Frames, graphs and erasures

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Abstract

Two-uniform frames and their use for the coding of vectors are the main subject of this paper. These frames are known to be optimal for handling up to two erasures, in the sense that they minimize the largest possible error when up to two frame coefficients are set to zero. Here, we consider various numerical measures for the reconstruction error associated with a frame when an arbitrary number of the frame coefficients of a vector are lost. We derive general error bounds for two-uniform frames when more than two erasures occur and apply these to concrete examples. We show that among the 227 known equivalence classes of two-uniform (36, 15)-frames arising from Hadamard matrices, there are 5 that give smallest error bounds for up to 8 erasures.

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1. Introduction

Uniform tight frames are a well-known means for the redundant encoding of vectors in terms of their expansion coefficients. Such frames were studied in [2,6,7] and
shown to be optimal in some sense for one erasure. In addition, further properties of these frames were developed, including their robustness to more than one erasure. In this paper we introduce some measures of how well a frame behaves under multiple erasures and then seek optimal frames in these contexts.

In an earlier paper [9], a family of so-called two-uniform frames was introduced. When they exist, two-uniform frames were demonstrated to be optimal for one and two erasures. Moreover, it was proved that a frame is two-uniform if and only if it is equiangular which is a family of frames that has been studied independently by Thomas Strohmer and Robert Heath [13]. The existence of such frames, over the reals, depends on the existence of a matrix of ±1’s which satisfies certain algebraic equations. Strohmer and Heath made the key discovery that these matrices had been studied earlier in the graph theory literature and correspond precisely to the Seidel adjacency matrices of a very special type of graph. In this paper, we derive explicit formulas that relate how well two-uniform frames behave under erasures to various connectivity problems of the related graphs.

This paper is organized as follows. After fixing the notation in Section 2, we discuss the idea of using frames as codes in Section 3 and introduce a family of numerical measures for the error when the coded information is partially deleted. Section 4 recalls two-uniform frames as the ones that perform best under one and two deletions. The construction of such frames is then related to a problem in graph theory. In Section 5, we derive error bounds from the graph-theoretic formulation. The general error bounds are illustrated with examples in Section 6.

2. Preliminaries and notation

We begin by recalling the basic definitions and concepts.

Definition 2.1. Let \( \mathcal{H} \) be a Hilbert space, real or complex, and let \( F = \{f_i\}_{i \in I} \subset \mathcal{H} \) be a subset. We call \( F \) a frame for \( \mathcal{H} \) provided that there are two constants \( C, D > 0 \) such that the norm inequalities

\[
C \cdot \|x\|^2 \leq \sum_{j \in I} |\langle x, f_j \rangle|^2 \leq D \cdot \|x\|^2
\]

hold for every \( x \in \mathcal{H} \). Here, \( \langle \cdot, \cdot \rangle \) denotes the inner product of two vectors, by convention conjugate linear in the second entry if \( \mathcal{H} \) is a complex Hilbert space.

When \( C = D = 1 \), then we call \( F \) a Parseval frame. Such frames are also called normalized, tight frames, but Parseval frame is, perhaps, becoming more standard.

A frame \( F \) is called uniform or equal-norm provided there is a constant \( c \) so that \( \|f\| = c \) for all \( f \in F \).

The linear map \( V : \mathcal{H} \to \ell_2(I) \) defined by

\[
(Vx)_i = \langle x, f_i \rangle
\]
is called the **analysis operator**. When $F$ is a Parseval frame, then $V$ is an isometry and its adjoint, $V^*$, acts as a left inverse to $V$.

For the purposes of this paper we will only be concerned with finite dimensional Hilbert spaces and frames for these spaces that consist of finitely many vectors. When the dimension of $H$ is $k$, then we will identify $H$ with $\mathbb{R}^k$ or $\mathbb{C}^k$ depending on whether we are dealing with the real or complex case, and for notational purposes regard vectors as columns.

When we wish to refer to either case, then we will denote the ground field by $\mathbb{F}$.

**Definition 2.2.** We shall let $\mathcal{F}(n, k)$ denote the collection of all Parseval frames for $\mathbb{F}^k$ consisting of $n$ vectors and refer to such a frame as either a real or complex $(n, k)$-frame, depending on whether or not the field $\mathbb{F}$ is the real numbers or the complex numbers. Thus, a uniform $(n, k)$-frame is a uniform Parseval frame for $\mathbb{F}^k$ with $n$ vectors. The ratio $n/k$ we shall refer to as the **redundancy** of the frame.

If we identify the analysis operator $V$ of an $(n, k)$-frame with an $n \times k$ matrix, using the standard basis, then the columns of $V^*$ are the frame vectors.

**Facts 2.3.** Using some basic operator theory, $F$ is an $(n, k)$-frame if and only if the Grammian (or correlation) matrix $VV^* = (\langle f_j, f_i \rangle)$ of the frame vectors is a self-adjoint $n \times n$ projection of rank $k$. Moreover, the rank of a projection is equal to its trace, so $\text{tr}(VV^*) = k$. Thus, when $F$ is a uniform $(n, k)$-frame, each of the diagonal entries of $VV^*$ must be equal to $k/n$, and hence each frame vector $f_j$ must be of length $\| f_j \| = \sqrt{k/n}$.

Conversely, given an $n \times n$ self-adjoint projection $P$ of rank $k$, we can always factor it as $P = VV^*$ with an $n \times k$ matrix $V$, by choosing an orthonormal basis for the range of $P$ as the column vectors of $V$. It follows that $V^*V = I_k$ and hence $V$ is the matrix of an isometry and so corresponds to an $(n, k)$-frame. Moreover, if $P = WW^*$ is another factorization of $P$, then necessarily $W^*W = I_k$ and there exists a unitary $U$ such that $W^* = UV^*$ and hence the two corresponding frames differ by multiplication by this unitary. Thus, $P$ determines a unique unitary equivalence class of frames. A self-adjoint projection $P$ corresponds to a uniform $(n, k)$-frame if and only if all of its diagonal entries are $k/n$.

**Definition 2.4.** In the following, we wish to identify certain frames as being equivalent. Given frames $F = \{ f_1, \ldots, f_n \}$ and $G = \{ g_1, \ldots, g_n \}$, we say that they are **type I equivalent** if there exists a unitary (orthogonal, in the real case) matrix $U$ such that $g_i = U f_i$ for all $i$. If $V$ and $W$ are the analysis operators for $F$ and $G$, respectively, then it is clear that $F$ and $G$ are type I equivalent if and only if $V = WU$ or equivalently, if and only if $VV^* = WW^*$. Thus, there is a one-to-one correspondence between $n \times n$ rank $k$ projections and type I equivalence classes of $(n, k)$-frames.
We say that two frames are type II equivalent if they are simply a permutation of the same vectors and type III equivalent if the vectors differ by multiplication with \( \pm 1 \) in the real case and multiplication by complex numbers of modulus one in the complex case.

Finally, we say that two frames are equivalent if they belong to the same equivalence class in the equivalence relation generated by these three equivalence relations. It is not hard to see that if \( F \) and \( G \) are frames with analysis operators \( V \) and \( W \), respectively, then they are equivalent if and only if \( UVV^*U^* = WW^* \) for some \( n \times n \) unitary \( U \) that is the product of a permutation and a diagonal unitary (or diagonal orthogonal matrix, in the real case).

We caution the reader that the equivalence relation that we have just defined is different from the equivalence relation that is often used, but it is the one studied in [9]. In other papers, often frames \( \{f_i\} \) and \( \{g_i\} \) are called equivalent provided that there is an invertible operator \( T \) such that \( T f_i = g_i \) for all \( i \), which is clearly a much coarser equivalence relation than the one used here.

3. Frames and erasures

The idea behind treating frames as codes is that, given an original vector \( x \) in \( \mathbb{F}^k \) and an \((n, k)\)-frame with analysis operator \( V \), one regards the vector \( Vx \in \mathbb{F}^n \) as an encoded version of \( x \), which might then be somehow transmitted to a receiver and then decoded by applying \( V^* \). Among all possible left inverses of \( V \), we have that \( V^* \) is the unique left inverse that minimizes both the operator norm and Hilbert–Schmidt norm.

Suppose that in the process of transmission some number, say \( m \), of the components of the vector \( Vx \) are lost, garbled or just delayed for such a long time that one chooses to reconstruct \( x \) with what has been received. In this case we can represent the received vector as \( EVx \), where \( E \) is a diagonal matrix of \( m \) 0’s and \( n - m \) 1’s corresponding to the entries of \( Vx \) that are, respectively, lost and received. The 0’s in \( E \) can be thought of as the coordinates of \( Vx \) that have been “erased” in the language of [6].

There are now two methods by which one could attempt to reconstruct \( x \). Either one is forced to compute a left inverse for \( EV \) or one can continue to use the left inverse \( V^* \) for \( V \) and accept that \( x \) has only been approximately reconstructed.

If \( EV \) has a left inverse, then the left inverse of minimum norm is given by \( T^{-1}W^* \) where \( EV = WT \) is the polar decomposition and \( T = |EV| = (V^*EV)^{1/2} \). Thus, the minimum norm of a left inverse is given by \( t_{\min}^{-1} \) where \( t_{\min} \) denotes the least eigenvalue of \( T \).

In the second alternative, the error in reconstructing \( x \) is given by

\[
x - V^*EVx = V^*(I - E)Vx = (I - T^2)x = V^*DVx,
\]
where $D$ is a diagonal matrix of $m$ 1’s and $n - m$ 0’s. Thus, the norm of the error operator is $1 - \ell^2_{\text{min}}$.

Hence we see that, when a left inverse exists, the problems of minimizing the norm of a left inverse over all frames and of minimizing the norm of the error operator over all frames are really equivalent and are both achieved by maximizing the minimal eigenvalue of $T$.

