Topological stability through extremely tame retractions

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A R T I C L E   I N F O

A B S T R A C T

Suppose that $F : (\mathbb{R}^n \times \mathbb{R}^d, 0) \to (\mathbb{R}^p \times \mathbb{R}^d, 0)$ is a smoothly stable, $\mathbb{R}^d$-level preserving germ which unfolds $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$; then f is smoothly stable if and only if we can find a pair of smooth retractions $r : (\mathbb{R}^{n+d}, 0) \to (\mathbb{R}^n, 0)$ and $s : (\mathbb{R}^{p+d}, 0) \to (\mathbb{R}^p, 0)$ such that $f \circ r = s \circ F$. Unfortunately, we do not know whether f will be topologically stable if we can find a pair of continuous retractions r and s.

The class of *extremely tame* (E-tame) retractions, introduced by du Plessis and Wall, are defined by their nice geometric properties, which are sufficient to ensure that f is topologically stable.

In this article, we present the E-tame retractions and their relation with topological stability, survey recent results by the author concerning their construction, and illustrate the use of our techniques by constructing E-tame retractions for certain germs belonging to the E- and Z-series of singularities.

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1. Introduction

This article presents results contained in the author's PhD thesis [4, Chapter 4], concerning the construction of E-tame retractions in order to detect topological stability. We shall focus on explaining the topological and geometric ideas underlying the construction, and refer to the thesis for proofs and technical details.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth germ with a stable, $\mathbb{R}^d$-level preserving unfolding $F : (\mathbb{R}^n \times \mathbb{R}^d, 0) \to (\mathbb{R}^p \times \mathbb{R}^d, 0)$. Then f is stable if and only if we can find a pair of smooth retractions $r : (\mathbb{R}^{n+d}, 0) \to (\mathbb{R}^n, 0)$ and $s : (\mathbb{R}^{p+d}, 0) \to (\mathbb{R}^p, 0)$ such that $f \circ r = s \circ F$, and in particular, the retraction (r, s) induces a smooth equivalence $((r, z \circ F), (s, z))$ between $f \times \mathrm{id}_{\mathbb{R}^d}$ and F.

If we replace stability with topological stability and smooth retractions with continuous ones in the statements above, then we can no longer make such conclusions. The concept of *tame retraction* was introduced by du Plessis and Wall [2] in order to prove topological stability using retraction arguments. Tameness refers to geometric conditions which ensure that the proof of stability carries over from the smooth case when we consider tame retractions from stable unfoldings onto germs which are, consequently, topologically stable [2, Proposition 4.2.3].

We study *extremely tame*, or *E-tame* retractions for short, which satisfy even stronger geometric properties, causing them to have nice functorial properties in the sense that E-tameness is often preserved when an original retraction induces new ones.

Our new toolbox consists of several techniques for constructing E-tame retractions, including, in particular, two methods for gluing retractions together either using local symmetries or taking advantage of situations where one of the retractions is smooth. These methods are presented in Sections 2.1.1 and 2.1.2. In Section 3 we illustrate the use of our tools in a couple of nontrivial examples, and in Section 4 we discuss how these methods can be used even in other situations.
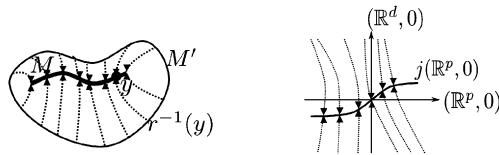
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Fig. 1. Left: The fibers of an E-tame retraction form a $C^{0,1}$ foliation. Right: A $C^{0,1}$ foliation induces an E-tame retraction onto any submanifold germ of complimentary dimension which is transverse to it.

1.1. Notation and terminology

Given a map $f : N \to P$ between smooth manifolds N and P, we will denote by $t(f)$ the target P and by $s(f)$ the source N. We say that a map or germ is *stable* when it is smoothly stable.

Let $y \in t(f)$. If there exists a subset $S \subset f^{-1}(y) \cap \Sigma f$ such that the germ f_S of f at S is $\mathscr{K}$-equivalent to the multigerm Δ, then we say that the singularity Δ is *presented* at y. If $S = f^{-1}(y) \cap \Sigma f$, then we say that Δ is *strictly presented* at y. We often partition targets into singularity sets, that is subsets where you find given singularities presented.

Given an embedding $i : M \to M'$, we say that a map $r : M' \to M$ is a *retraction* to i if $r \circ i = \mathrm{id}_M$.

2. Retractions

Given an unfolding $F : N' \to P'$ of $f : N \to P$, both smooth maps between smooth manifolds, we say that two retractions $r : N' \to N$ and $s : P' \to P$ define a retraction $(r, s) : F \to f$ if $s \circ F = f \circ r$ and the map $(F, r) : N' \to P' \times N$ is injective. We make the analogous definition for germs.

