The six-dimensional Delaunay polytopes

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Abstract

Given a lattice $L$, a polytope $P$ is called a Delaunay polytope if the set of its vertices is $S \cap L$ with $S$ being an empty sphere of the lattice. Extending our previous work (Available from http://il.arXiv.org/abs/math.MG/0108177) on the hypermetric cone $HYP_7$, we classify the six-dimensional Delaunay polytopes according to their combinatorial type. The list of 6241 combinatorial types is obtained by a study of the set of faces of the polyhedral cone $HYP_7$. The data and programs are available from http://www.liga.ens.fr/~dutour/DelaunaySix.

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1. Introduction

A distance vector $(d_{ij})_{0 \leq i < j \leq n} \in \mathbb{R}^N$ with $N = \binom{n+1}{2}$ is called an $(n+1)$-hypermetric if it satisfies the following hypermetric inequalities:

$$H(b)d = \sum_{0 \leq i < j \leq n} b_i b_j d_{ij} \leq 0$$

for any $b = (b_i)_{0 \leq i \leq n} \in \mathbb{Z}^{n+1}$ with $\sum_{i=0}^n b_i = 1$. (1)

The set of distance vectors satisfying (1) is called the hypermetric cone and denoted by $HYP_{n+1}$.

In fact, $HYP_{n+1}$ is a polyhedral cone (see [9, p. 199]). Lovasz (see [10, pp. 201–205]) gave another proof of this and the bound $\max |b_i| \leq n! 2^n \binom{2n}{n}^{-1}$ for any vector $b = (b_i)_{0 \leq i \leq n-1}$ defining a facet of $HYP_n$.

There is a many-to-many correspondence between Delaunay polytopes with $p$ vertices and the elements of the hypermetric cone $HYP_p$ (see [10]). So, the study of Delaunay...
polytopes is equivalent to the study of hypermetrics. Moreover, the dimension of the faces of $HYP_P$ allows one to define the notion of rank of a Delaunay polytope (see [7]).

A key ingredient of our enumeration method is that every six-dimensional Delaunay polytope has an affine basis (see Theorem 1). So, our enumeration of six-dimensional Delaunay polytopes is reduced to the enumeration of non-degenerate faces of $HYP_P$ under an equivalence relation called geometrical equivalence (see Definition 2 and Section 4).

We have the inclusion $CUT_n \subset HYP_n$, where $CUT_n$ (see Section 3 and Chapter 4 of [10]) is the cone generated by all cut semi-metrics on $n$ points. Both $CUT_n$ and $HYP_n$ have symmetry group $Sym(n)$ (see [8, 11]). One has $HYP_n = CUT_n$ for $n \leq 6$; so, the enumeration of combinatorial types of Delaunay polytopes of dimension $n$, with $n \leq 5$, corresponds to the study of faces of $CUT_{n+1}$. This study was done by Fedorov [15], Erdahl and Ryskhyov [22, 23] and Kononenko [4, 6, 17–19] in dimensions three, four and five, respectively.

In the case of dimension six, the inclusion $CUT_7 \subset HYP_7$ is strict, but the description of facets (i.e. faces of rank 20, corresponding to repartitioning Delaunay polytopes) and extreme rays (i.e. faces of rank 1, corresponding to extreme Delaunay polytopes) of the cone $HYP_7$ is still computationally possible (see [11] and Section 3). In Section 3 we present the number of combinatorial types of six-dimensional Delaunay polytopes for every given rank. In Section 4 we present our method for the computation of those combinatorial types using the face lattice of $HYP_P$.

Voronoi [24] defined a partition of the cone $PSD_n$ of positive semi-definite quadratic forms into $L$-type domains. Two forms of the same $L$-type domain have affinely equivalent Voronoi polytopes. Every vertex of a Voronoi polytope corresponds to a Delaunay polytope. All Delaunay polytopes form a partition of the space $R^n$, which is dual to the partition by Voronoi polytopes. It is proved in [9] that the hypermetric cone $HYP_n$ is the union of a finite number of $L$-type domains.

While Voronoi theory of $L$-type domains describes the combinatorial structure of lattices, the theory of hypermetrics [7, 10] describes the combinatorial structure of one Delaunay polytope in a lattice.

2. Delaunay polytopes and hypermetrics

Here we present the main notions for hypermetrics, which are needed to our study; the presentation is slightly simplified by the limitation to basic Delaunay polytopes and non-degenerate faces. For the complete theory, with proofs, see [7] and Chapters 13–16 of [10].

A family $v_0, \ldots, v_n$ of $n+1$ vertices of $R^n$ is called affinely independent if the family $(v_i - v_0)_{0 \leq i \leq n}$ has linear rank $n$. Let $L \subset R^n$ be an $n$-dimensional lattice and let $S = S(c, r)$ be a sphere in $R^n$ with center $c$ and radius $r$. Then $S$ is said to be an empty sphere in $L$ if the following two conditions hold:

(i) $\|v - c\| \geq r$ for all $v \in L$ and
(ii) the set $S \cap L$ contains an affinely independent set of size $n + 1$.

The center $c$ of $S$ is called a hole in [5]. The polytope $P$, which is defined as the convex hull of the set $S \cap L$, is called a Delaunay polytope, or (in original terms of Voronoi, who introduced them in [24]) $L$-polytope.
Definition 1. Let $P$ be an $n$-dimensional Delaunay polytope with vertex set $V$.

(i) A family $v_0, \ldots, v_n$ of vertices of $P$ is called an affine basis if for all $v \in V$ there exists a unique vector $b = (b_i)_{0 \leq i \leq n} \in \mathbb{Z}^{n+1}$, such that

$$
\sum_{i=0}^{n} b_i v_i = v \quad \text{and} \quad \sum_{i=0}^{n} b_i = 1.
$$

(ii) The Delaunay polytope $P$ is called basic if it has at least one affine basis. The vertices of an affine basis are called basic vertices.