It is important to note that a left inverse will exist if and only if the norm of the error operator $V^*DV$ is strictly less than 1.

In this paper, we study the norms of error operators rather than those of the left inverses, since this seems to lead to cleaner formulas and attempt to describe the frames for which the norms of these error operators are in some sense minimized, independent of which erasures occur. Of course there are many ways that one could define “best” in this setting and we only pursue a few reasonable possibilities.

The first quantity that we introduce represents the maximal norm of an error operator given that some set of $m$ erasures occurs and the second represents an $\ell^p$-average of the norm of the error operator over the set of all possible $m$ erasures.

**Definition 3.1.** We let $D_m$ denote the set of diagonal matrices that have exactly $m$ diagonal entries equal to 1 and $n - m$ entries equal to 0.

Given an $(n, k)$-frame $F$, we set

$$e^\infty_m(F) = \max \{\|V^*DV\| : D \in D_m\},$$

and for $1 \leq p$,

$$e^p_m(F) = \left(\sum_{D \in D_m} \|V^*DV\|^p\right)^{1/p},$$

where $V$ is the analysis operator of $F$ and by the norm of a matrix we always mean its operator norm.

**Remark 3.2.** Note that for a given frame $F$, a left inverse will exist for all possible sets of $m$ erasures if and only if $e^\infty_m(F) < 1$. Moreover, by the explanation preceding the definition of $e^p_m(F)$, whenever at most $m$ erasures occur, then a left inverse of $EV_L$ will exist satisfying $\|L\| \leq \frac{1}{\sqrt{1-e^\infty_m(F)}}$.

In [9] only the quantity $e^\infty_m(F)$ is considered and it is denoted $d_m(F)$.

Finally, we remark that the above quantities are invariant under the frame equivalence defined in the first section.

**Definition 3.3.** Since $\mathcal{F}(n, k)$ is a compact set for any $p$, $1 \leq p \leq \infty$, the value

$$e^p_1(n, k) = \inf\{e^p_1(F) : F \in \mathcal{F}(n, k)\}$$

is attained and we define the $(1, p)$-erasure frames to be the nonempty compact set $\delta^p_1(n, k)$ of frames where this infimum is attained, i.e.,

$$\delta^p_1(n, k) = \{F \in \mathcal{F}(n, k) : e^p_1(F) = e^p_1(n, k)\}.$$
Proceeding inductively, we now set, for \(1 \leq m \leq n\),
\[
e_{m}^{p}(n, k) = \inf\{e_{m}^{p}(F) : F \in \mathcal{F}_{m-1}^{p}(n, k)\}
\]
and define the \((m, p)\)-erasure frames to be the nonempty compact subset \(\mathcal{E}_{m}^{p}(n, k)\) of \(\mathcal{E}_{m-1}^{p}(n, k)\) where this infimum is attained.

In this fashion, we obtain a decreasing family of frames and we wish to describe and construct the frames in these sets. Since these sets are invariant under frame equivalence, we are really only interested in finding representatives for each frame equivalence class.

The results of [2] can be interpreted as characterizing \(\mathcal{E}_{\infty}^{1}(n, k)\). The following extends their result slightly.

\textbf{Proposition 3.4.} For \(1 < p \leq \infty\), the set \(\mathcal{E}_{1}^{p}(n, k)\) coincides with the family of uniform \((n, k)\)-frames, while \(\mathcal{E}_{1}^{1}(n, k) = \mathcal{F}(n, k)\). Consequently, for \(1 \leq p \leq \infty\),
\[
e_{1}^{p}(n, k) = \frac{k}{n}.
\]

\textbf{Proof.} Given an \((n, k)\)-frame \(F = \{f_1, \ldots, f_n\}\), if we regard the frame vectors as column vectors, then the analysis operator \(V\) is just the matrix whose \(j\)th row is \(f_j^*\).

Given \(D\) in \(\mathcal{D}_1\) which is 1 in the \(j\)th entry, we have that
\[
\|V^*DV\|^p = \|DVV^*D\|^p = \|f_jf_j^*\|^p = \|f_j\|^2p.
\]

Thus, we see that
\[
e_{1}^{\infty}(F) = \max\{\|f_j\|^2 : 1 \leq j \leq n\}
\]
and
\[
e_{1}^{p}(F) = \left(\frac{1}{n} \sum \|f_j\|^2p\right)^{1/p}.
\]

Since \(\sum_j \|f_j\|^2 = \text{tr}(VV^*) = k\), we see that these quantities are minimized when \(\|f_j\|^2 = k/n\) for all \(j\).

Note that when \(p = 1\), the quantity \(e_{1}^{1}(F) = k/n\) for every \((n, k)\)-frame and so the result follows. \(\Box\)

We now turn to finding frames that belong to \(\mathcal{E}_{2}^{p}(n, k)\). By Proposition 3.4, for \(1 < p\) these are the uniform \((n, k)\)-frames which achieve the infimum of \(e_{2}^{p}(n, k)\), while for \(p = 1\) these are just the \((n, k)\)-frames that minimize \(e_{1}^{p}(n, k)\).

If \(D\) is in \(\mathcal{D}_2\) and has a 1 in the \(i\)th and \(j\)th diagonal entries and \(V\) is the analysis operator for a uniform \((n, k)\)-frame \(F = \{f_1, \ldots, f_n\}\), then \(\|V^*DV\| = \|DVV^*D\| = k/n + |\langle f_i, f_j \rangle| = (1 + \cos(\theta_{i,j}))k/n\) where \(0 \leq \theta_{i,j} \leq \pi/2\) is the angle between the lines spanned by the \(i\)th and \(j\)th frame vector.
Thus, as observed in [9] the frames in $\mathcal{E}_{\infty}^2(n, k)$ are precisely the uniform $(n, k)$-frames for which the smallest angle between the lines generated by the frame vectors is maximized. In [13] these frames were introduced for other reasons and were called Grassmannian frames.

We now turn our attention to the frames that are the main topic of this paper.

**Definition 3.5.** We call $F$ a 2-uniform $(n, k)$-frame provided that $F$ is a uniform $(n, k)$-frame and in addition $\|V^*DV\|$ is a constant for all $D$ in $\mathbb{D}_2$.

Unlike uniform frames, 2-uniform frames do not exist for all values of $k$ and $n$. Later we will give a table that gives a complete list of all pairs $(n, k)$ for $n \leq 50$ for which 2-uniform $(n, k)$-frames exist over the reals, together with what is known about the numbers of frame equivalence classes. Each such frame is also a 2-uniform frame over the complex field, but a complete list of all 2-uniform frames over the complex field for $n \leq 50$ is still not known.

In [9] it is proved that a uniform $(n, k)$-frame $F$ is 2-uniform if and only if $|\langle f_j, f_i \rangle| = c_{n,k}$ is constant for all $i \neq j$, where

$$c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}}.$$

The families of frames with this property have been studied independently in [13], where they are called equiangular frames.

In [9] it is shown that if there exists a 2-uniform $(n, k)$-frame, then every frame in $\mathcal{E}_{\infty}^m(n, k)$ is 2-uniform for $2 \leq m$ and $e_{\infty}^2(n, k) = k/n + c_{n,k}$. If there does not exist a 2-uniform $(n, k)$-frame, then necessarily $e_{\infty}^2(n, k) > k/n + c_{n,k}$.

We now prove an analogous result for sufficiently large $p$.

**Theorem 3.6.** If a 2-uniform $(n, k)$-frame $F$ exists among the uniform frames, then for $p > 2 + \sqrt{\frac{5k(n-1)}{n-1}}$ and $m \geq 2$, every frame in $\mathcal{E}_{m}^p(n, k)$ is 2-uniform and $e_{m}^p(n, k) = k/n + c_{n,k}$. If there does not exist a 2-uniform $(n, k)$-frame, then $e_{m}^p(n, k) > k/n + c_{n,k}$ for the above values of $p$.

**Proof.** We recall that by definition, a frame $F$ in $\mathcal{E}_{m}^p(n, k) \subset \mathcal{E}_{\infty}^p(n, k)$ minimizes

$$2 \sum_{D \in \mathbb{D}_2} \|V^*DV\|^p = \sum_{i \neq j} (k/n + |p_{i,j}|)^p$$

among all $(n, k)$-frames in $\mathcal{E}_{m}^p(n, k)$, where $p_{i,j} = \langle f_i, f_j \rangle$. Any such frame $F \in \mathcal{E}_{m}^p(n, k)$ satisfies the constraint

$$\sum_{i \neq j} |p_{i,j}|^2 = \frac{n k - k^2}{n}.$$
because $F$ is uniform and therefore $k = \text{tr}(P) = \text{tr}(P^2)$. To simplify notation, we renumber the $N := n(n-1)$ quantities $\{|p_{i,j}|, i \neq j\}$ and denote them by $x_l, l \in \{1, 2, \ldots, N\}$. In addition, we abbreviate $a := k/n$ and $b := \sqrt{(nk-k^2)/n}$.

Our proof is a variational argument. It is complete if we show that the minimum of the function $\sigma_p(x) := \sum_{l=1}^N (a + x_l)^p$ subject to $x_l \geq 0$ for all $l$ and $\sum_{l=1}^N x_l^2 = b^2$ is attained if and only if all $x_l$ are identical.

As a first step we show that there is $d > 0$ such that for any $l$, either $x_l = d$ or $x_l = 0$. For $N = n = 2$, this is an exercise in calculus. We want to minimize the function $\sigma_p(u, v) = (a + u)^p + (a + v)^p$ subject to the constraints $u^2 + v^2 = b^2$, $u \geq 0$ and $v \geq 0$. Our claim is that the minimum occurs either when $u = 0$ or $v = 0$ or at $u = v$. At first we identify critical points of $\sigma_p$ on the arc $A_b = [u^2 + v^2 = b^2, u > 0, v > 0]$. By symmetry, the center $u = v = b/\sqrt{2}$ is a critical point and there must be an odd number of such points. The usual Lagrange multiplier argument shows that at a critical point, the equation $(a + u)^{p-1}/u = (a + v)^{p-1}/v$ holds. The set of points satisfying this equation in the domain $u, v > 0$ can be split in the three curves $C_1 = \{u = v \geq 0\}$, $C_2 = \{v > u > 0, v(a + u)^{p-1} = u(a + v)^{p-1}\}$, and $C_3 = \{u > v > 0, v(a + u)^{p-1} = u(a + v)^{p-1}\}$.

