BC$ *there is a unique solution* $U \in C([0, \infty) \mid B)$ *of* (2.1) *with initial and boundary data* (8.1). *Furthermore, there are constants* k *and* c, *independent of* g *and* h, *such that*

$$\| U(t) \|_\infty \leqslant k e^{\xi t} \left(\| g \|_\infty + \frac{\| h \|_\infty}{2\pi^{1/2}} \right).$$

EXAMPLE (8.1). If we apply Theorem (8.3) to the system:

$$\begin{aligned}
\phi_j &= \phi_{xx} - \alpha, \\
\alpha_j &= \sigma\phi - \gamma\alpha, \\
\phi(0, x) &= g_1(x), \\
\alpha(0, x) &= g_2(x), \qquad g = (g_1, g_2) \in C_0 , \\
\phi_x(t, 0) &= h(t).
\end{aligned}$$

We get $\| \alpha(t) \|_{C_0} + \| \phi(t) \|_{C_0} \leqslant \text{const} \exp(ct)$, where

$$\text{const} = \text{const}(\| h \|_\infty , \| g \|_\infty).$$

§9. *Global Existence for the Neumann Problem*

1. *General Theory*

Assume f has the qualitative behavior pictured in Figure 1 of the introduction; we have in mind the cubic

$$f(v) = v(v-a)(b-v) \qquad \text{with} \quad a, b > 0.$$

More precisely, we assume that f has the following properties:

$$f(v) > 0, \qquad \text{if} \quad v < 0. \tag{9.1}$$

$$\lim_{|v|\to\infty} \frac{f(v)}{v} = -\infty. \tag{9.2}$$

$$\lim_{|v|\to\infty} \frac{|f(v)|}{|v|} > \frac{\sigma}{\gamma}. \tag{9.3}$$

To show global existence of solutions of (1.1) with initial and boundary conditions

$$\begin{aligned} v(0, x) &= g_1(x), \qquad x \geqslant 0; \\ u(0, x) &= g_2(x), \qquad x \geqslant 0; \\ v_x(t, 0) &= h(t) \qquad t > 0. \end{aligned} \tag{9.4}$$

We need the following Lemma

LEMMA (9.1). i. *Let f satisfy* (9.1), (9.2), *and* (9.3).
For any β such that

$$\lim \frac{|f(v)|}{|v|} > \beta > \frac{\sigma}{\gamma},$$

$$v_\beta = \max\{|v| \mid |f(v)| \leqslant \beta |v|\}.$$

Let ϕ: $R^2 \to R$ be such that $\sup_{x,t\geqslant 0} |\phi(x, t)| \leqslant M$. *Let β' satisfy*

$$\liminf \left| \frac{f(v)}{v} \right| > \beta' > \beta > \sigma/\gamma.$$

If $|v| > \max\{v_\beta + M, M\beta'/(\beta' - \beta)\}$, *then there exists* $\epsilon > 0$, *such that the symmetric rectangle R with vertex at* $(|v|, (\sigma/\gamma)|v| + \epsilon)$ *is contracting for the vector field.*

$$F_\phi(U) = ((f(v) + \phi) - u, \sigma v - \gamma u).$$

Proof. Lemma (3.1) states that there is a contracting rectangle R for the linear vector field $F_L(U) = (-\beta v - u, \sigma v - \gamma u)$, $\gamma, \beta, \sigma > 0$ if and only if

$\beta > \sigma/\gamma$. Moreover R can be constructed symmetric with respect to the origin, with right upper vertex

$$(|\, v\,|, u) = \left(|\, v\,|, \frac{\sigma}{\gamma}\,|\, v\,| + \epsilon\right).$$

It is easy to check that R is also contracting for $F_\phi(U)$. $\square$

Lemma (9.1) leads to the following existence theorem.

THEOREM (9.1). *Let f be a smooth function which verifies* (9.1), (9.2) *and* (9.3). *Let $g_1, g_2 \in C_0$. If $h \in BC$, then there exists a unique solution $U = (v, u) \in C([\text{-}, \infty), C_0)$ to the Neumann problem* (1.1) *with initial and boundary condition* (9.4). *Furthermore, for any $T \geqslant 0$,*

$$\|\, U(t)\|_\infty \leqslant \theta(T, \|\, h\,\|_\infty, \|\, g_1\,\|_\infty, \|\, g_2\,\|_\infty), \qquad 0 \leqslant t \leqslant T,$$

where θ grows at most exponentially in T.