All known Delaunay polytopes are basic. We prove in Theorem 1 that all six-dimensional Delaunay polytopes are basic too.

For every family $\mathcal{A} = \{v_0, \ldots, v_m\}$ of vertices of a Delaunay polytope $P$ circumscribed by $S(c, r)$, one can define a distance function $d_{\mathcal{A}}$ by $(d_{\mathcal{A}})_{ij} = \|v_i - v_j\|^2$. The function $d_{\mathcal{A}}$ turns out to be a hypermetric by the following formula (see [1, 10, p. 195]):

$$
H(b)d_{\mathcal{A}} = \sum_{0 \leq i, j \leq m} b_i b_j (d_{\mathcal{A}})_{ij} = 2 \left( r^2 - \sum_{i=0}^{m} b_i v_i - c \right) \leq 0.
$$

On the other hand, Assouad has shown in [1] that every $d \in HYP_{n+1}$ can be expressed as $d_{\mathcal{A}}$ with $A$ being a family of vertices of a Delaunay polytope $P$ of dimension less or equal to $n$.

Moreover, the equality $H(b)d_{\mathcal{A}} = 0$ is equivalent to $\sum_{i=0}^{m} b_i v_i$ being a vertex of $P$; so, for a given ray $d \in HYP_{n+1}$, the annihilator is defined by

$$
\text{Ann}(d) = \left\{ b \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} b_i = 1 \text{ and } H(b)d = 0 \right\}.
$$

We call the vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{n+1}$ with $0 \leq i \leq n$ basic vectors.

Using the proposition below, these vectors $e_i$ are identified with basic vertices $v_i$; one has $H(b)d = 0$ for all $d$ if and only if $b = e_i$ for some $0 \leq i \leq n$.

Proposition 1. Let $P$ be an $n$-dimensional Delaunay polytope with vertex set $V$; let $d = d_{\mathcal{A}}$, with $\mathcal{A} = \{v_0, \ldots, v_n\}$ a subset of $V$. Then there is equivalence between following properties:

(i) the family $\mathcal{A}$ is affinely independent and
(ii) $\det [d_{0i} + d_{0j} - d_{ij}]_{0 \leq i, j \leq n} \neq 0$.

Also one has equivalence between the following properties:

(iii) $\mathcal{A}$ is an affine basis and
(iv) the following mapping is one-to-one:

$$
b \in \text{Ann}(d) \mapsto \sum_{i=0}^{n} b_i v_i \in V.
$$

Proof. The family $(v_i)_{0 \leq i \leq n}$ is affinely independent if and only if the matrix $((v_i - v_0) \cdot (v_j - v_0))_{0 \leq i, j \leq n}$ is positive definite. The formula $2((v_i - v_0), (v_j - v_0)) = \|v_i - v_0\|^2 + \|v_j - v_0\|^2 - \|v_i - v_j\|^2$ gives the first equivalence.

The second equivalence is obvious. \qed
A ray \( d \in HY P_{n+1} \) is called non-degenerate if \( d = d_B \) with \( B \) being an affine basis of an \( n \)-dimensional Delaunay polytope \( P \). By the above proposition, the set \( \text{Ann}(d) \) is finite if \( d \) is non-degenerate, since the hypermetric vectors \( b \in \text{Ann}(d) \) correspond to lattice points \( v \), which belong to a sphere.

In fact, given a basic Delaunay polytope \( P \) with vertex set \( V \) and an affine basis \( B \), one can easily reconstruct \( d_V \) in terms of \( d_B \). This fact, together with Theorem 1, allows us to restrict our enumeration to non-degenerate faces in the six-dimensional case.

Using the above mapping, one can identify vertices \( v \) of \( P \) with hypermetric vectors \( b \). For example, for the affine basis \( B = \{v_0 = (0, 0, 0), v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1)\} \) of the 3-cube, one has

\[
\begin{align*}
(1, 1, 0) &= v_1 + v_2 - v_0, \\
(0, 1, 1) &= v_2 + v_3 - v_0, \\
(1, 0, 1) &= v_1 + v_3 - v_0 & \text{and} & (1, 1, 1) &= v_1 + v_2 + v_3 - 2v_0.
\end{align*}
\]

Note that, while \((-2, 1, 1, 1) \in \text{Ann}(d_B)\), the inequality \( H(-2, 1, 1, 1)d \geq 0 \) does not define a facet of the cone \( HY P_4 \), since \( H(-2, 1, 1, 1) = H(-1, 1, 1, 0) + H(-1, 0, 1, 1) + H(-1, 0, 1, 1) \).

The face \( F(d) \), associated with a hypermetric \( d \in HY P_{n+1} \), is the minimal face, containing the vector \( d \). It can also be defined as

\[
F(d) = \{e \in HY P_{n+1} : H(b)e = 0 \text{ for all } b \in \text{Ann}(d)\}.
\]

The rank of an \( n \)-dimensional basic Delaunay polytope \( P \) is defined as the dimension of \( F(d_B) \), where \( B \) is an affine basis of \( P \) (see [10, p. 217]); its corank is, by definition, \((\frac{n+1}{2}) - \text{rank}(P)\). In fact, the notion of rank can be defined for any Delaunay polytope, not just the basic one; it is then interpreted as the topological dimension of the set of affine bijections \( f \) of \( \mathbb{R}^n \) (up to isometries) for which \( f(P) \) is again a Delaunay polytope (see [10, p. 225]).

If \( P \) is an \( n \)-simplex, then its unique affine basis defines a hypermetric \( d \), which has \( \text{Ann}(d) = \{e_0, \ldots, e_n\} \) and \( F(d) = HY P_{n+1} \). If \( P \) is an \( n \)-dimensional basic Delaunay polytope with \( n+2 \) vertices \( v_0, \ldots, v_{n+1} \) (such a polytope is called a repartitioning polytope) and affine basis \( B = \{v_0, \ldots, v_n\} \), then, by writing \( v_{n+1} = \sum_{i=0}^{n} b_i v_i \), one obtains \( F(d_B) = \{d \in HY P_{n+1} : H(b)d = 0\} \), which is a facet of \( HY P_{n+1} \). On the other hand, if \( d \in HY P_{n+1} \) with \( F(d) \) being the facet defined by a hypermetric inequality \( H(b) \), then \( \text{Ann}(d) = \{e_0, \ldots, e_n, b\} \) and \( d = d_B \) with \( B \) being an affine basis of a repartitioning polytope.