We parametrize the curve $C_2$ by $\lambda = (a + u)^{p-1}/u = (a + v)^{p-1}/v$ and show that this curve intersects only in one point with the arc $A_b$. Once we have proved this, we know this critical point cannot be a local minimum, because $\sigma_p(u, \sqrt{b^2 - u^2})$ is increasing for sufficiently small values of $u$. By the same argument, $C_3$ does not contain any local minima, and therefore they must occur either at $u = 0, v = 0$, or $u = v$.

To finish the argument for $N = 2$, let us assume that $C_2$ and $A_b$ intersect in more than one point and derive a contradiction. The function $f(t) = \frac{(a+t)^p}{t}$ has a global minimum at $t = \frac{a}{p-2}$ so assuming $f(u) = f(v) = \lambda$, $v > u > 0$, and $u^2 + v^2 = b^2$ implies the bounds $u < \frac{a}{p-2}$ and $v > \sqrt{b^2 - a^2/(p-2)}$ for all these intersection points. By the condition on $p$, $\frac{a}{p-2} < \frac{b}{\sqrt{2}}$, and therefore points in $C_2$ sufficiently close to $u = v = \frac{a}{p-2}$ are in the region bounded by the arc $A_b$ and the coordinate axes. Moreover, $C_2$ contains points outside of this region, because as the parameter $\lambda$ tends to infinity, so does $v$. All these facts are unchanged if we replace the radius $b$ of the arc by a sufficiently close radius $b'$. Choosing $b'$ appropriately, we can obtain intersection points of $C_2$ and $A_{b'}$ with coordinates $u_1 < v_1 < v_2$ such that at $(u_1, v_1)$, the radius $\sqrt{u^2 + v^2}$ increases as the parameter $\lambda$ increases, and at $(u_2, v_2)$ the radius decreases with $\lambda$. Consequently, $\frac{d}{dx}[u^2 + v^2]]_{u=u_1, v=v_1} \geq 0$ and $\frac{d}{dx}[u^2 + v^2]]_{u=u_2, v=v_2} \leq 0$. To derive the contradiction, we note that

$$\frac{d}{dx}[u^2 + v^2] = 2 \left[ \frac{u}{f'(u)} + \frac{v}{f'(v)} \right].$$
Lemma 3.7. An \((n, k)\)-frame \(F = \{f_1, \ldots, f_n\}\), is in \(\mathcal{E}_2^n(n, k)\) if and only if it is a uniform frame and minimizes \(\sum_{i \neq j} |\langle f_i, f_j \rangle|\).

Proof. The uniformity of \(F\) results from the inclusion \(\mathcal{E}_2^n(n, k) \subset \mathcal{E}_1^n(n, k)\).

Due to the constraint implicit in the uniformity as stated in the proof of the preceding theorem, \(e_2^1(F) = (A + B \sum_{i \neq j} |\langle f_i, f_j \rangle|)^{1/2}\) for some positive constants \(A\) and \(B\) that depend only on \(n\) and \(k\). Thus, \(e_2^1(F)\) is clearly minimized when \(\sum_{i \neq j} |\langle f_i, f_j \rangle|\) is minimized. \(\square\)
Proposition 3.8. If a 2-uniform \((n, k)\)-frame \(F\) exists among the uniform frames, then it maximizes the error \(e^2_2(F)\). If a uniform \((n, k)\)-frame exists that is not 2-uniform, then \(e^2_2(n, k)\) does not contain any 2-uniform frames.

Proof. This is again a consequence of the fact that all frames in \(e^2_2(n, k)\) are uniform, and thus the entries of the associated Grammians observe a constraint of the form \(\sum_{i\neq j} |p_{i,j}|^2 = b^2\) with a constant \(b\) that only depends on \(n\) and \(k\).

If 2-uniform frames exist, they are maximizers because, subject to the constraint \(\sum_j x_j^2 = b^2\), the function \(\sum_j |x_j|\) is maximized when all \(|x_j|\) are equal. Thus, given any uniform \((n, k)\)-frame that is not 2-uniform, it will perform better and \(e^2_2(n, k)\) cannot contain 2-uniform frames.

Example 3.9. By Example 4.4 below, we know that 2-uniform \((2k, k)\)-frames exist for infinitely many choices of \(k \geq 2\). One example of a uniform frame \(F'\) that outperforms any such 2-uniform frame is given by basis repetition. That is, we simply repeat the vectors of an orthonormal basis twice and rescale them by \(1/\sqrt{2}\) to construct the uniform frame \(F'\). For \(k \geq 2\), this is not 2-uniform, because the associated Grammian \(P'\) has off-diagonal elements that are zero. By the preceding proposition then \(e^2_2(2k, k)\) does not contain any 2-uniform frames.

4. Two-uniform frames and graphs

In this section we study the existence and construction of 2-uniform frames. For many possible values of \((n, k)\), there do not exist any 2-uniform frames. Moreover, when there do exist 2-uniform frames, then there are at most finitely many such frame equivalence classes and hence the problem of determining optimal frames in our sense, i.e., frames in \(E^p_m(n, k)\), is reduced to the problem of finding representatives for each equivalence class and determining which one of these finitely many equivalence classes is optimal.

Thanks to the discovery by [13] of the connection between equiangular frames and the earlier work of Seidel and his collaborators in graph theory, much of the work on existence, construction and determining frame equivalence classes for these frames is already known and exists in the literature.

We begin this section by summarizing this information.

Definition 4.1. Given a 2-uniform \((n, k)\)-frame \(F = \{f_1, \ldots, f_n\}\) the correlation matrix is a self-adjoint rank \(k\) projection that can be written in the form \(P = VV^* = aI + c_{n,k}Q\) where \(a = k/n\), \(c_{n,k}\) is given by the formula derived in the last section, and \(Q = (q_{i,j})\) is a self-adjoint matrix satisfying \(q_{i,i} = 0\) for all \(i\) and for \(i \neq j\), \(|q_{i,j}| = 1\). We call the \(n \times n\) self-adjoint matrix \(Q\) obtained above the signature matrix of \(F\).
We shall derive further properties that the signature matrix $Q$ must satisfy and then use solutions of these equations to generate 2-uniform frames. The fact that in the real case, $Q$ must be a matrix of 0’s, 1’s and $-1$’s shows that for fixed $(n, k)$ there are only finitely many possibilities for the Grammian matrix of a 2-uniform $(n, k)$-frame. Consequently, up to equivalence, there can be only finitely many 2-uniform $(n, k)$-frames for each pair $(n, k)$.

The key facts about signature matrices are summarized in the following theorem from [9].

**Theorem 4.2** [9]. Let $Q$ be a self-adjoint $n \times n$ matrix $Q$ with $q_{i,i} = 0$ for all $i$ and $|q_{i,j}| = 1$ for all $i \neq j$. Then the following are equivalent:

(i) $Q$ is the signature matrix of a 2-uniform $(n, k)$-frame,

(ii) $Q^2 = (n - 1)I + \mu Q$

for some necessarily real number $\mu$,

(iii) $Q$ has exactly two distinct eigenvalues, denoted as $\rho_1 > \rho_2$.

When any of these equivalent conditions hold, then the parameters $k$, $\mu$, $\rho_1$, and $\rho_2$ are related by the following equations,

$$
\mu = (n - 2k) \sqrt{\frac{n - 1}{k(n - k)}} = \rho_1 + \rho_2,
$$

$$
1 - \rho_1 \rho_2 = n,
$$

$$
k = \frac{n}{2} - \frac{\mu n / 2}{\sqrt{4(n - 1) + \mu^2}} = \frac{-n \rho_2}{\rho_1 - \rho_2} = \text{mult}(\rho_1),
$$

where $\text{mult}(\rho_1)$ indicates the multiplicity of the eigenvalue $\rho_1$. In particular, solutions of these equations can only exist for real numbers $\mu$ such that the formula for $k$ yields an integer. Moreover, in the case of real 2-uniform frames, the entries of $Q^2$ will all be integers and hence $\mu$ must also be an integer.

The above theorem reduces the construction of 2-uniform frames to producing matrices $Q$ satisfying the appropriate equations.

**Example 4.3** (The dimension and codimension 1 case). Let $J_n$ denote the $n \times n$ matrix all of whose entries are 1. Then $Q = I_n - I_n$ satisfies $Q^2 = J_n^2 = 2J_n + I_n = (n - 2)J_n + I_n = (n - 1)I_n + (n - 2)Q$ and so by our above formulas $\mu = n - 2$ and $k = 1$, yielding the rather uninteresting 2-uniform frame for $\mathbb{F}^1$.

However, $Q = I_n - J_n$ is also a signature matrix with $\mu = 2 - n$, $k = n - 1$, which shows that for each $k$ there exists a 2-uniform $(k + 1, k)$-frame.

This frame is described in detail in [2] and is in fact the only real uniform $(k + 1, k)$-frame, up to some natural equivalence. We shall refer to these examples, which exist for every $n$ as the trivial 2-uniform frames.
Example 4.4 (Conference matrices). The idea of using conference matrices to construct frames of this type originates in [13].

A real \( n \times n \) matrix \( C \) with \( c_{i,i} = 0 \) and \( c_{i,j} = \pm 1 \) for \( i \neq j \) is called a conference matrix [3] provided \( C^2 = (n - 1)I \).