If $F : (\mathbb{R}^{n+d}, 0) \to (\mathbb{R}^{p+d}, 0)$ is a stable unfolding of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, then the existence of a smooth retraction $(r, s) : F \to f$ is equivalent to f being stable. Similarly, we can prove that f is *topologically stable* if the retraction (r, s) is *tame*, where we say that a retraction s to an embedding $e : P \to P \times \mathbb{R}^d$ is *tame* whenever there exists a neighborhood of e in $C^\infty(P, P \times \mathbb{R}^d)$ such that all embeddings j in that neighborhood combine with s to form a homeomorphism $s \circ j$. The pair (r, s) is said to be tame whenever s is tame.

Definition 1 (*E-tame retraction*). A $C^{0,1}$-foliation of a smooth manifold N of dimension n is a partition $\mathscr{F}$ of N such that for any $y_0 \in N$ there exist open neighborhoods W of y_0 in N and U, V of 0 in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively, and a homeomorphism $\phi : U \times V \to W$ such that for each $u \in U$ there exists $F \in \mathscr{F}$ such that $\phi(u \times V) = W \cap F$, each leaf F is a C^1 submanifold of N, and the tangent space $T_y F$ varies continuously with $y \in N$.

Suppose $i : M \to M'$ is an embedding. We say that the retraction $r : M' \to M$ to i is *E-tame* if its fibers form a $C^{0,1}$-foliation transverse to $i(M)$, in a neighborhood of $i(M)$. See Fig. 1. A germ of retraction is E-tame if it has an E-tame representative.

Let $f : N \to P$; let $\{F : N' \to P'; \ i : N \to N', \ j : P \to P'\}$ be an unfolding of f, and let $(r, s) : F \to f$ be a retraction to (i, j), that is, $s \circ F = f \circ r$ and $(r, F) : N' \to N \times P'$ is injective. We say that (r, s) is *E-tame* if s is E-tame.

E-tame retractions from stable maps are universal in the sense that if you can find *one* such retraction, then you can find one for each stable unfolding of the same map or germ [4, Theorem 22], using the foliation and the fact that any two stable unfoldings of one map have $\mathscr{A}$-equivalent suspensions. Hence, given an E-tame retraction $F \to f$ from a stable unfolding F, it follows that f is topologically stable by [2, Proposition 4.2.3 and Lemma 9.3.22].

It is easy to see how E-tame retractions induce topological triviality of level-preserving unfoldings: If $(r, s) : F \to f$ is an E-tame retraction (between maps or germs), where $f : N \to P$ and $F : N \times U \to P \times U$ is a U-level preserving unfolding of f, then there exist neighborhoods W and V of $N \times \{0\}$ and $P \times \{0\}$ in $N \times U$ and $P \times U$ such that $F(W) \subset V$ and the projection of retraction fibers $W \cap r^{-1}(x)$ and $V \cap s^{-1}(y)$ onto U is submersive by [4, Lemma 24]. In particular, the maps $(r, \mathrm{pr}_U) \mid W$ and $(s, \mathrm{pr}_U) \mid V$ are open embeddings inducing a topological equivalence of $F \mid W$ with $(f \times \mathrm{id}_U) \mid W$ since we have

$$(s, \mathrm{pr}_U) \circ F \mid W = (f \times \mathrm{id}) \circ (r, \mathrm{pr}_U) \mid W.$$

See also Fig. 2.

It is also easy to see how E-tameness is often preserved when new retractions are induced from old ones:

Firstly, if $(r, s) : F \to f$ is an E-tame retraction between the germs $F : (\mathbb{R}^{n+d}, 0) \to (\mathbb{R}^{p+d}, 0)$ and $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, and F also unfolds a germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ together with embeddings $i : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+d}, 0)$ and $j : (\mathbb{R}^p, 0) \to (\mathbb{R}^{p+d}, 0)$, such that j is transverse to the fibers of s at $0 \in (\mathbb{R}^{p+d}, 0)$, then we can construct an E-tame retraction $(\tilde{r}, \tilde{s}) : F \to g$ by following the fibers of the retractions s and r. This is [4, Proposition 26]. In other words, we get E-tame retractions onto all germs transverse to the foliation by fibers. See Fig. 1 for an illustration.

This is a nice property because it allows us to think in terms of transverse foliations rather than retractions. In particular, if we had E-tame retractions both in source and target, then we would not have to consider a new foliation whenever the unfolded map is slightly perturbed – unlike the current situation, where any map transverse to the foliation needs its own
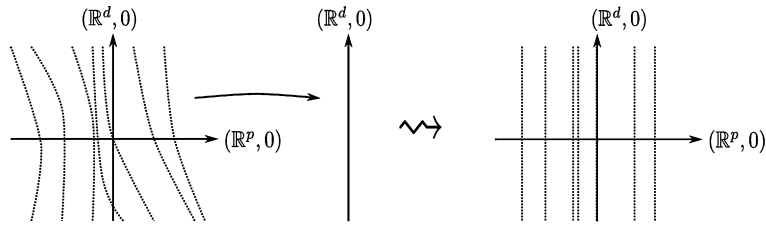
Fig. 2. A smooth retraction induces a smooth trivialization together with the projection onto the vertical coordinate. Since the fibers of E-tame retractions are transverse to the target, they similarly induce continuous trivializations.

retraction, even though the fibers are the same. Replacing retractions by foliations could effectively reduce the level of technicality in arguments.