Define two Delaunay polytopes to be affinely equivalent if there is an affine bijective mapping transforming one into the other; the equivalence classes for this relation are called combinatorial type. By Theorem 15.2.1 in [10, p. 222], two vectors \( d, d' \) of \( HY P_{n+1} \), such that \( F(d) = F(d') \), correspond to affinely equivalent Delaunay polytopes. Moreover, if two Delaunay polytopes \( P, P' \) are affinely equivalent, then any affine basis \( B \) of \( P \) induces an affine basis \( B' \) of \( P' \) such that \( F(d_B) = F(d_B') \).

A Delaunay polytope \( P \), circumscribed by a sphere \( S(c, r) \), is called centrally symmetric if the central symmetry of center \( c \) preserves it. By Lemma 13.2.7 in [10], a Delaunay polytope is centrally symmetric if and only if for one vertex \( v \) of \( P \) its centrally symmetric image is also a vertex of \( P \). This property is invariant by affine equivalence; so,
we call a combinatorial type centrally symmetric if the corresponding Delaunay polytopes are centrally symmetric.

A face of $HYP_{n+1}$ is called non-degenerate if one of its interior vectors $d$ is non-degenerate (by Theorem 15.2.1 of [10], this is equivalent to all interior vectors being non-degenerate). If $B$ is a set of vertices of a Delaunay polytope $P$, then the face $F(d_B)$ is non-degenerate if and only if $B$ is an affine basis.

Given a non-degenerate face $F$ of $HYP_{n+1}$, define

$$Ann(F) = \left\{ b \in \mathbb{Z}^{n+1} : \sum b_i = 1 \text{ and } H(b)d = 0 \text{ for all } d \in F \right\}.$$ 

Call $\{b^0, \ldots, b^n\} \subset Ann(F)$ an affine basis of the combinatorial type if $\det \{b^0, \ldots, b^n\} = \pm 1$. Any affine basis of $Ann(F)$ corresponds to an affine basis of a Delaunay polytope $P$ if $P$ has this combinatorial type.

**Definition 2.** Two non-degenerate faces $F, F'$ of $HYP_{n+1}$ are said to be geometrically equivalent if there exists an affine basis $B = \{b^0, \ldots, b^n\}$ of $Ann(F)$ such that the mapping

$$\phi_{F, F'} : Ann(F') \rightarrow Ann(F)$$

$$b \mapsto b_0b^0 + \cdots + b_nb^n$$

is bijective.

Two basic Delaunay polytopes $P$ and $P'$, with respective affine bases $B$ and $B'$, are affinely equivalent if and only if the faces $F(d_B)$ and $F(d_{B'})$ are geometrically equivalent. So, the study of combinatorial types of $n$-dimensional basic Delaunay polytopes is reduced to the study of non-degenerate faces of the hypermetric cone $HYP_{n+1}$ under geometrical equivalence.

If $\pi$ is a permutation of $\{0, \ldots, n\}$ and $F$ a non-degenerate face of $HYP_{n+1}$, then $\{e_{\pi(0)}, \ldots, e_{\pi(n)}\}$ is an affine basis of $Ann(F)$; so, two non-degenerate faces $F$ and $F'$ which are equivalent by a symmetry of $HYP_{n+1}$ are also geometrically equivalent. The reverse is not true, in general.

**Remark 1.** Let $P$ be an $n$-dimensional basic repartitioning polytope with an affine basis $B = \{v_0, \ldots, v_1\}$; we write $v_{n+1} = \sum_{i=0}^n b_i v_i$. The family $B' = (v_0, \ldots, v_{i-1}, v_{n+1}, v_{i+1}, \ldots, v_n)$ is an affine basis if and only if $|b_i| = 1$.

If $b_i = -1$, then $F(d_B) = F(d_{B'}) = \{d \in HYP_{n+1} : H(b)d = 0\}$, while if $b_i = 1$, then $F(d_B) \neq F(d_{B'}) = \{d \in HYP_{n+1} : H(b')d = 0\}$ with $b' = (-b_0, \ldots, -b_{i-1}, b_i, -b_{i+1}, \ldots, -b_n)$.

Consider, as an example, the two-dimensional case. We have $HYP_3 = CUT_3$ and $CUT_3$ has three facets, which correspond to $H(-1, 1, 1)$ and its permutations. So, a triangle $T = \{v_0, v_1, v_2\}$ satisfies $d_T = (d_{01}, d_{02}, d_{12}) \in HYP_3$ if and only if it is an acute triangle (i.e. all three angles are less than or equal to $\pi/2$). The vector $d_T$ is non-degenerate if and only if three vertices of $T$ are not aligned. Moreover, $d_T$ is incident on a hypermetric facet, say, $H(-1, 1, 1)$, if and only if the vertex $v_0$ has angle $\pi/2$, in which case the Delaunay polytope has four vertices: $v_0, v_1, v_2$ and $v_1 + v_2 - v_0$. So, there are two combinatorial types of Delaunay polytope in dimension two: acute triangles and rectangles.
Take a non-degenerate face $F$ of the hypermetric cone $HY_{n+1}$. With every $b = (b_i)_{0 \leq i \leq n} \in \text{Ann}(F)$ we associate a vertex $(b_1, \ldots, b_n) \in \mathbb{Z}^n$; denote the set of such vertices by $\mathcal{V}$. Every distance vector $d$ in the interior of $F$ corresponds to an $n \times n$ Gram matrix $G$, which is positive definite, since $F$ is non-degenerate. $\mathcal{V}$ is the vertex set of a Delaunay polytope for the scalar product defined by $G$. So, one can encode, in our computations, the combinatorial type of basic Delaunay polytope by the set $\text{Ann}(F)$.