Thus, every symmetric conference matrix is a signature matrix with \( \mu = 0 \) and \( k = n/2 \). So, in particular such matrices must be of even size and they yield real 2-uniform \((2k, k)\)-frames, for certain values of \( k \).

Conference matrices are known to exist for many values of \( n \). Paley [10] constructs symmetric conference matrices, for every \( n = p^j + 1 \equiv 2 \mod 4 \) with \( p \) prime. For further examples, see [4].

If \( C = -C^t \) is a skew-symmetric conference matrix, then setting \( Q = iC \) yields a complex 2-uniform \((2k, k)\)-frame. Similarly, examples of skew-symmetric conference matrices can be found in many places in the literature. See, for example [14, 4]. Note that conference matrices yield 2-uniform frames with redundancy 2. Conversely, it is not hard to see that the signature matrix of any real 2-uniform frame of redundancy 2 is a conference matrix.

Example 4.5 (Hadamard matrices). Using Hadamard matrices to construct 2-uniform frames has been discussed in [9]. A real \( n \times n \) matrix \( H \) is called a Hadamard matrix [3] provided that \( h_{i,j} = \pm 1 \) and \( H^*H = nI \). If \( H = H^* \) is a symmetric Hadamard matrix and in addition, \( h_{i,i} = 1 \) for all \( i \), then \( H \) is called a graph Hadamard. In this case \( Q = H - I \) is a signature matrix for a real 2-uniform frame with \( \mu = -2 \) and \( k = \frac{n+\sqrt{n}}{2} \).

Similarly, \( Q = I - H \) is a signature matrix for a real 2-uniform frame with \( \mu = 2 \) and \( k_1 = n - k = \frac{n-\sqrt{n}}{2} \).

Graph Hadamards are known to exist for many values of \( n \). Given two graph Hadamard matrices, their Kronecker tensor product gives rise to another graph Hadamard matrix. Thus, starting with the easily constructed \( 4 \times 4 \) graph Hadamard, one obtains graph Hadamards of order \( 4^m \) for every \( m \).

A Hadamard matrix \( H \) is called a skew Hadamard if \( H + H^* = 2I \). Note that such a matrix is not actually skew, but is as nearly skew as a Hadamard matrix can be.

If \( H \) is a skew Hadamard matrix, then \( Q = \pm i(H - I) \) are signature matrices for complex 2-uniform frames with \( k = n/2 \).

Skew Hadamards are known to exist for all \( n \equiv 0 \mod 4 \) and \( n \leq 100 \) with the exception of \( n = 72 \).

Note that for the 2-uniform frames derived from graph Hadamards, as \( n \) tends to infinity, the redundancy tends to 2, while for all the skew Hadamards the redundancy is equal to 2.

For further examples of 2-uniform frames, we need to turn to some results in graph theory that were first introduced to the frame theory community by [13].
Definition 4.6. Given a graph $G$ on $n$ vertices, the **Seidel adjacency matrix** of $G$ is defined to be the $n \times n$ matrix $A = (a_{i,j})$ where $a_{i,j}$ is defined to be $-1$ when $i$ and $j$ are adjacent, $+1$ when $i$ and $j$ are not adjacent, and $0$ when $i = j$.

Two graphs on $n$ vertices are called **switching equivalent** exactly when their Seidel adjacency matrices are equivalent via conjugation by an orthogonal matrix that is the product of a permutation and a diagonal matrix of $\pm 1$'s.

The following result, summarized from the results in [9], explains the significance of this connection.

**Theorem 4.7** [9]. An $n \times n$ matrix $Q$ is the signature matrix of a real $2$-uniform $(n, k)$-frame if and only if it is the Seidel adjacency matrix of a graph with $2$ eigenvalues and in this case, $k$ is the multiplicity of the largest eigenvalue. Moreover, if $\{F_i\}, i \in \{1, 2\}$, is a set of real $2$-uniform frames, with associated signature matrices $\{Q_i\}$ and the corresponding graphs $\{G_i\}$, then $F_1$ and $F_2$ are frame equivalent if and only if $G_1$ and $G_2$ are switching equivalent graphs.

There is a considerable literature in graph theory dedicated to finding graphs with two eigenvalues and classifying these graphs up to switching equivalence. By referring to this literature, we can give a complete list of all integers $n \leq 50$ for which such graphs (and hence $2$-uniform frames) are known to exist, together with information about how many frame equivalence classes there are in each case.

This information is gathered together in Table 1. When an integer $j$ appears in the column labeled, “frame equivalence classes”, it indicates that exactly $j$ inequivalent real $2$-uniform $(n, k)$-frames exist. When the symbol $j+$ appears, it indicates that at least $j$ inequivalent real $2$-uniform $(n, k)$-frames are known to exist, but it is not known yet if this exhausts all equivalence classes. The letters $C$, $H$ and $G$ in the column labeled “type” indicate that the corresponding frames are all constructed using conference matrices, graph Hadamards, or only arise from certain graphs, respectively.

So for example, using Table 1, and looking at $n = 36$, we see that there exist at least 227 switching inequivalent graph Hadamard matrices and these can be used to construct at least 227 frame inequivalent $2$-uniform $(36, 15)$-frames and at least 227 frame inequivalent $2$-uniform $(36, 21)$-frames. For $n = 276$, there exists a graph whose Seidel adjacency matrix has exactly 2 eigenvalues, but it is neither a conference matrix nor graph Hadamard matrix, and this matrix can be used to construct a $2$-uniform $(276, 23)$-frame that up to frame equivalence is the unique such frame.

The number of equivalence classes are often computed by using the theory and enumeration of two-graphs. A **two-graph** $(\Omega, \Delta)$ is a pair consisting of a vertex set $\Omega$ and a collection $\Delta$ of three element subsets of $\Omega$ such that every four element subset of $\Omega$ contains an even number of the sets from $\Delta$. A two-graph is regular, provided that every two element subset of $\Omega$ is contained in the same number, $\alpha$, of sets in $\Delta$. 
Given \( n \), Seidel [11] exhibits a one-to-one correspondence between the two-graphs on the set of \( n \) elements and the switching equivalence classes of graphs on \( n \) elements and gives a concrete means, given the two-graph, to construct a graph from the corresponding switching class. Thus, a two-graph can be regarded as a switching equivalence class of ordinary graphs. In [13], it was noted that signature matrices of real 2-uniform frames are always Seidel adjacency matrices of regular two-graphs. The following result from [9] more fully summarizes this connection.

**Theorem 4.8** [9]. An \( n \times n \) matrix \( Q \) is the signature matrix of a real 2-uniform \((n, k)\)-frame if and only if it is the Seidel adjacency matrix of a graph on \( n \) vertices whose switching equivalence class is a regular two-graph on \( n \) vertices with parameter \( \alpha \). This relationship defines a one-to-one correspondence between frame equivalence classes of real 2-uniform frames and regular two-graphs.

The relationship between the parameter \( \alpha \) and earlier parameters is given by the equations,

\[-2\alpha = (1 + \rho_1)(1 + \rho_2) = 2 + \mu + n.\]

Thus, by the above theorem every regular two-graph produces a real 2-uniform frame. For a given \( n \) these could just be the trivial, known examples corresponding to \( k = n - 1, 1 \). In [11] many of the known regular two-graphs are listed and it is elementary to use the formulas given above to determine the pairs \((n, k)\) for which they yield a real 2-uniform frame.
In particular, the two-graph denoted in [11] (see Definition 9.5 and Theorem 9.7) as $Q^-(6, 2)$ yields a 2-uniform $(28, 7)$-frame and a 2-uniform $(28, 21)$-frame. Since by Table 1 there is only one such equivalence class, the frames derived from this two-graph will be frame equivalent to the frames generated by the $28 \times 28$ signature matrix (and its negative) found by Holmes [8].

Problem 4.9. One fundamental question that we have not been able to answer is whether or not regular two-graphs exist that give rise to 2-uniform frames of arbitrarily large redundancy. The examples that come from conference matrices all have redundancy 2 and those coming from Hadamard matrices have redundancy approaching 2. The existence of two-graphs with arbitrarily large redundancy may possibly be a consequence of Ramsey theory.

5. Graphs and error bounds

In this section we derive estimates and formulas for $e^p_m(F)$ when $F$ is a real 2-uniform frame, using connectivity properties of the graph associated to the signature matrix $Q$ of $F$.

Recall that if $F$ is a real 2-uniform $(n, k)$-frame, then the Grammian $P$ of $F$ is an $n \times n$ matrix that is a projection of rank $k$ and has the form $P = k/nI + c_{n,k}Q$, where $c_{n,k} = \sqrt{k(n-k)/n(n-1)}$ and $Q$ is the Seidel adjacency matrix of a certain graph that we shall denote $G_F$.

We also need to recall a few concepts from graph theory.

Definition 5.1. A graph on $m$ vertices is called complete bipartite provided that the vertex set can be divided into two disjoint subsets, $V_1$ and $V_2$, of sizes, say $m_1$ and $m_2$ with $m_1 + m_2 = m$, such that no pair of vertices in $V_1$ or in $V_2$ are neighbors, but every vertex in $V_1$ is adjacent to every vertex in $V_2$. We shall denote this graph by $B(m_1, m_2)$. In our definition of complete bipartite graph, we allow the possibility that one of the sets is empty, so that the complete bipartite graph, $B(0, m)$, is the graph on $m$ vertices with no edges. If $G$ is a graph with vertex set $V$ and $W \subset V$ then by the induced subgraph on $W$, we mean the graph with vertex set $W$ and two vertices in $W$ are adjacent if and only if they were adjacent in $G$.

Note that if $F$ is a real 2-uniform $(n, k)$-frame with signature matrix $Q$ and graph $G_F$, then the Seidel adjacency matrix of an induced subgraph on $m$ vertices is just the $m \times m$ matrix obtained by compressing $Q$ to the corresponding entries.