Moreover, seeing how the foliation induces trivializations lends some geometric intuition as to why E-tame retractions $F \to f$ from stable germs F imply topological stability of f.

Secondly, since the product of $C^{0,1}$-foliations is a $C^{0,1}$ foliation, E-tameness is preserved in products, and in particular, we get a natural way of forming E-tame retractions from stable multigerms using their product presentation. Up to a choice of local coordinates, any stable multigerm can be written in the form

$$F = \bigsqcup_{i=1}^{k} \sigma_i \circ (F_i \times \mathrm{id}_{\prod_{j \neq i} \mathbb{R}^{p_j + d_j}}) : \bigsqcup_{i=1}^{k} \mathbb{R}^{n_i + d_i} \times \prod_{j \neq i} \mathbb{R}^{p_j + d_j} \to \prod_{i=1}^{k} \mathbb{R}^{p_i + d_i},$$

where σ_i is the permutation $\mathbb{R}^{p_i + d} \times \prod_{j \neq i} \mathbb{R}^{p_j + d} \to \prod_{j=1}^{k} \mathbb{R}^{p_j + d}$. If F_i unfolds f_i for each $i \in \{1, \ldots, k\}$, then F unfolds

$$f = \bigsqcup_{i=1}^{k} \tilde{\sigma}_i \circ (f_i \times \mathrm{id}_{\prod_{j \neq i} \mathbb{R}^{p_j}}) : \bigsqcup_{i=1}^{k} \mathbb{R}^{n_i} \times \prod_{j \neq i} \mathbb{R}^{p_j} \to \prod_{i=1}^{k} \mathbb{R}^{p_i},$$

where $\tilde{\sigma}_i$ is the permutation $\mathbb{R}^{p_i} \times \prod_{j \neq i} \mathbb{R}^{p_j} \to \prod_{j=1}^{k} \mathbb{R}^{p_j}$, and if there is an E-tame retraction $(r_i, s_i) : F_i \to f_i$ for each $i = 1, \ldots, k$, then

$$\left(\bigsqcup_{i=1}^{k} r_i \times \prod_{j \neq i} s_j, \prod_{j=1}^{k} s_j \right) \tag{2}$$

is an E-tame retraction $F \to f$. In particular, if the (r_j, s_j) are smooth, then so is (2).

2.1. The retraction toolbox

We would like to emphasize two strategies for combining local E-tame retractions. We start out considering how local symmetries can be used to combine local retractions along singularity sets, and next we present a method for combining an E-tame retraction with a smooth one.

2.1.1. Combining retractions using local symmetries

A frequent situation when studying stable maps F is one where a certain singularity type (for instance, a $\mathcal{K}$-multigerm class) Δ is found presented along a submanifold S of the target $t(F)$ of a map, such that the germ F_S of F at S is trivialized over S. That is, in a neighborhood W of S, the restriction $F \mid F^{-1}(W) \to W$ is $\mathcal{A}$-equivalent to a fibered map $\bigsqcup_{i=1}^{k} \tilde{H}_i \times \mathrm{id}_S : \bigsqcup_{i=1}^{k} U_i \times S \to V \times S$ where the U_i and V are neighborhoods of 0 in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, and where the fiber germ $H = \bigsqcup_{i=1}^{k} H_i : \bigsqcup_{i=1}^{k} (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ defined by the $\tilde{H}_i$ is stable and has the singularity type Δ.

Suppose that H unfolds a multigerm h, and that we know how to construct E-tame retractions $H \to h$. Using the trivialization we construct E-tame retractions $H \times \mathrm{id}_S \to h \times \mathrm{id}_S$, which can be transported back to a retraction $F_S \to f$ for some map-germ f which is a restriction of F_S.

Suppose, furthermore, that at points of S near the boundary ∂S of S, we have a fixed choice of local coordinates for the germ of F, also taking the germ of $F \mid$ fiber to the form H – perhaps from the construction of some other retraction near ∂S, which we wish to combine with a retraction on S. We now ask: Can we extend this choice of fiber coordinates near ∂S smoothly over S, in order to construct E-tame retractions near all of S using the trivialization, which agree with something pre-defined at ∂S?