**Proposition 2.** If $P$ is a basic Delaunay polytope and $B, B'$ are two affine bases of $P$, then the faces $F(d_B), F(d_{B'})$ are equal, up to a linear mapping. This linear mapping preserves the non-degeneracy.

**Proof.** If $(v_0, \ldots, v_n)$ and $(v'_0, \ldots, v'_n)$ are two affine bases of $P$, then one can express $v'_i$ in terms of $(v_j)_{0 \leq j \leq n}$ as follows:

$$v'_i = \sum_{j=0}^{n} a_{ij} v_j \quad \text{with} \quad \sum_{j=0}^{n} a_{ij} = 1.$$ 

One can express $\|v'_i - v'_j\|^2$ in terms of $\|v_j - v'_j\|^2$. This induces a linear mapping $\phi$ from $F(d_B)$ to $F(d_{B'})$; expressing $v_j$ in terms of $(v'_i)_{0 \leq i \leq n}$, one gets the reverse mapping $\phi^{-1}$ and, so, the linear equivalence.

If $d \in F(d_B)$, then $d$ is non-degenerate if and only if $\phi(d)$ is non-degenerate. \hfill □

**Definition 3.** Let $B$ be an affine basis of an $n$-dimensional basic Delaunay polytope $P$; then:

(i) If $\text{rank}(P) = 1$, then $P$ is called extreme.

(ii) If all sub-faces of the face $F(d_B) \subset HY_{n+1}$ are degenerate, then $P$ is called maximal.

The above definition of maximality is independent of the choice of affine basis $B$, since, by the above proposition, the linear equivalence between two faces preserves the non-degeneracy. Obviously, any extreme Delaunay polytope is maximal. We present in Corollary 1 the list of all maximal six-dimensional Delaunay polytopes.

Let $\gamma_n = \{0, 1\}^n$ be the vertex set of the Delaunay polytope of the lattice $\mathbb{Z}^n$ and let

$$h\gamma_n = \left\{ x \in \gamma_n : \sum_{i=1}^{n} x_i \text{ is even} \right\}$$

be the vertex set of the Delaunay polytope with center $c = (\frac{1}{2}, \ldots, \frac{1}{2})$ (this polytope is called the half $n$-cube) of the root lattice

$$D_n = \left\{ x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \text{ is even} \right\}.$$ 

The half 3-cube is the 3-simplex and the half 4-cube is the four-dimensional cross-polytope. $\gamma_n$ is centrally symmetric, while $h\gamma_n$ is centrally symmetric if and only if $n$ is even.

**Proposition 3.** (i) $\gamma_n$ is a maximal Delaunay polytope and $\text{Rank}(\gamma_n) = n$. 
(ii) \( h\gamma_n \) is a maximal Delaunay polytope if and only if \( n \geq 5 \); moreover, one has

\[
\text{Rank}(h\gamma_n) = \begin{cases} 
6 & \text{if } n = 3 \\
7 & \text{if } n = 4 \\
n & \text{if } n \geq 5 .
\end{cases}
\]

**Proof.** Clearly, \( h\gamma_n \) (respectively, \( \gamma_n \)) has an affine basis, say, \( B_n \); denote by \( F_n \) the corresponding face belonging to \( HY P_{n+1} \). Any \( d \) belonging to the relative interior of \( F_n \) can be extended as a distance on \( h\gamma_n \) (respectively, \( \gamma_n \)), which we still denote by \( d \).

This metric \( d \) relates to a positive definite Gram matrix \((g_{ij})_{1 \leq i,j \leq n}\) a vector \((c_i)_{1 \leq i \leq n}\) and a constant \( C \) such that

\[
\sum_{1 \leq i,j \leq n} g_{ij} x_i x_j + \sum_{i=1}^{n} c_i x_i = C
\]

for all \( x \in h\gamma_n \) (respectively, \( \gamma_n \)).

Let us now consider the case of \( h\gamma_n \). We already know that \( \text{Rank}(h\gamma_3) = 6 \) and \( \text{Rank}(h\gamma_4) = 7 \) (see [10]). We want to prove, by induction, that if \( n \geq 5 \), then the metric \( d \) can be expressed as \( \sum_{i=1}^{n} \lambda_i d_i \) with \( \lambda_i > 0 \) and \( d_i(x, x') = (x_i - x'_i)^2 \). The result for \( n = 5 \) is proved by computer. Let us assume \( n \geq 6 \).

If one restricts the above equation to the case \( x_n = 0 \), then one obtains

\[
\sum_{1 \leq i,j \leq n-1} g_{ij} x_i x_j + \sum_{i=1}^{n-1} c_i x_i = C
\]

for all \( x \in h\gamma_{n-1} \). By induction, this yields \( g_{ij} = 0 \) for \( i < j < n \). By sectioning other coordinates, we would also get the Delaunay polytope \( h\gamma_{n-1} \) and, so, one has \( g_{ij} = 0 \) for \( i \neq j \). Thus, one gets the following equality:

\[
\sum_{1 \leq i \leq n} g_{ii} x_i^2 + c_i x_i = C
\]

for all \( x \in h\gamma_n \). By taking \( x = 0 \), one obtains \( C = 0 \), and by taking the vector \( x \) with component 1 at coordinates \( i \) and \( j \), one gets \( g_{ii} + c_i + g_{jj} + c_j = 0 \) for all \( i < j \). Clearly, this is possible only if \( g_{ii} + c_i = 0 \); so, one gets

\[
\sum_{1 \leq i \leq n} g_{ii} (x_i - \frac{1}{2})^2 = \frac{1}{4} \sum_{i=1}^{n} g_{ii}.
\]

This corresponds to \( d = \sum_{i=1}^{n} g_{ii} d_i \) with \( g_{ii} > 0 \)

This computation proves that the rank of \( h\gamma_n \) is \( n \) if \( n \geq 5 \), since there are \( n \) parameters \( g_{ii} \) describing the face \( F_n \). Moreover, if \( F' \) is a face contained in \( F_n \), then one or more coefficients \( g_{ii} \) become zero and the face \( F' \) corresponds to \( \gamma_p \) with \( p \leq n - 1 \); so, it is degenerate. Therefore, \( h\gamma_n \) is maximal if \( n \geq 5 \).