Proof. The uniqueness follows from the local uniqueness theorem.

We have already proven the local existence of solutions; to obtain global existence we need only establish an apriori bound for the solution. It suffices to show that there is an apriori bound on the interval $[0, T]$, $T > 0$ arbitrary.

To do this we construct a comparison function $\Phi(t, x) = (\phi(t, x), \alpha(t, x))$, bounded on $[0, T]$, which has the same initial and boundary values as the solution $U(t, x)$ of (1.1) and (9.4).

The difference, $\tilde{U} = U - \Phi$, satisfies a system of equations similar to the Fitz Hugh-Nagumo equations and has zero Neumann boundary conditions. This last fact aloows us to construct a family of rectangles, $R(t)$, depending on time, such that $\tilde{U}(t, x) \in R(T)$ for $0 \leqslant t \leqslant T$, $x \geqslant 0$. Furthermore, the size of $R(t)$ grows exponentially with t. These facts give us the required apriori bound for the solution $U(t, x)$.

The details of the proof are as follows. Let $\Phi(t, x) = (\phi(t, x), \alpha(t, x))$ be the solution of:

$$\phi_t = \phi_{xx} - \alpha;$$

$$\alpha_t = \sigma\phi - \gamma\alpha,$$

with initial and boundary conditions

$$\phi(0, x) = g_1(x), \qquad \alpha(0, x) = g_2(x),$$

$$\phi_x(t, 0) = h(t).$$

By example (8.1) we know that Φ has at most exponential growth. Thus there exist $N > 0$ such that $\|\, \Phi(t)\| \leqslant N$ for $0 \leqslant t \leqslant T$. Now we are going to estimate $\tilde{U} = U - \Phi$.

Let

$$\tilde{v} = v - \phi,$$
$$\tilde{u} = u - \alpha.$$

Then:

$$\begin{aligned} \tilde{v}_t &= \tilde{v}_{xx} + f(\tilde{v} + \phi) - \tilde{u}, \qquad && x \geqslant 0, \quad t \geqslant 0; \\ \tilde{u}_t &= \sigma\tilde{v} - \gamma\tilde{u}, && x \geqslant 0, \quad t \geqslant 0; \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} \tilde{v}(0, x) &= 0 = \tilde{u}(0, x), \qquad && x \geqslant 0; \\ \tilde{v}_x(t, 0) &= 0, && t \geqslant 0. \end{aligned} \tag{9.6}$$

By Lemma (9.1) we can construct a rectangle, $R(T)$, $0 \in \operatorname{int} R(T)$, which for $0 \leqslant t \leqslant T$, is contracting for the vector field $F_\phi(\tilde{U}) = (f(\tilde{v} + \phi) - \tilde{u}, \sigma\tilde{v} - \gamma\tilde{u})$.

We want to show that $(\tilde{v}, \tilde{u})$ stays bounded for $0 \leqslant t \leqslant T$. Furthermore, since $(\tilde{v}(0, x), \tilde{u}(0, x))$ is in $R(T)$, we will show that, for $0 \leqslant t \leqslant T$, $(\tilde{v}(t, x), \tilde{u}(t, x)) \in R(T)$. We prove by contradiction that it is impossible for $(\tilde{v}, \tilde{u})$ to reach the boundary of $R(T)$.

Suppose, on the contrary, that $(\tilde{v}, \tilde{u})$ reaches the boundary of $R(T)$. Since the initial condition is in C_0 there is a first time t_0 such that, there exists a finite x_0, for which $U(t_0, x_0) \in \partial R$. Observe that x_0 is not zero, since $0 \in \operatorname{int} R(T)$. Suppose we are on the right hand side of $R(T)$, then we have $v(t_0, x_0) \in \partial R$. Since t_0 is the first time, we have

$$\tilde{v}_t(t_0, x_0) \geqslant 0. \tag{9.7}$$

By construction of $R(T)$ we know that

$$(f(\tilde{v} + \phi) - \tilde{u}))\,|_{(t,x)=(t_0,x_0)} < 0.$$

Since $\tilde{v}(t, x) \leqslant \tilde{v}(t_0, x_0)$ for all $x \geqslant 0$, $t \leqslant t_0$, we see that $\tilde{v}(t_0, \cdot)$ has a local maximum at x_0, so $\tilde{v}_{xx}(t_0, x_0) \leqslant 0$.