We are grateful to Ryan Pepper for the following observation, which can also be found in the work of Seidel.
Lemma 5.2. A graph on \( m \) vertices is switching equivalent to the graph with no edges if and only if it is complete bipartite.

Proof. Given any complete bipartite graph corresponding to a preselected partition of vertices into two sets, we show that it can be obtained by switching the empty graph on \( m \) vertices. Without loss of generality we may order the vertices \( \{v_1, v_2, \ldots, v_m\} \) such that the partition is given by \( \{v_j\}_{j \leq r} \) and \( \{v_j\}_{j > r} \), with \( 0 \leq r \leq m \). Let us choose the switching matrix

\[
S = \begin{pmatrix}
+I_r & 0 \\
0 & -I_{m-r}
\end{pmatrix}.
\]

The empty graph on \( m \) vertices has the Seidel adjacency matrix \( J_m - I_m \), so that of the switched graph is then

\[
S(J - I)S = \begin{pmatrix}
J_r - I_r & -J_{r,m-r} \\
-J_{m-r,r} & J_{m-r} - I_{m-r}
\end{pmatrix},
\]

which by inspection belongs to the preselected complete bipartite graph. Moreover, switching the empty graph always leads to complete bipartite graphs. Again, the empty (edgless) graph is in our sense bipartite, corresponding to a partition \( \emptyset \) and \( \{v_1, v_2, \ldots, v_m\} \). □

5.1. Error estimates for \( e^\infty_m(F) \)

Theorem 5.3. Let \( F \) be a real \( 2 \)-uniform \((n, k)\)-frame. Then \( e^\infty_m(F) \leq k/n + (m - 1)c_{n,k} \) with equality if and only if \( GF \) contains an induced subgraph on \( m \) vertices that is complete bipartite.

Proof. The proof has three parts. First we show that the operator norm \( \|DVV^*D\| \) is equal to the largest eigenvalue of the compression of \( VV^* \) to the rows and columns where \( D \) has 1’s. Then we bound the largest possible eigenvalue. Finally we characterize the case when this bound is saturated.

To begin with, we note that \( DVV^*D \) is a positive operator, and so is its compression \( (VV^*)_m \) to the rows and columns where \( D \) has 1’s. Consequently, the operator norm of \( DVV^*D \) is equal to the largest eigenvalue of \( (VV^*)_m \). This eigenvalue, in turn, follows from the largest eigenvalue of the compression of \( Q \), because \( (VV^*)_m = \frac{1}{2}I_m + c_{n,k}Q_m \). So we can reduce the calculation to that of the largest eigenvalue of \( Q_m \). In fact, to simplify the argument, we will look for the largest possible eigenvalue of \( Q_m + I_m \) and later adjust for the added constant.

We now claim that the largest eigenvalue of \( Q_m + I_m \) occurs when \( Q_m + I_m = J_m \), that is, the matrix of all 1’s. This follows from considering that for any given \( x \in \mathbb{R}^m \) and \( Q_m \), changing signs to make all their entries positive increases \( (x, (Q_m + I_m)x)/\|x\|^2 \).
By inspection, the largest eigenvalue of $J_m$ is $m$, so that of $J_m - I_m$ is $m - 1$, and the claimed error bound follows:

$$\|DVV^*D\| \leq \frac{k}{n} + (m - 1)c_{n,k}.$$ 

To characterize cases of equality, suppose $G$ contains an induced subgraph that is switching equivalent to the graph with no edges. If we choose $D$ to have 1’s in the places on the diagonal corresponding to the vertices of this subgraph and 0’s everywhere else, then $D(I + Q)D$ is switching equivalent to $DJ_nD$ and so the error is $e^\infty_m(F) = k/n + (m - 1)c_{n,k}$. Conversely, assume that equality holds in the error estimate. Then, $\|D(I + Q)D\| = m$. Given an eigenvector $x$ corresponding to eigenvalue $\pm m$, we may choose a switching matrix $S$ such that all of the entries of $Sx$ are positive. Similarly to the above reasoning, all the entries in $S(I + Q)S$ must be 1’s in the rows and columns where $D$ has 1’s on the diagonal, otherwise it would be possible to increase the largest eigenvalue of $SD(I + Q)DS$ by flipping signs in $Q$, contradicting that the inequality is saturated. Hence, the induced subgraph on these vertices is switching equivalent to the edgeless graph. □

**Corollary 5.4.** For a given $m$, a real 2-uniform frame $F$ maximizes the error $e^\infty_m(F)$ iff $G_F$ has an induced subgraph on $m$ vertices that is complete bipartite.

**Corollary 5.5.** Let $F$ be a real 2-uniform $(n, k)$-frame. Then either $G_F$ contains an induced complete bipartite graph on 3 vertices or it is switching equivalent to the complete graph on $n$ vertices. Consequently, if $k < n - 1$ we have $e^\infty_3(F) = k/n + 2c_{n,k}$.

**Proof.** Let us assume that $G_F$ has no induced complete bipartite subgraph on 3 vertices. We may choose one vertex and switch the others if necessary in order to have edges between this one and all others. Then any two vertices must be adjacent, otherwise there would be an induced complete bipartite subgraph on 3 vertices. Thus, the resulting graph is a switched version of $G_F$ that is complete. This corresponds to $F$ being equivalent to the uniform $(n, n - 1)$-frame. □

**Corollary 5.6.** If $F_1$ and $F_2$ are real 2-uniform $(n, k)$-graphs, then $e^\infty_3(F_1) = e^\infty_3(F_2)$.

By analogy with our earlier definitions, we call a Parseval frame $F$ 3-uniform if it is 2-uniform and if the error $\|DVV^*D\|$ associated with a deletion is constant for every $D \in D_3$.

**Corollary 5.7.** The trivial 2-uniform $(n, k)$-frames, corresponding to $k = 1$ or $k = n - 1$, are 3-uniform. Conversely, if $F$ is a real 3-uniform $(n, k)$-frame, then either $k = 1$ or $k = n - 1$ and it is equivalent to the corresponding trivial frame.
Proof. It is clear from their definition that the trivial frames are 3-uniform. What we need to show is that if \( F \) is 3-uniform then \( G_F \) is either switching equivalent to the complete or to the edgeless graph.

To begin with, we pick a vertex and switch the others if necessary in order to isolate it. Any two additional vertices are either adjacent or not, and computing the norm of \( DV^*D \), where \( D \) is associated with these 3 vertices, distinguishes these two cases. However, 3-uniformity then implies that every additional pair of vertices must behave the same way. Thus, if there is one edgeless induced subgraph on 3 vertices, then all of \( G_F \) is edgeless. On the other hand, if there is one neighboring pair, then all pairs of vertices except those including the isolated one are neighbors. Switching this one vertex then yields the complete graph. □

We now discuss how non-existence of complete bipartites gives rise to refined error bounds.

Definition 5.8. Let \( \mathcal{G}_m(s) \) denote the set of graphs on \( m \) vertices such that \( s \) is the minimum number of edges occurring among graphs in the equivalence class associated with each \( G \in \mathcal{G}_m(s) \).

We note that \( \mathcal{G}_m(0) \) are the complete bipartites on \( m \) vertices, and \( \mathcal{G}_m(1) \) is the equivalence class of graphs that may be reduced to one edge by switching. However, for larger values of \( s \), \( \mathcal{G}_m(s) \) may contain more than one equivalence class.

Lemma 5.9. Let \( Q(0), Q(1) \) and \( Q(s) \), \( s \geq 1 \), be Seidel adjacency matrices belonging to graphs \( G(0) \in \mathcal{G}_m(0) \), \( G(1) \in \mathcal{G}_m(1) \), and \( G(s) \in \mathcal{G}_m(s) \), respectively, for some common number of vertices \( m \geq 3 \). Denote by \( \lambda(0) \), \( \lambda(1) \) and \( \lambda(s) \) the largest eigenvalue of \( Q(0) \), \( Q(1) \) and \( Q(s) \). Then \( \lambda(0) \geq \lambda(1) \geq \lambda(s) \).

Proof. By appropriate switching, we can make \( G(0) \), \( G(1) \) and \( G(s) \) have a minimal number of edges in their respective equivalence class. In particular, then \( G(0) \) is the edgeless graph. Permuting the vertices if necessary, we have \( G(0) \subset G(1) \subset G(s) \). To simplify notation, we can choose this permutation in such a way that there is an edge between the \( m \)th and \( m-1 \)th vertex in \( G(1) \) and in \( G(s) \). Since switching corresponds to a change of basis in \( \mathbb{R}^n \), the maximal eigenvalues of the Seidel adjacency matrices \( Q(0) \), \( Q(1) \) and \( Q(s) \) are unchanged. After switching, the components of \( Q(1) \) and \( Q(s) \) observe \( q_{m,m-1}^{(1)} = q_{m-1,m}^{(1)} = q_{m,m-1}^{(s)} = q_{m-1,m}^{(s)} = -1 \). The inequality between the largest eigenvalues of \( Q(0) \) and \( Q(1) \) follows by explicit computation,

\[
\lambda(0) = m - 1 \geq \lambda(1) = \frac{m^2}{2} - 2 + \sqrt{\frac{m^2}{4} + m - 3} \quad \text{for} \quad m \geq 3.
\]

To establish the inequality \( \lambda(1) \geq \lambda(s) \), we use a variational argument similar to that in the proof of Theorem 5.3. We consider a normalized eigenvector \( x \) belonging to the largest eigenvalue of \( Q^{(s)} \). We show there is a normalized vector \( p \) such that \( \lambda(1) \geq \langle p, Q^{(1)} p \rangle \geq \langle x, Q^{(s)} x \rangle = \lambda(s) \). The construction of \( p \) proceeds in several parts:
Part 1. First let us assume that not all of \( |x_i|_{i=1}^{m-2} \) are negative, otherwise we multiply \( x \) by \(-1\). Now we set \( x'_i = |x_i| \) for \( i \leq m - 2 \) and replace \( q_{i,j} \) by \( q'_{i,j} := |q_{i,j}| \) for the block given by \( i, j \leq m - 2 \). At the same time, we modify the last two rows and columns of \( Q^{(s)} \) as follows. If \( x_{m-1} \) and \( x_m \) are both positive or both negative, we set \( p_i = |x_i| \) for all \( i \), let \( q'_{m-1,m} = q'_{j,m-1} = 1 \) and \( q'_{m,j} = q'_{j,m} = 1 \) for \( j \leq m - 2 \), and thus obtain \( Q^{(1)} = Q' \) from \( Q^{(s)} \) while only increasing \( \langle x, Q^{(s)}x \rangle \) to \( \langle p, Q'p \rangle \), which finishes the proof for this case.