For an \( n \)-cube, the proof is similar, except that we begin at \( n = 1 \). □
3. The case of dimension six

Name as a cut cone and denote by $\text{CUT}_{n+1}$ the cone generated by all cuts $\delta_S \in \mathbb{R}^N$ (where $S$ is a subset of $\{0, \ldots, n\}$), defined by

$$(\delta_S)_{ij} = 1 \text{ if } |S \cap \{i, j\}| = 1 \quad \text{and} \quad (\delta_S)_{ij} = 0, \text{ otherwise.}$$

Clearly, $H(b)\delta_S = b(S)(1 - b(S))$ with $b(S) = \sum_{x \in S} b_x$; this proves that all hypermetric inequalities are valid on $\text{CUT}_{n+1}$. So, $\text{CUT}_{n+1} \subset HYP_{n+1}$. Moreover, a cut $\delta_S$ is incident on the face, defined by $H(b)$, if and only if $b(S) = 0$ or 1. It is easy to see (see [10, p. 228]) that a non-degenerate face $F$ of $HYP_{n+1}$ corresponding to an affine basis of $\gamma_n$ or $h\gamma_n$ satisfies $F \subset \text{CUT}_{n+1}$.

The list of 3773 facets of $HYP_7$ was found by Baranovskii [3] using the method described in [2]; i.e. he found, by hand, that, for all other hypermetric vectors $b$, one can express $H(b)$ as a sum of terms $H(b')$ with $b'$ belonging to his list of 3773 elements. While this result was announced in [3], the detailed computations were not published. In [11], another method was proposed: if Baranovskii’s list was not complete, then, in our computation [11] of the extreme rays of $HYP_7$, we should find some extreme rays which are not hypermetric. But this was not the case; so, the list is complete.

Under the action of the group $\text{Sym}(7)$, the set of facets is partitioned into 14 orbits, whose representations are given below:

$$
\begin{align*}
\mathbf{b}^1 &= (1, 1, -1, 0, 0, 0, 0) & \mathbf{b}^2 &= (1, 1, 1, -1, -1, 0, 0) \\
\mathbf{b}^3 &= (1, 1, 1, 1, -1, -2, 0) & \mathbf{b}^4 &= (-1, -1, -1, 1, 1, 2, 0) \\
\mathbf{b}^5 &= (1, 1, 1, 1, -1, -1, -1) & \mathbf{b}^6 &= (2, -2, 1, 1, 1, -1, -1) \\
\mathbf{b}^7 &= (2, 2, 1, -1, -1, -1) & \mathbf{b}^8 &= (-2, -2, 1, 1, 1, 1, 1) \\
\mathbf{b}^9 &= (3, 1, 1, -1, -1, -1) & \mathbf{b}^{10} &= (-3, -1, 1, 1, 1, 1, 1) \\
\mathbf{b}^{11} &= (2, 2, 1, 1, -1, -1, -1) & \mathbf{b}^{12} &= (-2, -2, 1, 1, 1, -1, 1) \\
\mathbf{b}^{13} &= (3, 2, 1, -1, -1, -1, -2) & \mathbf{b}^{14} &= (-3, -2, 1, 1, 1, 1, 1)
\end{align*}
$$

The first ten orbits are the orbits of hypermetric facets of the cut cone $\text{CUT}_7$; the first four of them come as the zero-extension of facets of the cone $HYP_6 = \text{CUT}_6$ (see [10, Chapter 7]). The last four orbits consist of some 19-dimensional simplex-faces of $\text{CUT}_7$, becoming 20-dimensional, i.e. simplex-facets in $HYP_7$.

Using Remark 1, we obtain that the list of 14 orbits of facets falls into nine equivalence classes $\mathbf{b}^1, \mathbf{b}^2, \{\mathbf{b}^3, \mathbf{b}^4\}, \mathbf{b}^5, \mathbf{b}^6, \{\mathbf{b}^7, \mathbf{b}^8\}, \{\mathbf{b}^9, \mathbf{b}^{10}\}, \{\mathbf{b}^{11}, \mathbf{b}^{12}\}, \{\mathbf{b}^{13}, \mathbf{b}^{14}\}$. So, there are nine combinatorial types of six-dimensional Delaunay polytope of rank 20 (i.e. repartition polytopes).

In [10, p. 229] another notion, switching by root, is defined: if $b = (b_0, \ldots, b_n)$ with $\sum_i b_i = 1, A \subset \{0, \ldots, n\}$ and $b(A) = 0$, then define $b^A$ by

$$
b^A_i = -b_i \quad \text{if } i \in A \quad \text{and} \quad b^A_i = b_i \quad \text{if } i \notin A.
$$

The switching by root of a facet-defining vector of $HYP_{n+1}$ is again a facet-defining vector (see [10, p. 229]). The three following vectors define facets of $HYP_7$, which are switching...
by root equivalent:

\[ b^6 = (2, -2, 1, 1, -1, -1), \quad b^7 = (2, 2, 1, -1, -1, -1), \quad b^8 = (-2, -2, 1, 1, 1, 1) . \]

On the other hand, \( b^7 \) and \( b^8 \) are geometrically equivalent by Remark 1, while \( b^6 \) and \( b^7 \) are not geometrically equivalent.