Thus at (t_0, x_0)

$$\tilde{v}_t = \tilde{v}_{xx} - f(\tilde{v} + \phi) - \tilde{u} < 0,$$

which contradicts (9.7).

On the left side of $\partial R(T)$ all the inequalities are reversed. For the top we note that if t_0 is the first time such that for some x_0, $\tilde{u}(t_0, x_0) \in \partial R(T)$, we have $\tilde{u}_t \geqslant 0$. But by the construction of R we know that at (t_0, x_0), $\sigma\tilde{v} - \gamma\tilde{u} < 0$ and hence $\tilde{u}_t = \sigma\tilde{v} - \gamma\tilde{u} < 0$.

At the bottom the inequalities are reversed. Hence the solution $\tilde{U} = (\tilde{v}, \tilde{u})$ remains in $R(T)$ for $0 \leqslant t \leqslant T$. Now we have,

$$\| U(t) - \Phi(t)\|_\infty \leqslant \text{const} \qquad (0 \leqslant t \leqslant T).$$

Since the growth of Φ is at most exponential we get

$$\| U(t)\|_\infty \leqslant \text{const} \exp(kT), \qquad 0 \leqslant t \leqslant T. \quad \square$$

In the next three sections we study the threshold problem and exponential decay for solutions of the FitzHugh-Nagumo equations with Neumann boundary conditions $v_x(t, 0) = h(t)$, where $h(t)$ vanishes outside a finite interval $[0, t_0]$.

§10. *The Threshold Problem for the Neumann Problem*

THEOREM (10.1). *Let f be a smooth function which satisfies* (9.1), (9.2), (9.3), $f(0) = 0$, $-f(0) \geqslant \sigma/\gamma$. *Suppose $h \in BC$, satisfies*

1. $h(t) = 0, \quad t \geqslant t_0$;
2. $\| h \|_\infty \leqslant M$;

let $U \in C([0, \infty) \mid C_0)$ be the unique solution of the Fitz Hugh-Nagumo equations (1.1) *with zero initial data and Neumann boundary condition $v_x(t, 0) = h(t)$.*

Denote by $F = (F_1, F_2)$ the vector field $(f(v) - u, \sigma v - \gamma u)$. Then for any $T > t_0 \geqslant 0$ there exists a constant $K = k(T, t_0, M, F)$, such that $\| U(t)\|_\infty \leqslant k \| h \|_1$ for all $t \geqslant T$. Note: k grows at most like $\max\{1/(T - t_0), \exp T\}$.

Proof. We know that for $t \leqslant T$, the solution $U(t, x)$ is bounded. Therefore there exists a constant $r = r(T, M, F)$ such that

$$| FU(t, x)| \leqslant r \mid U(t, x)|, \qquad 0 \leqslant t \leqslant T, \quad 0 \leqslant x. \tag{10.1}$$

Recall $U = (v, u)$, and

$$v(t, x) = \frac{-1}{\pi^{1/2}} \int_0^t \frac{h(s)}{(t - s)^{1/2}} \exp\left(\frac{-x^2}{4(t - s)}\right) ds$$
$$+ \int_0^t \int_0^\infty K(t - s, z, x) F_1 U(s, z)\, dz\, ds = I(t) + II(t) \tag{10.2}$$

$$u(t, x) = \int_0^t F_2 U(s, x)\, ds = \sigma \int_0^t v(s, x)\, e^{-\gamma(t-s)}\, ds. \tag{10.3}$$

Thus by (10.1) (10.2) and (10.3), we have

$$| U(t, x)| \leqslant \frac{1}{\pi^{1/2}} \int_0^t \frac{h(s)}{(t - s)^{1/2}} \exp\left(\frac{-x^2}{t - s}\right) ds$$
$$+ r \int_0^t \int_0^\infty | \bar{K}(t - s, z, x)\, U(s, z)|\, dz\, ds$$
$$+ r \int_0^t | U(s, x)|\, ds. \tag{10.4}$$