Part 2. If one component of \( x \), say \( x_m \), is negative, we set \( x'_i = |x_i| \) only for \( i \leq m - 1 \) and retain \( x'_m = x_m \), while letting \( q'_{i,m} = q'_{m,i} = -1 \) if \( x_i < 0 \) and \( q'_{i,m} = q'_{m,j} = 1 \) if \( x_i > 0 \), for \( i \leq m - 2 \). This ensures that \( \langle x', Q'x' \rangle \geq \langle x, Q^{(s)}x \rangle \), while we have reduced the occurrence of \(-1\)'s in \( Q' \) to the last row and column. Note that by our assumption that not all \( \{x_i\}_{i=1}^{m-2} \) are negative, according to this procedure, there remains at least one entry \( q''_{i,m} = q''_{m,i} = 1 \) in the last row and column of \( Q' \).

Part 3. Now define \( x'' \) by \( x''_i = |x_i| \) and let \( Q'' \) be obtained from switching signs in the last row and column of \( Q' \). Then \( \langle x'', Q''x'' \rangle = \langle x', Q'x' \rangle \). Since \( x'' \) has all positive entries and \( Q'' \) contains at least one pair of \(-1\)'s in the last row and column, setting all entries to 1 but this one pair \( q''_{i,m} = q''_{m,i} = -1 \) only increases \( \langle x'', Q''x'' \rangle \) and transforms \( Q'' \), together with a permutation of indices, to \( Q^{(1)} \). Applying the same permutation to the entries of \( x'' \) yields \( p \) satisfying \( \langle p, Q^{(1)}p \rangle \geq \langle x, Q^{(s)}x \rangle \). \( \square \)

**Theorem 5.10.** Given a real \( 2 \)-uniform \((n, k)\)-frame \( F \) such that for some \( m \geq 3 \), the associated graph \( G_F \) does not have an induced complete bipartite subgraph on \( m \) vertices. Then we have the improved error bound

\[
e_m^\infty(F) \leq \frac{k}{n} + c_{n,k} \left( \frac{m}{2} - 2 + \sqrt{\frac{m^2}{4} + m - 3} \right).
\]

If \( G_F \) contains an induced subgraph on \( m \) vertices that differs from a complete bipartite by one edge, then equality holds.

**Proof.** The improved error bound results from the fact that by the preceding lemma, in the absence of complete bipartites on \( m \) vertices, the graphs in \( \mathcal{G}_m^{(s)} \) maximize the error.

To prepare the argument, we recall that \( VV^* = \frac{k}{n} I_n + c_{n,k} Q \) is a projection, so the compression of \( DVV^*D \) to the rows and columns where \( D \) has 1’s, henceforth denoted as \( \frac{k}{n} I_m + c_{n,k} Q_m \), is a non-negative operator. Consequently, the norm of \( DVV^*D \) equals that of \( \frac{k}{n} I_m + c_{n,k} Q_m \) and is given by its largest eigenvalue. To obtain this eigenvalue, it is enough to consider \( Q_m \).

In the absence of complete bipartites, we know that any matrix \( Q_m \), obtained from the compression of \( Q \) to \( m \) rows and columns, corresponds to a graph \( G \in \mathcal{G}_m^{(s)} \) with \( s \geq 1 \). By the inequality in the preceding lemma, to bound the largest possible
eigenvalue we only need to consider $s = 1$. Since we may switch and permute $G$ without loss of generality, we again choose $Q_m$ to be the $m \times m$ matrix of all 1’s, except the 0’s on the diagonal and the two elements $q_{m-1,m} = q_{m,m-1} = -1$. By inspection, $Q_m$ has an eigenvector $(0, 0, \ldots, 1, -1)$ with eigenvalue 1, a set of $m - 3$ linearly independent eigenvectors of eigenvalue $-1$ given by $(1, -1, 0, \ldots, 0)$ and those obtained when exchanging its second entry with all others except the first one and the last two entries. The larger of the remaining two eigenvalues is $\lambda = m^2 - 2 + \sqrt{(m^2 - 1)^2 + 2m - 4}$ which is seen to be greater than or equal to 1 for $m \geq 3$. The claimed error bound follows. □

The next five results can be deduced by converting results in [5], especially Remark 2.8 and Theorem 2.7, into statements about frames and re-deriving their formulas in terms of the parameter $k$, but it is perhaps clearer to deduce them directly. A main part of the results consists of sufficient conditions that rule out the existence of induced complete bipartite subgraphs on certain numbers of vertices. Theorem 5.17 summarizes these conditions.

**Proposition 5.11.** Let $F$ be a real 2-uniform $(n, k)$-frame, and suppose $1 + \sqrt{(n-k)(n-1)} < m$. Then the associated graph $G_F$ does not contain an induced subgraph on $m$ vertices that is complete bipartite.

**Proof.** Since $e_m^\infty(F) \leq 1$ and $\frac{k}{n} + (m-1)c_{n,k} > 1$ for $1 + \sqrt{(n-k)(n-1)} < m$, we have $e_m^\infty(F) < \frac{k}{n} + (m-1)c_{n,k}$. This excludes an induced complete bipartite subgraph, because otherwise equality would hold. □

**Corollary 5.12.** Let $F$ be a real 2-uniform $(n, k)$-frame. If $n - k + 1 \leq m \leq n$, then no induced subgraph of $G_F$ on $m$ vertices can be complete bipartite.

**Proof.** Since $1 + \sqrt{(n-k)(n-1)} < n - k + 1$, there cannot be any induced complete bipartite subgraphs of $m$ vertices when $n - k + 1 \leq m \leq n$. □

**Proposition 5.13.** Let $F$ be a real 2-uniform $(n, k)$-frame. If $m > k$, then the Seidel adjacency matrix of any induced subgraph of $G_F$ on $m$ vertices has an eigenvalue $-\frac{k}{m}c_{n,k}$.

**Proof.** Take $m > k$ and consider $P = \left( \begin{array}{cc} \frac{k}{n} I_m + c_{n,k} Q_m & \ast \\ \ast & \ast \end{array} \right)$, where $\ast$ denotes the remaining entries of $P$ outside of the first $m$ rows and columns. Since $\text{rk}(P) = k$, these $m$ columns are linearly dependent and

$$0 \in \sigma\left( \frac{k}{n} I_m + c_{n,k} Q_m \right).$$
Thus

$$0 = \frac{k}{n} + \lambda c_{n,k}$$

for some eigenvalue $\lambda$ of $Q_m$. □

As a precursor to the next consequence, we recall that since $VV^*$ is an $n \times n$ matrix and a projection of rank $k$, it has eigenvalues 1 with multiplicity $k$ and 0 with multiplicity $n - k$. Hence $Q$ has eigenvalues $\rho_2 < \rho_1$ with $\rho_1 = \frac{n-k}{nc_{n,k}} = \sqrt{\frac{(n-k)(n-1)}{k}}$ of multiplicity $k$ and $\rho_2 = -\frac{k}{nc_{n,k}} = -\sqrt{\frac{k(n-1)}{n-k}}$ of multiplicity $n - k$.

**Corollary 5.14.** Given a real 2-uniform $(n, k)$-frame $F$, then $G_F$ has no induced subgraph on $m > k$ vertices that is complete bipartite.

**Proof.** If it had, then the signature matrix $Q_m$ associated with the subset of vertices that form the induced complete bipartite subgraph would have eigenvalues $\sigma(Q_m) = \sigma(J_m - I_m) = \{-1, m-1\}$, so $\frac{k}{nc_{n,k}} = 1$, contradicting $k/n > c_{n,k}$. □

**Proposition 5.15.** Let $F$ be a real 2-uniform $(n, k)$-frame. If $m > n - k$, then the Seidel adjacency matrix of every induced subgraph on $m$ vertices has an eigenvalue $\frac{n-k}{nc_{n,k}}$.

**Proof.** The projection onto the complement of the range of $P$, $I - P$, has rank $n - k$. So $0 \in \sigma(J_m - (\frac{k}{n}I_m + c_{n,k}Q_m))$ and $0 = 1 - \frac{k}{n} - \lambda c_{n,k}$ for some eigenvalue $\lambda$ of $Q_m$. □

**Corollary 5.16.** Let $F$ be a real 2-uniform $(n, k)$-frame. If $m > n - k$, then no induced subgraph of $G_F$ on $m$ vertices is complete bipartite.

**Proof.** $m > n - k$ gives $m - 1 \geq n - k$. But $nc_{n,k} = \sqrt{\frac{k(n-k)}{n-1}} > 1$ so $\lambda = \frac{n-k}{nc_{n,k}} < n - k \leq m - 1$. Thus $\lambda \notin \{m, m-1\}$. □

**Theorem 5.17.** Let $F$ be a real 2-uniform $(n, k)$-frame. If $G_F$ contains an induced subgraph on $m$ vertices that is complete bipartite then $m \leq \min\{k, n - k, 1 + \sqrt{(n-k)(n-1)/k}\}$.