Recall that \( E_6, E_7, E_8 \) are root lattices defined by

\[
E_6 = \{ x \in E_7 : x_1 + x_2 = 0 \}, \quad E_7 = \{ x \in E_8 : x_1 + \cdots + x_8 = 0 \},
E_8 = \left\{ x \in \mathbb{R}^8 : x \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 \text{ and } \sum_i x_i \in 2\mathbb{Z} \right\} .
\]

The unique type of Delaunay polytope \( E_6 \) is called the Schlafli polytope (see [6]) and denoted by \( Sch \). Its skeleton graph is a 27-vertex (strongly regular) graph, called the Schlafli graph, whose symmetry group has size 51 840 and is isomorphic to the group of isometry preserving the Schlafli polytope. This group is denoted by \( Aut(Sch) \).

In [7] there were found 26 orbits of affine basis of the Schlafli polytope of the root lattice \( E_6 \) under the action of \( Aut(Sch) \). This yields 26 orbits of non-cut extreme rays of \( HY P_7 \) under the action of \( Sym(7) \), which are called Schlafli extreme rays. For every non-cut extreme ray \((\mathbb{R}^+, v)\) of \( HY P_7 \), there exists a facet-inducing inequality \( f(x) \geq 0 \) of \( CUT_7 \), which is non-hypermetric and satisfies \( f(v) < 0 \). This property establishes a bijection between the 26 orbits of non-hypermetric facets of the cut cone \( CUT_7 \) under the action of \( Sym(7) \) (see [13]) and the 26 orbits of the non-cut extreme rays of \( HY P_7 \) and proves that \( HY P_7 \) has 29 orbits of extreme rays under the action of \( Sym(7) \): three orbits of non-zero cuts and 26 orbits coming from \( Sch \) (see [11]).

**Proposition 4.** Let \( B \) be an affine basis of the Schlafli polytope; then it holds that:

i) The distance vector \( d_B \) is incident on 20 hypermetric faces of \( HY P_7 \), which are all facets of \( HY P_7 \).

ii) If \( F \) is a face of \( HY P_7 \), containing the vector \( d_B \), then it is non-degenerate, \( |Ann(F)| = 7 + \text{corank}(F) \) and the corresponding combinatorial type is not centrally symmetric.

**Proof.** The Schlafli polytope is six-dimensional and has 27 vertices. So, for every affine basis \( B \), the vector \( d_B \) satisfies \( H(b)d_B = 0 \) for 27 different \( b \in Ann(d_B) \); we write \( Ann(d_B) = \{ e_0, \ldots, e_6 \} = \{ b^1, \ldots, b^{20} \} \). On the other hand, it is known ([7, 10, p. 239]) that the Schlafli polytope has rank 1. Therefore, the rank of the matrix \( [H(b^i)_1 \leq i \leq 20] \) must be \((8+1)/2 - 1 = 20\). So, the family \( [H(b^j)]_{1 \leq j \leq 20} \) is linearly independent. If one of the \( H(b^j) \) is not a facet of \( HY P_7 \), then it can be expressed in terms of \( [H(b^j)]_{j\neq i} \); this contradicts linear independence and, so, (i) holds.

The extreme ray \( d_B \) is non-degenerate and is a sub-face of \( F \); so \( F \) is also non-degenerate. Every hypermetric face containing \( F \) contains \( d_B \); so, one has \( Ann(F) = \{ e_0, \ldots, e_6 \} = \{ b^1, \ldots, b^k \} \subset \{ b^1, \ldots, b^{20} \} \). The linear independence of the family \( [H(b^j)]_{1 \leq j \leq 20} \) implies that \( k = \text{corank}(F) \). Every \( b \in Ann(F) \) corresponds to a vertex of the Schlafli polytope, which is not centrally symmetric. If the combinatorial type of \( F \)
is centrally symmetric, then this implies by Lemma 13.2.7 of [10] that \( Sch \) is also centrally symmetric. \( \square \)

The above proposition is not true for the 56-vertex Gosset polytope \( Gos [6] \): the Gosset polytope has 374 orbits of affine bases under the action of the symmetry group \( Aut(Gos) \). Each extreme ray, corresponding to an affine basis \( \{v_0, \ldots, v_l\} \) of the Gosset polytope, is incident on 48 (= 56 – 8) hypermetric faces of \( HY P_8 \). But amongst these 48 face-defining inequalities, the number of facets varies from \( 27 \left(\frac{7+1}{2} - 1\right) \) to 41. See [10, p. 230], for general lower bounds (on the number of vertices of a Delaunay polytope) as a function of its rank.

**Theorem 1.** All six-dimensional Delaunay polytopes are basic.

**Proof.** Take \( P \) as a Delaunay polytope of a lattice \( L \) generated by the vectors \( w_1, \ldots, w_6 \). If \( P \) is a simplex, then it is a basic polytope, since there are 7 vertices and they form an affine basis; so, one can assume that \( P \) is not a simplex.

Denote by \( V \) the volume of the simplex formed by the vectors \( 0, w_1, \ldots, w_6 \). Take a family of 7 affinely independent vertices \( A = \{v_0, \ldots, v_6\} \) in the vertex set of \( P \) and denote the volume of the corresponding simplex by \( V' \).

In [21], it was proved that the relative volume \( k = \frac{V'}{V} \) is 1, 2 or 3. If \( k = 1 \), then \( A \) is an affine basis and we are done. Assume now that \( k > 1 \), i.e. that \( A \) is not an affine basis. Then, there exists a vertex \( v \) of \( P \), which is written uniquely as \( v = \sum_{i=0}^{6} b_i v_i \) with \( b \) being fractional and \( \sum_{i=0}^{6} b_i = 1 \).