As in the proof of Theorem (5.1) we first establish the weaker estimate

$$\| U(T)\|_\infty \leqslant \text{const}(T, t_0, M, F) \| h \|_1 . \tag{10.5}$$

Unlike the Dirichlet problem in this case, we do not have an immediate version of Theorem (4.2): instead we use the following approach. Observe that since $v_x(t, 0) = 0$ for $t \geqslant T$, $U(t, x)$ has an even extension in x, $\bar{U}(t, x)$, which is a solution of the Cauchy problem (1.1) with initial data at (T, x),

$$U(T, x) = \begin{cases} U(T, x) & x \geqslant 0 \\ U(T, -x) & x \leqslant 0, \end{cases}$$

which allows to replace Theorem (4.2) by the following sup norm estimate for the Cauchy problem

Let $U \in C([0, \infty) \mid C_0)$ be a solution to the Cauchy problem for the Fitz Hugh-Nagumo equations, with initial data $U(0, x) = U_0(x)$. Then

$$\| U(t)\|_\infty \leqslant \text{const}(\| U_0 \|_\infty).$$

This estimate follows from observation 1 of Section 3.

Thus it is sufficient to show (10.5) or equivalently

$$\begin{aligned} &\text{a.} \quad \| v(T)\|_\infty \leqslant \text{const}(T, t_0, M, F) \| h \|_1 \\ &\text{b.} \quad \| u(T)\|_\infty \leqslant \text{const}(T, t_0, M, F) \| h \|_1 \end{aligned} \tag{10.6}$$

To establish (10.6).a we use the integral representation (10.2).

Since

$$| I(T)| \leqslant \frac{\text{const}}{(T - t_0)^{1/2}} \| h \|,$$

and

$$| II(T)| \leqslant \text{const}(T, M, F) \int_0^T \frac{1}{(T - s)^{1/2}} \int_0^\infty | U(s, z)| \, dz \, ds.$$

We need to bound

$$\int_0^\infty | U(s, z)| \, dz \qquad \text{for} \quad 0 \leqslant s \leqslant T.$$

We get this bound by applying Gronwall's inequality to the following.

$$\begin{aligned} \int_0^\infty | U(t, x) \, dx &\leqslant \frac{1}{\pi^{1/2}} \int_0^\infty \int_0^t \frac{| h(s)|}{(t - s)^{1/2}} \exp\left(\frac{-x^2}{t(4 - s)}\right) ds \, dx \\ &\quad + \text{const}(T, M, F) \int_0^\infty \int_0^t | U(s, x)| \, ds \, dx \\ &\leqslant \text{const} \| h \|_1 + \text{const}(T, M, F) \int_0^t \int_0^\infty | U(s, z)| \, dz \, ds \end{aligned}$$

and we are done with (10.6).a.

To show (10.6).b we use (10.3) and the integral representation of v (10.4)

$$u(T,x) \leqslant \sigma \int_0^T |v(s,x)|\,ds \leqslant \sigma \int_0^T \int_0^s \frac{|h(q)|}{(s-q)^{1/2}} \exp\left(\frac{-x^2}{4(s-q)}\right) dq\,ds$$

$$+ \operatorname{const}(T, M, F) \int_0^T \int_0^s \int_0^\infty |\bar{K}(s-0, z, x)\, U(q,z)|\,dz\,dq\,ds$$

$$\leqslant \sigma \int_0^T |h(\sigma)| \int_q^T \frac{1}{(s-q)^{1/2}} \exp\left(\frac{-x^2}{4(s-q)}\right) ds\,dq$$

$$+ \sigma \int_0^T \int_0^s \frac{1}{(s-q)^{1/2}} \int_0^\infty |U(q,z)|\,dz\,dq\,ds$$

$$\leqslant \sigma(T)^{1/2} \| h \|_1 + \operatorname{const}(T, M, F) \int_0^T \int_0^s \frac{1}{(s-q)^{1/2}}\,dq\,ds\, \| h \|_1$$

and (10.6).b follows completing the proof of the theorem. □

§11. *Stability by Contracting Rectangles for the Neumann Problem*

THEOREM (11.1). *Suppose $U = (v, u)$ satisfies the Fitz Hugh-Nagumo equations* (1.1), *with zero, initial data and Neumann boundary data $v_x(t, 0) = h(t)$. Let f and h be as in Theorem* (10.1). *There exist constants, c, k and λ, such that if*

$$\| h \|_1 \leqslant \lambda,$$

then

$$\| U(t, \cdot)\|_\infty \leqslant k \exp(-ct) \qquad t \geqslant 0$$

where k and λ depend on T, t_0, M and F, c depend only on F.