**Proof.** Follows from preceding Proposition 5.11, Corollaries 5.14 and 5.16. □

The next result shows which number of erasures may cause a 2-uniform frame to lose all information contained in some encoded vector.
Proposition 5.18. For \( n - k + 1 \leq m \leq n \) and any (real or complex) 2-uniform \((n, k)\)-frame \( F \), \( e_m^\infty(F) = 1 \).

Proof. This follows from an eigenvalue interlacing theorem and the multiplicity \( k \) of the eigenvalue one of \( P = VV^* \). If \( k \geq 2 \) and \( m = n - 1 \), then the \( k - 1 \) largest eigenvalues of \( DVV^*D \) must lie between the \( k \) largest eigenvalues of \( VV^* \), which are all one. By iteration, the eigenvalue one will remain up to \( m = n - k + 1 \). □

5.2. Computation of the error \( e_3^2(F) \)

We now turn our attention to \( e_3^2(F) \). Recall that switching equivalence leads to two different equivalence classes \( \Gamma_e \) and \( \Gamma_o \) for 3-vertex graphs, those with an even number and those with an odd number of edges, respectively. We observe that \( \Gamma_e \) contains exactly the complete bipartite graphs with 3 vertices.

Lemma 5.19. The number of complete bipartite 3-vertex subgraphs \( E_3(G) \) in a graph \( G \) that corresponds to a real 2-uniform \((n, k)\)-frame depends only on \( n \) and \( k \). It is given by

\[
E_3(G) = \binom{n}{3} - \frac{v(n-1)c}{6} - \frac{(n-2v+c)(n-1)v}{2},
\]

with

\[
v = \frac{1}{2} \left( n - 2 - \sqrt{\frac{(n-k)(n-1)}{k}} + \sqrt{\frac{k(n-1)}{n-k}} \right)
\]

and

\[
c = v - 1 - \frac{1}{4} \left( \sqrt{\frac{(n-k)(n-1)}{k}} - 1 \right) \left( \sqrt{\frac{k(n-1)}{n-k}} + 1 \right).
\]

Proof. By Seidel, if \( G \) is a graph in the switching class of a regular two-graph and if \( G \) has an isolated vertex, then the induced graph \( \tilde{G} := G \setminus \{\omega\} \) is strongly regular [11, Theorems 6.11 and 7.2]. Thus, \( \tilde{G} \) is characterized by the tuple \((n-1, v, p, q)\) which represent, respectively, the total number of vertices \( n-1 \), the common valency \( v \) of each vertex, whenever two vertices are adjacent there are \( p \) vertices adjacent to one vertex and not the other, and whenever two vertices are non-adjacent there are \( q \) vertices adjacent to one vertex and not the other. If we let \( c \) denote the number of common neighboring vertices of two adjacent vertices, then \( c + p + 1 = v \).

As a first step, we count the number of odd-edged induced subgraphs in \( \tilde{G} \), denoted as \( \Omega_3(\tilde{G}) \). The total number of edges in \( \tilde{G} \) is \( \frac{(n-1)v}{2} \). Each edge belongs to \( c \) 3-edged graphs. Therefore, \( \frac{(n-1)v}{2} \) counts each 3-edged graph three times and so there are \( \frac{v(n-1)c}{6} \) 3-edged subgraphs. To arrive at the number of 1-edged subgraphs,
we recall that two connected vertices are each connected to \( v - 1 \) other vertices, and have \( v \) of these as common neighbors. Thus, these vertices are connected to \( 2(v - 1) - c \) other vertices, and not connected to \( (n - 1) - 2v + c \). Hence, there are \( (n - 1 - 2v + c)(n-1) \) 1-edged subgraphs. The number of 3-vertex subgraphs in \( \widetilde{G} \) with odd edges is consequently

\[
O_3(\widetilde{G}) = \frac{v(n-1)c}{6} + (n - 1 - 2v + c)\frac{(n-1)v}{2}.
\]

The number of odd-edged 3-vertex subgraphs in \( G \) is then

\[
O_3(G) = O_3(\widetilde{G}) + \frac{(n - 1)v}{2}
\]
due to adding an 1-edged subgraph for every edge in \( \widetilde{G} \) when including the isolated vertex. Thus \( O_3(G) \) and \( E_3(G) = \binom{n}{3} - O_3(G) \) follow once \( v \) and \( c \) are given. What remains is to deduce their values from the 2-uniformity of the frame belonging to \( G \).

By [11, Theorem 7.5] if we pick \( G \) from the switching class of a regular two-graph such that \( G \) has an isolated vertex \( \omega \), then the Seidel adjacency matrix of \( \widetilde{G} = G \setminus \{\omega\} \) has eigenvalues \( \rho_0 = \rho_1 + \rho_2 \), \( \rho_1 \), and \( \rho_2 \), where \( \rho_1 = \sqrt{\frac{(n-k)(n-1)}{k}} \) and \( \rho_2 = -\sqrt{\frac{k(n-1)}{n-k}} \) are the eigenvalues associated with \( G \). By Seidel (pp. 155–156), the valency is \( v = \frac{1}{2}(n - 2 - \rho_0) \) and \( 4p = -(\rho_1 - 1)(\rho_2 - 1) \), which concludes the proof.

**Proposition 5.20.** For a real 2-uniform \((n, k)\)-frame \( F \), the error \( e_3^p(F) \) for \( 2 \leq p < \infty \) is given by

\[
e_3^p(F) = \left( \binom{n}{3}^{-1} \left[ \frac{k}{n} + 2c_{n,k} \right]^p E_3(G) + \left( \frac{k}{n} + c_{n,k} \right)^p O_3(G) \right)^{1/p},
\]

where \( E_3(G) \) and \( O_3(G) \) are the constants that were calculated in the preceding lemma.

**Proof.** For an induced 3-vertex subgraph in \( \Gamma_{\omega} \) associated with a 2-uniform \((n, k)\)-frame \( F \) and \( D \in \mathcal{D}_3 \), an explicit computation gives \( \|DVV^*D\| = \frac{k}{n} + 2c_{n,k} \), whereas if the subgraph is in \( \Gamma_0 \) then \( \|DVV^*D\| = \frac{k}{n} + c_{n,k} \). Consequently, the definition of \( e_3^p(F) \) simplifies to the claimed expression.

**Corollary 5.21.** For any two real 2-uniform \((n, k)\)-frames \( F_1 \) and \( F_2 \), \( e_3^p(F_1) = e_3^p(F_2) \) for \( 2 \leq p < \infty \).

**Proof.** Follows from Lemma 5.19 and Proposition 5.20.
6. Error estimates for concrete frames

In this section, we use the inequalities and methods of the previous section to explicitly compute the error estimates, \( e_m^p(F) \) for various 2-uniform \((n, k)\)-frames. In addition, we investigate how the error estimates compare to an explicit, computer-aided calculation of the error.

We begin with an example of frames constructed with the help of conference matrices.

**Example 6.1.** When \( n = 26 \) and \( k = 13 \), there are 4 equivalence classes of real 2-uniform \((n, k)\)-frames based on conference matrices. From Corollary 5.12 and \( 1 + \sqrt{(26-13)(26-1)} = 6 \), we deduce that the graphs of these frames cannot contain any induced complete bipartite subgraphs with \( m > 6 \) vertices.

**Theorem 6.2.** Let \( F \) be a real 2-uniform \((26, 13)\)-frame, then

\[
e_m^\infty(F) = \begin{cases} 
  m + 4/10 & \text{if } m \leq 6, \\
  1 & \text{if } 7 \leq m \leq 26.
\end{cases}
\]

(1)

Consequently, if \( F \) is any 2-uniform \((26, 13)\)-frame, then there exists a subset of six frame vectors, \( E = \{f_1, \ldots, f_6\} \), such that \( F \setminus E \) no longer spans \( \mathbb{R}^{13} \). If a set of five or fewer erasures occur, then there exists \( L : \mathbb{R}^{26} \to \mathbb{R}^{13} \) such that \( LEVx = x \) for all \( x \in \mathbb{R}^{13} \) with \( \|L\| \leq \sqrt{10} \).

**Proof.** By [1] there exist exactly four switching equivalence classes of graphs, which by our earlier results give rise to exactly four frame equivalence classes of 2-uniform \((26, 13)\)-frames. Thus, it will sufficient to compute \( e_m^\infty(F) \) for the frames generated by these four graphs. In [1, p. 101], representative graphs from each of these four equivalence classes are given. A careful inspection of these graphs shows that each graph contains a set of six vertices such that the induced subgraph is empty and the result follows. Combining this fact with our earlier formulas leads to the formula for \( e_m^\infty(F) \) and the estimate on \( \|L\| \). \( \square \)

**Remark 6.3.** Generally, for a \((26, 13)\)-frame \( F \), given any set \( E \) of 13 or fewer frame vectors, the set \( F \setminus E \) will still span \( \mathbb{R}^{13} \) and hence still be a frame. To see this fact, identify \( F \) with its \( 26 \times 13 \) isometric analysis operator \( V \). The set of all \( 26 \times 13 \) matrices such that any collection of 13 or more rows spans \( \mathbb{R}^{13} \) can easily seen to be dense in the set of all \( 26 \times 13 \) matrices. If we polar decompose such a matrix, then it follows that the isometric part of the polar decomposition inherits this property. Hence, it follows that the set of \( 26 \times 13 \) isometries such that any set of 13 or more rows spans \( \mathbb{R}^{13} \) is dense in the set of all isometries.

If we choose any such frame \( F_0 \) and let \( F_1 \) denote a 2-uniform frame, then \( e_m^\infty(F_0) < 1 \) for all \( m \leq 13 \). Hence, \( e_m^\infty(F_0) < e_m^\infty(F_1) \) for \( 6 \leq m \leq 13 \), while
necessarily, $e^\infty_2(F_0) \geq e^\infty_2(F_1)$. Thus, we see that by minimizing the error for two erasures, we have necessarily increased the error for some larger number of erasures.