Consider, for example, the situation in Fig. 3, where S is an open curve along which we find the singularity type H, and where we find some other singularities presented at the endpoints of the curves. Assume that we know how to construct retractions at the endpoints, and these retractions also induce retractions in the fibers near the end points, which stem from a certain choice of coordinates taking the germ of $F \mid$ fiber to H. We can push the choice of coordinates on the left over to
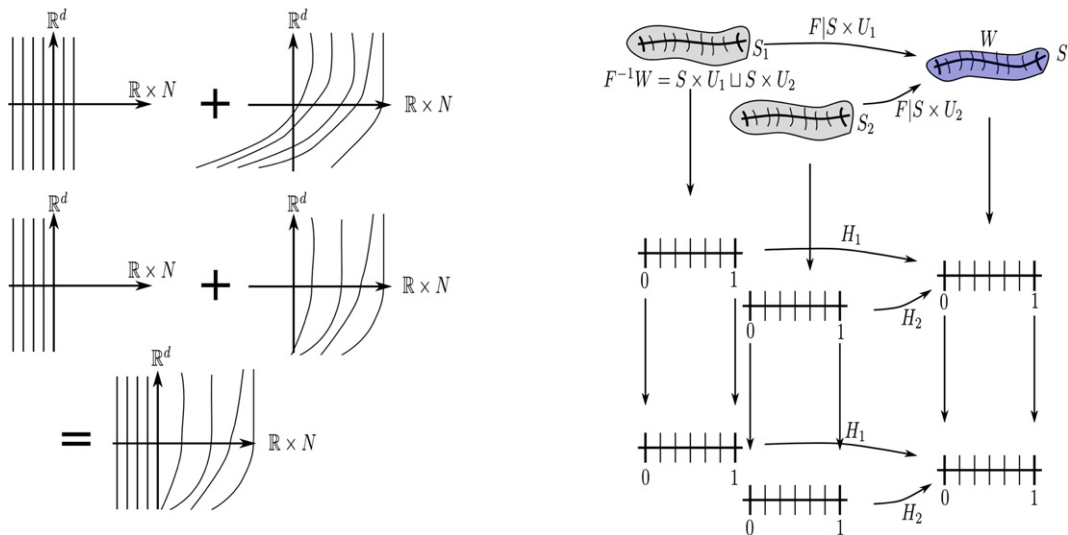
Fig. 3. Left: Combining a smooth retraction with an E-tame one along a line by "stretching" the fibers of the E-tame retraction. Right: Combining E-tame retractions in the fibers of a fibered germ over a 1-dimensional manifold by interpolating between symmetries of the fiber germ (i.e. interpolating between different possible choices of local coordinates for the fiber germ).

the right using the fibered structure of $F \mid F^{-1}W = H \times \mathrm{id}_S$, but the new coordinate change on the right (i.e. induced from that on the left via the trivialization) may differ from the one given by the fixed local coordinates at the right endpoint. Can we twist the induced coordinate choice along S so that the induced local coordinates on the right coincide with the given ones?

The choices of fiber coordinates are elements of the group $\mathscr{A}_H$, where $\mathscr{A}$ is the group of $\mathscr{A}$-equivalences $(\psi_1, \ldots, \psi_k, \phi)$ of germs $\bigsqcup_k (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, and $\mathscr{A}_H$ is the subgroup of $\mathscr{A}$ consisting of $\mathscr{A}$-equivalences which take H to H: $H = \phi^{-1} \circ (\bigsqcup H_k \circ \psi_k)$. We have shown in [4, Theorems 50 and 56] that when H is finitely $\mathscr{A}$-determined, $\mathscr{A}_H$ has a maximal compact subgroup G (that is, a subgroup which is conjugate in $\mathscr{A}$ to a compact linear group) and $\mathscr{A}_H/G$ is contractible in a generalized sense. Generalized contractibility means that any smooth map $\alpha : \partial M \to \mathscr{A}_H/G$ admits a smooth extension $\bar{\alpha} : M \to \mathscr{A}_H/G$, where smoothness means that at any point in M, there is a local lift $W_M \to \mathscr{A}_H$ over the projection $\mathscr{A}_H \to \mathscr{A}_H/G$ such that the induced map $W_M \times \bigsqcup_k U_i \to W_M \times V$ is smooth. In our setting, this means that we can find a path in $\mathscr{A}_H$ from the choices of coordinates on the left to those on the right, and we may assume that the local choice of coordinates which is part of the tubular structure at S agrees with the pre-defined choice of coordinates near ∂S up to action of an element of G. Consequently, we can define an E-tame retraction germ at S which agrees with the pre-defined one near ∂S.

2.1.2. Smooth and E-tame retractions

The general problem of combining local E-tame retractions is very difficult, as it basically amounts to combining continuous distributions whose integral manifolds are smooth, but whose generating hyperplane fields are not necessarily simultaneously tangent to something smooth. It may seem like a crude simplification to study the combination of an E-tame retraction with a smooth one, but in fact, this is vastly useful:

Suppose that we want to show that a map or germ f is topologically stable by constructing an E-tame retraction $F \to f$ from the stable unfolding F. Denote by $I(f)$ the *instability locus* of f, consisting of those points y in the target $t(f)$ such that the germ f_y of f at $f^{-1}(y) \cap \Sigma f$ is not stable. At any point y_0 *not in* $I(f)$, we can find a local smooth retraction $F_{y_0} \to f_{y_0}$, and it is easy to combine such smooth local retractions, for instance by combining their inducing vector fields using a partition of unity. If we can find an E-tame retraction $F \to f$ in a neighborhood of the instability locus $I(f)$, then we can use a combination lemma for smooth and E-tame retractions to glue this together with the smooth retraction.

The idea is to find coordinates in which the smooth retraction is just a projection, and stretch out the fibers of the E-tame retraction as we approach the smooth one. See Fig. 3. As a technical detail, we combine the smooth and E-tame retractions along a line. This could correspond to combining along a submersive distance function measuring the distance from (a neighborhood of) $I(f)$, and does not cause problems in practise.