The distance vector \( d_A \) satisfies \( H(b)d_A = 0 \) with \( b = (b_0, \ldots, b_6) \) being a fractional hypermetric vector. But one can express \( H(b) \) as \( \sum_{i=1}^{N} \lambda_i H(b^i) \) with \( \lambda_i > 0 \) and \( b^i \) being a permutation of one of the following vectors (see [21]):

<table>
<thead>
<tr>
<th>Case ( k = 2 )</th>
<th>Case ( k = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}(1, 1, 1, 1, -1, -1, 0) )</td>
<td>( \frac{1}{2}(2, 1, 1, 1, -1, -1, 1) )</td>
</tr>
<tr>
<td>( \frac{1}{2}(2, 1, 1, 1, -1, -1, -1) )</td>
<td>( \frac{1}{2}(1, 1, 1, 1, -1, -1, 1) )</td>
</tr>
<tr>
<td>( \frac{1}{2}(3, 1, 1, 1, -1, -1, -2) )</td>
<td>( \frac{1}{2}(2, 1, 1, 1, 1, -1, -2) )</td>
</tr>
<tr>
<td>( \frac{1}{2}(1, 1, 1, 1, 1, -1, -2) )</td>
<td>( \frac{1}{2}(3, 2, 1, -1, -1, -1, 0) )</td>
</tr>
<tr>
<td>( \frac{1}{2}(2, 1, 1, 1, 1, -1, -3) )</td>
<td>( \frac{1}{2}(1, -1, 0, 0, 0, 0, 0) )</td>
</tr>
</tbody>
</table>

Since \( \lambda_i > 0 \), one has \( H(b^i)d = 0 \); i.e. the vectors \( w'_i = \sum_{i=0}^{6} b^i v_i \) with \( 1 \leq l \leq N \) are vertices of the Delaunay polytope \( P \).

Since \( b \) is fractional, at least one of \( b^i \) is fractional, say, \( b^{i_0} \). But all fractional hypermetric vectors of the above table have one coordinate with absolute value equal to \( \frac{1}{k} \), say, \( |b^{i_0}_l| = \frac{1}{k} \). So, the family \( \{v_0, \ldots, v_{i_0-1}, v'_{i_0}, v_{i_0+1}, \ldots, v_6\} \) defines a simplex of relative volume \( k|b^{i_0}_l| = 1 \); i.e. it is an affine basis. \( \square \)
Theorem 2. The 6241 combinatorial types of Delaunay polytopes are partitioned by their rank in the following way:

<table>
<thead>
<tr>
<th>Rank</th>
<th>No. in HYP7</th>
<th>No. in CUT7</th>
<th>No. of centrally symmetric ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>1 (simplex)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>9 (repart.)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>30</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>95</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>233</td>
<td>28</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>500</td>
<td>95</td>
<td>1 (7-cross-polytope)</td>
</tr>
<tr>
<td>15</td>
<td>814</td>
<td>241</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>1092</td>
<td>434</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>1145</td>
<td>527</td>
<td>7</td>
</tr>
<tr>
<td>12</td>
<td>984</td>
<td>481</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>686</td>
<td>325</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>417</td>
<td>183</td>
<td>10</td>
</tr>
<tr>
<td>9</td>
<td>218</td>
<td>83</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>108</td>
<td>35</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>52</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>21</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1 (Schläfi)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Proof. The proof is purely computational and the method is described in the next section. Detailed results, as well as programs, are available from [12]. □

Corollary 1. All maximal six-dimensional Delaunay polytopes are: the Schlafli polytope, the 6-cube, the half 6-cube and the direct product of the half 5-cube with the 1-cube.

Proof. This result follows directly from the computation of the above theorem. □

4. Computational methods

In this section we explain our computational techniques; the actual implementation, using [20] and [14], is available from [12].

Our computation of combinatorial types of six-dimensional Delaunay polytopes used the face lattice of HYP7; combinatorial types of Delaunay polytopes of corank $i + 1$ were found from combinatorial types of Delaunay polytopes of corank $i$. We start from the list of combinatorial types of corank 1, i.e. the nine repartitioning polytopes. The plan of the computation was as follows:

(i) Take the list of combinatorial types of Delaunay polytopes of corank $i$.
(ii) For each of them, find all sub-faces, using our knowledge of facets and extreme rays of HYP7; we obtain faces of corank $i + 1$.
(iii) For every face $F$ of corank $i + 1$, find extreme rays $(\mathbb{R}_+, f_i)_{1 \leq i \leq M}$, contained in it, and define $d = \sum_{i=1}^{M} f_i$. The distance vector $d$ is in the interior of $F$; so, using Proposition 1, one can test whether this ray is non-degenerate or not, and this tells us whether the face is non-degenerate or not.
(iv) Find the classes of geometrical equivalence amongst the non-degenerate faces and, so, the list of combinatorial types of corank \( i + 1 \).

The above procedure finds all combinatorial types of Delaunay polytopes from corank 1 (i.e. repartitioning polytopes) up to corank 20 (i.e. the Schlafli polytope). But in order to describe our method completely, we need to be precise about how we find the classes under geometrical equivalence.

The first algorithm for the problem of geometrical equivalence is the following: given two faces \( F \) and \( F' \), find all affine bases of \( \text{Ann}(F) \) and test whether one yields an equivalence \( \phi_{F,F'} \) (see Definition 2). This method works for corank 1 or 2 and was used in the five-dimensional case (see [18]). But the number of affine bases becomes too large to be workable in corank 3.

So, one needs another, more efficient method. The first idea is to partition the set of faces of \( HY_{P_7} \) into two classes: faces which contain (one or more) Schlafli extreme rays (i.e. of the form \( d_B \) with \( B \) an affine basis of \( \text{Ann}(S) \)) and those which are generated only by cuts.

**Definition 4.** Let \( F \) be a face of \( HY_{P_7} \) which contains a Schlafli extreme ray \( d_B \) corresponding to an affine basis \( B = \{v_0, \ldots, v_6\} \) of \( \text{Sch} \). Then, every \( b \in \text{Ann}(F) \) defines a vertex \( v = \sum_{i=0}^{b} b_i v_i \) of \( \text{Sch} \). The set of such vertices is denoted by \( S(F,d_B) \) and contains \( B \).