We continue with an example derived from a graph that is neither of the conference nor Hadamard type.

**Example 6.4 (The 2-uniform frame of highest redundancy).** Among the known graphs giving rise to 2-uniform frames, the frame with highest redundancy is the 2-uniform $(276, 23)$-frame that arises from the unique regular two-graph on 276 vertices [5]. This frame has redundancy 12. Applying the inequalities of the previous section we see that $e^\infty_m(F) \leq \frac{m+4}{60}$ for all $m$ and since the graph cannot contain any induced complete bipartite subgraphs on 23 or more vertices, this inequality must be strict for $m > 23$. From this formula it follows that if any set of 55 or fewer erasures occurs, then a left inverse for $EV_L$ can be constructed with $\|L\| \leq \sqrt{60}$.

From these inequalities it follows that given any subset $E$ of $F$ containing at most 56 frame vectors, the set $F \setminus E$ will still span $\mathbb{R}^{23}$. Since we do not precisely know the value of $e^\infty_m(F)$ it is possible that this frame can handle much larger sets of erasures. By comparison, if we had produced a frame by simply repeating an orthonormal basis 12 times, then that frame would be able to handle at most subsets of 11 erasures. On the other hand, by the argument given in the above remark, a generic uniform $(276, 23)$-frame should be able to handle sets of up to 253 erasures, but at the expense of having a larger value for $e^\infty_2(F)$.

We now turn our attention to the special case of graph Hadamards. Suppose $H$ is a graph Hadamard, which means $H = H^*_n, H^2 = nI$. $H$ contains elements $h_{ij} = \pm 1$ only, and the diagonal is fixed by $h_{jj} = 1$. The following two results about error bounds of frames related to Hadamard matrices are based on an argument of Penny Haxell.

**Proposition 6.5.** Any real 2-uniform $(n, k)$-frame $F$ belonging to a signature matrix $Q = H - I$ with a graph Hadamard $H$ satisfies $e^\infty_m(F) = \frac{k}{2} + (m - 1)c_{n,k}$ if $n \geq 48$ and $m \leq 5$.

**Proof.** By conjugating $H$ by a diagonal matrix of $\pm 1$’s, we may always assume that the first row and column of $H$ consist entirely of $+1$’s. Then $H^2 = nI$ implies that the column vectors of $H$ are orthogonal and thus every additional row and column has to have an equal number of $+1$’s and $-1$’s. Moreover, for any two columns other than the first, say $i$ and $j$, their orthogonality forces them to have $n/2$ entries in common where both are $+1$’s, $n/4$ entries in common where both columns are $-1$’s, $n/4$ entries in common where column $i$ is $+1$’s and column $j$ is $-1$’s and $n/4$ entries in common where column $i$ is $-1$’s and column $j$ is $+1$’s.

The claimed values for the error now follows from showing the existence of induced complete bipartite subgraphs on $m$ vertices in the graph $G$ associated with the signature matrix $Q = H - I$. After switching as described above, we see that $G$
contains one isolated vertex and that all other vertices have \( n/2 \) neighbors. So let us pick as \( v_1 \) the isolated vertex and as \( v_2 \) any other, and as \( v_3 \) one vertex from those \( n/2 - 2 \) that are not adjacent with \( v_1 \) or \( v_2 \). Then by the orthogonality argument, \( v_3 \) has \( n/4 \) neighbors that are adjacent with \( v_2 \) and a set of \( n/4 \) neighbors in the set of \( n/2 - 2 \) that are not adjacent with \( v_1 \) or \( v_2 \). Thus, there remains a set \( A \) of \( n/4 - 3 \) vertices that are not adjacent with \( v_1 \), \( v_2 \), or \( v_3 \). If in this set there is a pair of vertices that are not adjacent then we have found an induced edgeless subgraph on 5 vertices, the one consisting of \( v_1 \), \( v_2 \), \( v_3 \) and the additional two non-adjacent vertices in \( A \). Thus, before switching, the subgraph induced by these vertices was complete bipartite.

If there is no non-adjacent pair in \( A \), then the induced subgraph on the vertices in \( A \) is a complete graph. We want to argue that this is impossible for \( n \) sufficiently large. Note that \( H \) has eigenvalues \( \pm \sqrt{n} \) because \( H^2 = nI \). If \( Q \) contains an induced complete subgraph of \( s = n/4 - 3 \) vertices, then the associated signature matrix \( Q_s = I_s - J_s \) has eigenvalues 1 and \( 1 - s \) if \( A \) induces a complete subgraph of \( s \) vertices in \( G \), and then necessarily \( 1 - \sqrt{n} \leq 1 - s \leq 1 - \sqrt{n} - 1 \). Thus, it is impossible that \( A \) induces a complete subgraph in \( G \) when \( n > 28 + 8\sqrt{6} > 47.5 \). □

Proposition 6.6. Any real 2-uniform \((n, k)\)-frame \( F \) belonging to a signature matrix \( Q = I - H \) with a graph Hadamard \( H \) satisfies \( e_{m}^\infty(F) = \frac{k}{n} + (m - 1)c_{n,k} \) if \( n \geq 30 \) and \( m \leq 5 \).

Proof. The first steps of the proof parallel the one for the preceding proposition, the only difference being that after switching to obtain the isolated vertex \( v_1 \), the valency of the other vertices is \( n/2 - 2 \). Having chosen a vertex \( v_2 \) and a vertex \( v_3 \) that is not adjacent with \( v_1 \) or \( v_2 \), we observe that there remains a set \( A \) of \( s = n/2 - 2 \) vertices that are not adjacent with any of \( v_1 \), \( v_2 \), and \( v_3 \). As before, we obtain that \( Q_s = I_s - J_s \) has eigenvalues 1 and \( 1 - s \) if \( A \) induces a complete subgraph of \( s \) vertices in \( G \), and then necessarily \( 1 - \sqrt{n} \leq 1 - s \leq 1 - \sqrt{n} - 1 \), thus there cannot be such an induced complete subgraph if \( n > 16 + 8\sqrt{3} > 29.8 \). □

Remark 6.7. The smallest possible values of \( n \) for graph Hadamards are \( n \in \{4, 16, 36, 64\} \). The preceding results imply that the graphs related to 2-uniform \((n, k)\)-frames of Hadamard type are guaranteed to contain induced complete bipartites on 5 vertices for \( n = 36 \), \( k = 15 \) and also for any \( n \geq 64 \).

Example 6.8. The 227 known switching equivalence classes of graph Hadamards with \( n = 36 \). For \( n = 36 \), \( k = 21 \), the argument in Proposition 6.5 does not guarantee the existence of induced complete bipartite subgraphs on 5 vertices. However, by having a computer search all 227 known equivalence classes [12], one finds that all members have at least one induced complete bipartite subgraph on 6 vertices. Thus, the \( m \)-deletion error is the same for all 2-uniform \((36, 21)\)-frames,
Table 2
Signature matrices for “good” 2-uniform (36, 15)-frames
Thus, if five or fewer erasures occur, then there exists a left inverse of norm at most $\sqrt{12}$.

If $n = 36$, $k = 15$, then Proposition 6.6 shows that each graph $G_F$ contains an induced complete bipartite on 5 vertices. Moreover, an explicit search finds that the maximal number of vertices that induce a complete bipartite subgraph varies from 6 to 8 among the 227 switching-equivalent classes: There are 217 switching-equivalent classes that have an induced complete bipartite subgraph on 8 vertices, 5 classes that have one on 7 vertices but not on 8, and 5 classes that have one on 6 but not on more than 6 vertices. Thus, for the group of 217, we have that $e_m^\infty(F) = \frac{5+(m-1)}{12}$ for $m \leq 8$ and $e_8^\infty(F) = 1$ for $m \geq 8$. For the next group of 5 equivalence classes that have an induced 7-vertex complete bipartite subgraph, we have $e_m^\infty(F) = \frac{5+(m-1)}{12}$ for $m \leq 7$, while $e_8^\infty(F) \leq \frac{7}{12} + \frac{1}{12} \sqrt{21} \approx 0.965$. The last bound follows from Theorem 5.10. Finally, for those having a maximal number of 6 vertices that induce a complete bipartite subgraph, we have $e_m^\infty(F) = \frac{5+(m-1)}{12}$ for $m \leq 6$, $e_7^\infty(F) = \frac{13}{12} + \frac{1}{24} \sqrt{65} \approx 0.878$ and $e_8^\infty(F) \leq \frac{7}{12} + \frac{1}{12} \sqrt{21} \approx 0.965$. Again, the results for the cases $m = 7$ and $m = 8$ follow from Theorem 5.10, because for each member of the group, one finds induced subgraphs on 7 vertices that differ from complete bipartites by only one edge, and we know that there are no induced complete bipartites on 8 vertices. The induced subgraphs giving the largest 8-deletion error are all found to be switching equivalent and related to complete bipartites by flipping two edges. Accordingly, the numerical value for the error $e_8^\infty(F) \approx 0.927$ for the members of this group is below the error bound derived from the absence of complete bipartites.

Thus, if 7 or fewer erasures occur, we know that a left inverse of $EV$ with norm at most $2\sqrt{6}/\sqrt{11} - \sqrt{65} \approx 2.86$ exists, compared to $\sqrt{12} \approx 3.46$ for the other 222 switching-equivalent classes. If 8 erasures occur, we know a left inverse exists of norm at most $\sqrt{12}/\sqrt{5 - \sqrt{21}} \approx 5.36$.

To summarize, any 2-uniform $(36, 15)$-frame belonging to the last group of 5 equivalence classes is somewhat preferable to the other 222, because it will have smaller error bounds, but we cannot guarantee that it can handle any more than 8 erasures.

We list a representative of the signature matrices belonging to each of these “good” 5 equivalence classes in Table 2.

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