To be precise, suppose that we are retracting onto source and target submanifolds N_0 and P_0, which are open subsets of $N \times \mathbb{R}$ and $P \times \mathbb{R}$ for smooth manifolds N and P, respectively, and we pass from one retraction to the other while moving along the $\mathbb{R}$-component.

Assume (as we may, up to a reparametrization of $\mathbb{R}$) that

$$N \times [0, 3] \subset N_0, \qquad P \times [0, 3] \subset P_0,$$

and that we are retracting onto a smooth map $f : N_0 \to P_0$ where $f(N \times \{t\}) \subset P \times \{t\}$ for all t, from a d-parameter, $\mathbb{R}^d$-level preserving unfolding $F : \tilde{N} \to \tilde{P}$ of f, where $\tilde{N}$ and $\tilde{P}$ are open neighborhoods of $N_0 \times \{0\}$ and $P_0 \times \{0\}$ in $(N \times \mathbb{R}) \times \mathbb{R}^d$ and $(P \times \mathbb{R}) \times \mathbb{R}^d$, respectively.

Assume that the retractions which we want to combine along $\mathbb{R}$ are given by

$$(r_i, s_i) : F \to f, \quad i = 1, 2, \tag{3}$$

where (r_1, s_1) is smooth and (r_2, s_2) is E-tame.

After a smooth change of coordinates (shrinking $\tilde{N}$ and $\tilde{P}$) given by

$$\Phi_N : \tilde{N} \to N_0 \times \mathbb{R}^d, \qquad \Phi_N(y, t, u) = \big(s_1(y, t, u), u\big),$$

$$\Phi_P : \tilde{P} \to P_0 \times \mathbb{R}^d, \qquad \Phi_P(x, t, u) = \big(r_1(x, t, u), u\big),$$

we see that we may assume that $(r_1, s_1) = (\mathrm{pr}_{N_0}, \mathrm{pr}_{P_0})$ and $F = f \times \mathrm{id}_{\mathbb{R}^d}$.

The following lemma from [4, Lemma 32] proves that we can, indeed, combine the smooth projection with an E-tame retraction along $\mathbb{R}$, by "straightening out" the fibers of the E-tame retraction (r_2, s_2) as we approach $0 \in \mathbb{R}$:

Lemma 4. *Suppose that as described above, we have defined retractions* $(r_i, s_i) : F \to f$, *where* $(r_1, s_1) = (\mathrm{pr}, \mathrm{pr})$ *and* (r_2, s_2) *is E-tame, and where* $F = f \times \mathrm{id}_{\mathbb{R}^d}$. *Then, allowing for shrinking* $\tilde{N}$ *and* $\tilde{P}$, *we can find an E-tame retraction* $(R, S) : F \to f$ *such that*

$$R = r_1 \quad \text{in } R^{-1}\big(N_0 \cap N \times [0, 1]\big), \qquad S = s_1 \quad \text{in } S^{-1}\big(P_0 \cap P \times [0, 1]\big),$$

$$R = r_2 \quad \text{in } R^{-1}\big(N_0 \cap N \times [2, 3]\big), \qquad S = s_2 \quad \text{in } S^{-1}\big(P_0 \cap P \times [2, 3]\big).$$

If (r_2, s_2) *is smooth, then so is* (R, S).

What seems like (*but, as we have seen, is not*) *a more general result, is that we can similarly combine the retractions* (r_i, s_i) *given in* (3).

3. Example constructions of E-tame retractions from the E- and Z-series

We study representatives of the $E_{4,0}(*)$ and $Z_{3,0}(*)$ singularities, showing that we can find an E-tame retraction $F \to F^+$, where F is the standard weighted homogeneous ministable unfolding of the $E_{4,0}(\lambda)$ and $Z_{3,0}(\lambda)$ singularities given by $(x, y) \mapsto y^3 + \lambda y x^8 + x^{12}$ and $(x, y) \mapsto x(y^3 + \lambda y x^6 + x^9)$, $\lambda \neq 0$, and where F^+ is its positively weighted part, as defined below. We assume that $4\lambda^2 + 27 \neq 0$ so that F has FST, since otherwise there is no stable unfolding germ. The assumption $\lambda \neq 0$ is necessary because du Plessis and Wall [3] have shown that germs corresponding to $\lambda = 0$ admit deformations which are topologically different from those which we find for $\lambda \neq 0$; hence there is no E-tame retraction $F \to F^+$ when $\lambda = 0$.

Since F has non-positively weighted unfolding variables, its positively weighted part F^+ is not smoothly stable, but we can prove that it is topologically stable by constructing an E-tame retraction $(r, s) : F \to F^+$. This shows, moreover, that the topological type of smoothly stable $E_{4,0}(\lambda')$ germs is constant for λ' near λ, and that the strata $E_{4,0}(*)$ and $Z_{3,0}(*)$ in jet space are *civilized* as defined in [2, Chapter 9].