**Proposition 5.** Let \( F \) and \( F' \) be two faces which contain at least one Schlafli extreme ray; then it holds that:

(i) If \( F \) and \( F' \) are geometrically equivalent, then for every Schlafli extreme ray \( d_B \) in \( F \), there exists an affine basis \( B' \) of \( \text{Sch} \) such that \( S(F,d_B) = S(F',d_{B'}) \).

(ii) If \( F \) and \( F' \) contain Schlafli extreme rays \( d_B \) and \( d_{B'} \) such that \( S(F,d_B) \) is identical to \( S(F',d_{B'}) \) up to an element of \( \text{Aut}(\text{Sch}) \), then \( F \) and \( F' \) are geometrically equivalent.

**Proof.** For every distance vector \( d_B \in F \), one gets an identification of \( \text{Ann}(F) \) with \( S(F,d_B) \). If \( F' \) is geometrically equivalent to \( F \), then the vectors \( e_0, \ldots, e_6 \) in \( \text{Ann}(F') \) are identified with vertices \( v_0', \ldots, v_6' \) in \( S(F,d_B) \). These vertices form an affine basis \( B' \) and one has \( S(F',d_{B'}) = S(F,d_B) \); so, (i) holds.

Since \( S(F,d_B) \) is isomorphic to \( S(F',d_{B'}) \) by an element of \( \text{Aut}(\text{Sch}) \), one can, without loss of generality, assume that they are identical. Now, \( B \) is identified with vertices \( v_0, \ldots, v_6 \) of \( S(F,d_B) \) and \( B' \) is identified with vertices \( v_0', \ldots, v_6' \) of \( S(F,d_B) \). Then, the expression for \( v_i' \) in terms of \( v_i \) determines an affine basis of \( \text{Ann}(F) \) for which the mapping \( \phi_{F,F'} \) is well-defined and bijective. \( \square \)

The above proposition expresses the geometrical equivalence in terms of the existence of an element of \( \text{Aut}(\text{Sch}) \) mapping a set of vertices into another set of vertices. Those sets of vertices are identified with corresponding sets of vertices of the Schlafli graph (the Schlafli graph and the Schlafli polytope have the same symmetry group \( \text{Aut}(\text{Sch}) \)). So, the problem is expressed in graph-theoretic terms and can be solved, using, for example, the Nauty program [20]. Therefore, one can build the geometrical equivalence classes.

Now, we extend the above method to the case of faces generated by cuts. Let \( F \) be a face generated by cuts \( \{\delta_j\}_{1 \leq j \leq M} \); first, one needs to find \( \text{Ann}(F) \), i.e. all vectors \( b \in \mathbb{Z}^{n+1} \) with \( \sum_{i=0}^{n} b_i = 1 \) having \( H(b)\delta_j = 0 \).
These equations can be rewritten as $\sum_{x \in S_j} b_x = x_j$ with $x_j = 0$ or $1$, i.e. as a linear system in $b$. This linear system has rank $n + 1$, because of the proposition below; so, one can find the set $\text{Ann}(F)$ for every face generated by cuts.

**Proposition 6.** Let $F$ be a face of $HY P_{n+1}$ generated by cuts $(\delta S_j)_{1 \leq j \leq M}$. Then the following properties are equivalent:

(i) the face $F$ is non-degenerate and

(ii) the linear system formed by the equations $\sum_{x \in S_j} \alpha_x = 0$ and $\sum_{i=0}^{n} \alpha_i = 0$ has solution set $\{0\}$.

**Proof.** If the face $F$ is degenerate, then there exists a vertex $v$ which can be expressed in two different forms: $v = \sum_{i=0}^{n} b_i v_i = \sum_{i=0}^{n} b_i' v_i$. So, writing $\alpha_i = b_i - b_i'$, one gets $v = \sum_{i=0}^{n} (b_i + k \alpha_i) v_i$ with $k \in \mathbb{Z}$ and $b + k \alpha$ belongs to $\text{Ann}(F)$. Therefore, one gets $\sum_{x \in S_j} b_x + k \alpha_x = 0$ or $1$, and $\sum_{i=0}^{n} b_i + k \alpha_i = 1$ for all $k \in \mathbb{Z}$. This is possible only if $\sum_{x \in S_j} \alpha_x = 0$ and $\sum_{i=0}^{n} \alpha_i = 0$.

Let the solution set be non-zero, i.e. suppose that one can find an integer-valued non-zero solution $\alpha$. This implies that the vectors $e_0 + k \alpha$ belong to $\text{Ann}(F)$ for every $k \in \mathbb{Z}$. So, $\text{Ann}(F)$ is infinite and $F$ is degenerate. □

Every cut $\delta S_j \in F$ with $1 \leq j \leq M$ defines a semi-metric on the set $\{e_0, \ldots, e_n\}$. If one writes $\text{Ann}(F) = \{b^1, \ldots, b^p\}$, then this semi-metric can be uniquely extended to $\text{Ann}(F)$ by $d_j(b, b') = |b(S_j) - b'(S_j)|$.

So, with every face, generated by cuts, one can associate a set of semi-metrics on $\text{Ann}(F)$ which are, in fact, cut semi-metrics.

A combinatorial type of Delaunay polytope encodes all possible embeddings of $\text{Ann}(F)$ into the vertex set $V$ of a Delaunay polytope of a lattice. These embeddings are completely described by the distance vector $d_V$ on their vertices. This distance vector is expressed as $\sum_{j=1}^{M} \lambda_j d_j$ with $\lambda_j > 0$.

Therefore, the combinatorial type of a face $F$ generated by cuts corresponds to a set of cut semi-metrics on the set $\text{Ann}(F)$. This combinatorial information can be expressed in graph-theoretic terms; so, we can again test whether two faces generated by cuts are isomorphic, using the Nauty program [20]. So, again one can build the geometrical equivalence classes and our method is completely described.

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**References**