Proof. Let $\bar{U}(\lambda, x)$ be the even extension of U.

$$\bar{U}(t,x) = \begin{cases} U(t,x), & x \geqslant 0; \\ U(t,x), & x \leqslant 0. \end{cases}$$

For $t \geqslant t_0$ we have $v_x(t, 0) = 0$, and as we have remarked in Theorem (10.1), $\bar{U}$ is a solution to the Fitz Hugh-Nagumo equations for $-\infty < x < \infty$.

For the Cauchy problem we know that the following theorem is true. (See [7].)

THEOREM (11.2). *For the Fitz Hugh-Nagumo equations, suppose $-f'(0) > \sigma/\gamma$, and let R_c be the critical rectangle described in Lemma* (3.5). *If $U^0 \in C_0(R)$ and $U^0(x) \in \operatorname{int}(R_c)$ for all $x \in R$, then there are non-negative constants c, k, such that*

$$\| U(t)\|_\infty \leqslant k e^{-ct}, \qquad \textit{for all} \quad t \geqslant T,\ T = t_0 + \delta,\ \delta > 0.$$

In our case let

$$U^0(x) = (v(T, x), u(T, x)).$$

By Theorem (10.1), *we know that*

$$\| U^0 \|_\infty \leqslant \text{const}(T, M, t_0, F) \| h \|_1 = c \| h \|_1 .$$

If we choose ν *so small that* $\| U^0 \|_\infty \leqslant c\nu$ *implies*

$$U^0(x) \in \text{int}(R_c),$$

by Theorem (11.2) *we get*

$$\| U(t)\|_\infty \leqslant k \exp(-ct), \qquad t \geqslant 0.$$

§12. *The Energy Method*

In §11 we obtain exponential decay in norm L_∞ for the solution $U(t, x)$ of the Neumann problem with zero initial data, with compactly supported boundary data $h(t)$, whose L_1 norm $\| h \|_1$ is so small that $U(t, x) \subset R_c$ for t large. (R_c is described in Lemma (3.2).)

In this section we show that under the weaker hypothesis, v less than the smallest positive zero of $f(v)$ for all $t \geqslant 0$, the solution $U(t, x)$ decays exponentially in $L_2 \cap L_\infty$. More precisely we have the following theorem.

THEOREM (12.1). *Let* $\alpha = \inf\{v > 0 \mid f(v) = 0\}$. *Suppose* $U = (v, u) \in C([0, \infty) \mid C_0)$ *satisfies the Fitz Hugh-Nagumo equations* (1.1) *with zero initial data and Neumann boundary condition* $v_x(t, 0) = h(t)$ *where* f *is a smooth function which satisfies*

1. $f(v) > 0$, *for* $v < 0$, $f'(0) < 0$, $f(0) = 0$.
2. $\lim_{|v|\to\infty} \dfrac{f(v)}{v} = -\infty$.
3. $\liminf_{|v|\to\infty} \dfrac{|f(v)|}{|v|} > \sigma/\gamma$,

and h *satisfies*

a. $h \in BC$;

b. $h(t) = 0$ *for* $t \geqslant t_0$.

If $v(t, x) \leqslant \alpha_0 < \alpha$ *for all* t, $x \geqslant 0$, *then there exist positive constants* k *and* c *such that*

$$\| U(t)\|_\infty + \| U(t)\|_2 \leqslant k \exp(-ct), \qquad t \geqslant 0.$$

We omit the proof since it is virtually the same as for the Dirichlet problem. □

Lastly we observe that by Theorem (10.1) the hypothesis $v \leqslant \alpha_0 < \alpha$ for $t \geqslant T$ is satisfied if $\| h \|_1$ is small. Thus we have the following corollary.

COROLLARY (12.1). *Suppose* $U \in C([0, \infty) \mid C_0(\bar{R}_+))$ *satisfies the Fitz Hugh-Nagumo equations* (1.1), *with zero initial data and Neumann boundary data* $v_x(t, 0) = h(t)$.