Topological triviality of F over F^+ was shown for $E_{3,0}(\lambda)$ $(\lambda \neq 0)$ by Damon and Galligo [1]; however, their trivialization does not provide topological stability. Topological triviality and stability for $E_{2,0}(\lambda)$ $(\lambda \neq 0)$ was shown by Looijenga [6] a long time ago. Tame retractions for $E_{2,0}(\lambda)$, $E_{3,0}(\lambda)$, $Z_{1,0}(\lambda)$ and $Z_{2,0}(\lambda)$ are constructed in the book by du Plessis and Wall [2, p. 504–507], and in [4] we have given detailed constructions of E-tame retractions for $E_{p,0}(\lambda)$ and $Z_{q,0}(\lambda)$ when $p \leqslant 4$ and $q \leqslant 3$. Here we present the geometric ideas behind the constructions for $E_{4,0}(\lambda)$ and $Z_{3,0}(\lambda)$.

3.1. Preliminaries

The series of singularities denoted by $E_{p,0}(*)$ and $Z_{p,0}(*)$ denote the $\mathscr{R}$-classes of germs represented by the polynomial germs $f_{(p,0)}$ given by

$$y^3 + yx^{2p}W_{p-1}(x) + x^{3p},$$

$$x\big(y^3 + yx^{2p}W_p(x) + x^{3p}\big) \tag{5}$$

respectively, where $W_p(x) = w_0 + w_1 x + \cdots + w_p x^p$, $w_0 \neq 0$ and $4w_0^2 + 27 \neq 0$. It is actually enough to construct E-tame retractions $F \to F^+$ for the case with $W_p = w_0$, as we can use the weighted homogeneity of F to show that we have E-tame retractions for all the choices of W_p.

Now $f_{(p,0)}$ has a standard ministable unfolding $F_{(p,0)}$ given by

$$
\begin{cases}
(x, y, \underline{u}, \underline{v}) \mapsto \left(y^3 + \lambda y x^{2p} + x^{3p} + y \sum_{j=0}^{3p-2} v_j x^j + \sum_{i=1}^{3p-2} u_i x^i, \underline{u}, \underline{v} \right), \\[4mm]
(x, y, \underline{u}, \underline{v}, c) \mapsto \left(x\left(y^3 + \lambda y x^{2p} + x^{3p}\right) + y \sum_{j=0}^{3p-2} v_j x^j + \sum_{i=1}^{3p-2} u_i x^i + c y^2, \underline{u}, \underline{v}, c \right).
\end{cases}
\tag{6}
$$

A map $f : \mathbb{R}^n \to \mathbb{R}^p$ is *weighted homogeneous* if it is equivariant with respect to $\mathbb{R}^+$-actions ($\mathbb{R}^+ = (]0, \infty[, \cdot)$) on source and target given by

$$
t \cdot (x_1, \ldots, x_n) = \left(t^{a_1} x_1, \ldots, t^{a_n} x_n\right), \qquad t \cdot (y_1, \ldots, y_p) = \left(t^{b_1} y_1, \ldots, t^{b_p} y_p\right),
$$

that is, if $f(t \cdot x) = t \cdot f(x)$ for all $t \in \mathbb{R}^+$ and all $x \in \mathbb{R}^n$. Here the a_i are called *source weights* and the b_i are called *target weights*.

It is easy to see that the polynomials in (6) are weighted homogeneous, and moreover, if we give the weights $2p - 2 - j$ to v_j and $3p - 2 - j$ to u_i and p to c, then $F_{(p,0)}$ is weighted homogeneous.

We define weighted distance functions $\sigma : \mathbb{R}^n \to \mathbb{R}$ and $\rho : \mathbb{R}^p \to \mathbb{R}$ on source and target as follows:

$$
\sigma(x) = \sum \left\{ x_i^{A/a_i} \mid 1 \leqslant i \leqslant n \text{ s.t. } a_i > 0, \ A = \operatorname{lcm}\{2a_i \mid 1 \leqslant i \leqslant n, \ a_i > 0\} \right\},
$$
$$
\rho(y) = \sum \left\{ y_i^{B/b_i} \mid 1 \leqslant i \leqslant p \text{ s.t. } b_i > 0, \ B = \operatorname{lcm}\{2b_i \mid 1 \leqslant i \leqslant p, \ b_i > 0\} \right\}.
$$

The functions σ and ρ are zero exactly on the non-positively weighted subspaces $\{(x_i) \in \mathbb{R}^n \mid x_i \neq 0 \Rightarrow a_i \leqslant 0\}$ and $\{(y_i) \in \mathbb{R}^p \mid y_i \neq 0 \Rightarrow b_i \leqslant 0\}$ of source and target, respectively, and they define a distance from them.

We are in particular interested in the positively weighted part of $F_{(p,0)}$, namely the germ $F_{(p,0)}^+$ given by eliminating the non-positively weighted unfolding parameters in $F_{(p,0)}$. This map is not stable, since $F_{(p,0)}$ was stable of minimal dimension – but we would like to prove that $F_{(p,0)}^+$ is topologically stable by constructing a so-called *E-tame* retraction $F_{(p,0)} \to F_{(p,0)}^+$, as introduced in the next section. It turns out that the main challenge is to find the E-tame retraction near the instabilities of $F_{(p,0)}^+$, namely the points $y \in t(F_{(p,0)}^+)$, such that the germ of $F_{(p,0)}^+$ at $(F_{(p,0)}^+)^{-1}(y) \cap \Sigma F_{(p,0)}^+$ is unstable. We denote the set of such points y by $I(F_{(p,0)}^+)$ and call it the *positive instability locus* of $F_{(p,0)}$.

3.2. Topology of the positive instability locus

The positive instability loci for $F_{(p,0)}$ (defined in (6)) belonging to the E- and Z-series were parametrized by du Plessis and Wall in [3]. For example, in the $E_{p,0}(\lambda)$ case the parametrization $\chi : \mathbb{R}^{p-2} \to t(F_{p,0}^+)$ is given by the polynomial coefficients as functions of ξ_i in the deformation

$$
y^3 + \lambda y \prod_{i=1}^{p-2} (x - \xi_i)^2 \cdot \left(x + \frac{1}{2} \sum_{i=1}^{p-2} \xi_i \right)^4 + \prod_{i=1}^{p-2} (x - \xi_i)^3 \cdot \left(x + \frac{1}{2} \sum_{i=1}^{p-2} \xi_i \right)^6
\tag{7}
$$

of $f_{(p,0)}$ (defined in (5) with $W_{p-1} = \lambda$), when the deformation is considered as a polynomial in the variables x and y.

This is not the whole parametrization given by du Plessis and Wall, but in [4, Chapter 4.1.2] we show that all other parts of the parametrization given in [3] actually appear as restrictions of (7). Note that the deformation (7) is symmetric in the variables ξ_i; hence, so is the parametrization χ. In particular, χ is not injective; it is not immersive everywhere, and for $Z_{3,0}(\lambda)$, for instance, the positive instability locus has self-intersections. The instability loci of $F_{(p,0)}^+$ for $E_{p,0}(*)$ and $Z_{p,0}(*)$ and the parametrizations from [3] are investigated in detail in [4, Chapters 4.1.2, 4.2.1 and 4.2.4], and we prove:

Theorem 8. *([4, Theorem 132 and Proposition 160]) The partition of the positive instability locus $I(F_{(p,0)}^+)$ into singularity sets I_Δ which are strict presentations of the singularities Δ by $F_{p,0}^+$, is a stratification, and the components of the strata are smooth and contractible. In fact, the strata are components of the intersection $Y_\Delta \cap t(F_{(p,0)}^+)$, where Y_Δ is the full strict presentation by $F_{(p,0)}$ of Δ in $t(F_{(p,0)})$. Moreover, the preimage $H(F_{(p,0)}^+) = (F_{(p,0)}^+)^{-1} I(F_{(p,0)}^+) \cap \Sigma F_{(p,0)}^+$ is a stratified set as well, with smooth strata $H_\Delta = (F_{(p,0)}^+)^{-1}(I_\Delta)$, and for each component of H_Δ for each Δ appearing in $I(F_{(p,0)}^+)$, the restriction*

$$
F_{(p,0)}^+ \mid : \operatorname{comp}(H_\Delta) \to \operatorname{comp}(I_\Delta)
$$

onto the target component is a diffeomorphism. See Fig. 5.

If $F_{(p,0)}$ belongs to $E_{p,0}(*)$, then all the singularities found in $I(F_{(p,0)}^+)$ belong to $E_{q,0}(*)$ for $q \leqslant p$. If $F_{(p,0)}$ belongs to $Z_{(p,0)}(*)$, then all the singularities found in $I(F_{(p,0)}^+)$ belong to $E_{q,0}(*)$ and $Z_{q,0}(*)$ for $q \leqslant p$.
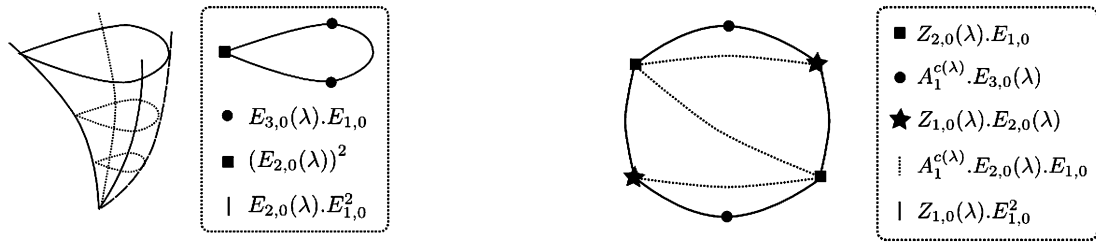
Fig. 4. Left: The positive instability locus of $E_{4,0}(\lambda)$ and its intersection with a level set $\rho^{-1}(\epsilon)$. Right: The intersection with $\rho^{-1}(\epsilon)$ for the positive instability locus of $Z_{3,0}(\lambda)$. Here $c(\lambda)$ is 1 or 3 when λ is $>$ or $< -\sqrt[3]{27/4}$.
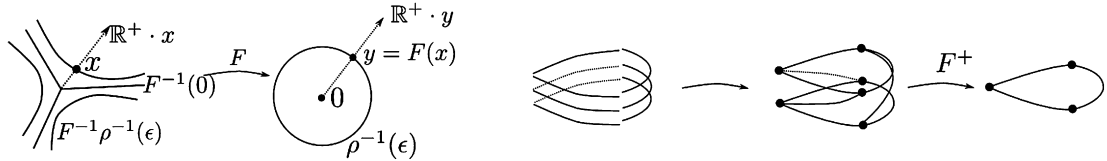

Fig. 5. Left: The E-tame retraction (r_ϵ, s_ϵ) on the slices $(F^{-1}\rho^{-1}(\epsilon), \rho^{-1}(\epsilon))$ induces a global E-tame retraction, extended through the $\mathbb{R}^+$-action and a deformation. Right: The instability locus of F_ϵ^+ is a stratified set, as is its preimage, and F_ϵ^+ restricts to a diffeomorphism on components of the source strata.