Let f, α *and* h *be as in Theorem* (12.1), *and furthermore* $\| h \|_1$ *is sufficiently small depending on* $\| h \|_\infty$ *then there exist positive constant* k *and* c *such that*

$$\| U(t)\|_2 + \| U(t)\|_\infty \leqslant ke^{-cj} \qquad \textit{for} \quad t \geqslant 0.$$

III. A SHORT NOTE ON THE HODGKIN HUXLEY EQUATION

§13. *Threshold Results for the Dirichiet Problem*

We shall apply the ideas of the previous section to the Hodgkin and Huxley equations. We write them in the form found in Chueh, Conley and Smoller [2].

$$\begin{aligned} cu_t &= R^{-1}u_{xx} + g(u, v, w, z) \\ v_t &= g_1(u)(h_1(u) - v) \\ w_t &= g_2(u)(h_2(u) - w) \\ z_t &= g_3(u)(h_3(u) - z) \end{aligned} \tag{13.1}$$

where

$$j(u, v, w, z) = k_1v^3w(c_1 - u) + k_2zt(c_2 - u) + k_3(c_3 - u). \tag{13.2}$$

Here c, R, $k_i(i = 1, 2, 3)$ are positive constants, the c_i are constants satisfying $c_1 > c_3 > 0 > c_2$. Furthermore the functions g_i and h_i are c^1 functions satisfying $g_i > 0$ and $1 > h_i > 0$. Set $U = (u, v, w, z)$. For further details (including explicit values of the constants and descriptions of the g_i and h_i we refer the reader to Hodgkin and Huxley [6]. The existence and uniqueness of solutions $U \in C[0, \infty) \mid C_0 \cap B)$ for the Dirichlet problem, with initial data in $B \cap C_0$ and boundary data in BC follows by showing

1. Existence and uniqueness of local solutions (standard).
2. Apriori sup norm estimates which are obtained from the existence of large invariant rectangles (for a proof see [2]).

To conclude we get threshold results, analogous to those in Sections 5 and 6 for the FitzHugh-Nagumo equations. Following [6], we shall assume that (13.1) has a unique critical point, U^0.

THEOREM (13.1). *Suppose* $h \in BC$ *satisfies*

1. $h(t) = h(0) = 0$ *for all* $t \geqslant t_0 \geqslant 0$
2. $\| h \|_\infty \leqslant M$.

Let $U = (u, v, w, z) \in C([0, \infty) \mid C_0)$ *be the unique solution of* (13.1) *with zero initial data and Dirichlet boundary data*

$$u(t, 0) = h(t) \qquad t \geqslant 0.$$

Then for any $T > t_0 \geqslant 0$ *there exists a constant* $k = k(T, t_0, M, g)$, *such that*

$$\| U(t, \cdot) - U^0 \|_\infty \leqslant k \| h \|_1 \qquad \text{for all} \quad t \geqslant T.$$

Recall that g *was defined in* (13.2).

Proof. As in Theorem (5.1) it suffices to show that

$$\| U(t) - U^0 \|_\infty \leqslant \text{const}(T, t_0, M, g) \| h \|_1 \qquad t \geqslant T. \tag{13.4}$$

With the same arguments as before one establishes

$$\| U(T) - U^0 \|_\infty \leqslant \text{const}(T, t_0, M, g) \| h \|_1 . \tag{13.5}$$

To pass from (13.5) to (13.4) one needs the following estimate for solutions within compactly supported boundary data in $[0, T]$,

$$\| U(t) - U^0 \|_\infty \leqslant \text{const} \| U(T) - U^0 \|_\infty . \tag{13.6}$$

For the specific model considered in Hodgkin and Huxley [6], John Evans has given numerical evidence in [3] which indicates that if we linearize the equation (13.1) about its critical point, U^0, the resulting linear system is asymptotically stable. This together with Theorem (2.4) of [2] (which tells us that a critical point of systems of the form (2.1) is stable if it is stable for the linearized system) implies U^0 is stable, which with the existence of large invariant rectangles [5] is sufficient to establish (13.6), from which the theorem follows. □

Observation. One can show that zero is an attractor for the Fitz Hugh-Nagumo equations (see [7]) thus to prove Theorems (5.1) and (6.1) we do not need the existence of small invariant rectangles.

THEOREM (13.2). *Under the hypothesis of Theorem* (13.1), *there exist constants* c, k, *and* λ *such that if*

$$\| h \|_1 \leqslant \lambda,$$

then $\| U(t, \cdot) - U^0 \|_\infty \leqslant k \exp(-ct)$ $t \geqslant 0$ *where* $k = k(T, t_0, M, g)$, $c = c(g)$, $\lambda = \lambda(g)$.

Proof. Follows from Theorem (13.1) and the stability of the critical point U^0. □

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