3.3. The $E_{4,0}(*)$-singularity

Using previous results by Wall and du Plessis [2, p. 504–507] (consult [4, Chapter 4.1] for details) it is not difficult to construct E-tame retractions $(r_{(p,0)}, s_{(p,0)}) : F_{(p,0)} \to F_{(p,0)}^+$, where $F_{(p,0)}$ is the ministable unfolding for $E_{p,0}(*)$, when $p = 1, 2, 3$. However, as p increases, the topology of the positive instability locus rapidly becomes more complex, and the methods of [2] are no longer enough to make a construction.

We describe the construction of an E-tame retraction $F \to F^+$ for F a stable unfolding of the $E_{4,0}(\lambda)$-singularity, with $\lambda \neq 0$ and $4\lambda^3 + 27 \neq 0$.

Note that by [4, Lemma 36], which is an E-tame version of [2, Lemma 9.6.4], it suffices to find a retraction $F_\epsilon \to F_\epsilon^+$, where F_ϵ is the restriction of F to the level sets $(F^{-1}(\rho^{-1}(\epsilon)), \rho^{-1}(\epsilon))$, see Fig. 5. By [2, Lemma 9.6.2] the instability locus of F_ϵ^+ is $I(F^+) \cap \rho^{-1}(\epsilon)$, which is the 1-dimensional stratified set in Figs. 4 and 5.

Denote by F^k the unfolding obtained from F by removing the k unfolding variables of lowest weight. Then, the restriction of F^1 to $(F^{-1}(\rho^{-1}(\epsilon)), \rho^{-1}(\epsilon))$ is stable, and it suffices to find an E-tame retraction $F_\epsilon^1 \to F_\epsilon^+$. Away from the positive instability locus, the germ of F_ϵ^1 is stable; thus we can find local smooth retractions $(F_\epsilon^1)_y \to (F_\epsilon^+)_y$ for any $y \in t(F_\epsilon^1) \setminus I(F_\epsilon^+)$. These retractions are induced by smooth vector fields, which can easily be combined using a partition of unity, hence we obtain a global smooth retraction $F_\epsilon^1 \to F_\epsilon^+$ off $I(F_\epsilon^+)$. Using the techniques from Section 2.1.2, we see that it is now enough to find a retraction $F_\epsilon^1 \to F_\epsilon^+$ in a neighborhood of $I(F_\epsilon^+)$.

By Theorem 8, the instability locus of F^+ is a stratified smooth set with respect to the partition by presented singularity types. Because $\rho^{-1}(\epsilon)$ is transverse to the orbits of the $\mathbb{R}^+$-action, with respect to which F^+ is smoothly fibered on $t(F^+) \setminus \{0\}$, we conclude that all the claims of Theorem 8 hold for $I(F_\epsilon^+)$ as well.

The positive instability locus $I(F_\epsilon^+)$ consists of two points where we find an $E_{3,0}(\lambda).E_{1,0}$-singularity presented, one point where we find an $(E_{2,0}(\lambda))^2$-singularity presented, and three 1-dimensional submanifolds along which we find $E_{2,0}(\lambda).E_{1,0}^2$-singularities presented. The 0-dimensional strata form the boundary of the 1-dimensional strata, and in particular, the $(E_{2,0}(\lambda))^2$ and $E_{3,0}(\lambda).E_{1,0}$-singularities are formed when one of the $E_{1,0}$-singularities in $E_{2,0}(\lambda).E_{1,0}^2$ approaches and combines with the other $E_{1,0}$-singularity or the $E_{2,0}(\lambda)$-singularity, respectively.

At points y_0 of the 0-dimensional stratum we can thus choose local coordinates which take the germ of F_ϵ^1 at y_0 to the germs $F_{(2,0)^2}$ and $F_{(3,0).(1,0)}$, as in Fig. 6.

In particular, the 1-dimensional stratum in $t(F_\epsilon^1)$ with $E_{2,0}(\lambda).E_{1,0}^2$-singularities presented by F_ϵ^1 are taken to the presentations of $E_{2,0}(\lambda).E_{1,0}^2$ by $F_{(2,0)^2}$ and $F_{(3,0).(1,0)}$, respectively, in the target of $F_{(2,0)^2}$ and $F_{(3,0).(1,0)}$. We can form a distance function ρ_Δ, for $\Delta = (2,0)^2$ or $(3,0).(1,0)$, such that $\rho_\Delta^{-1}(\epsilon)$ intersects the presentation of $E_{2,0}(\lambda).E_{1,0}^2$ in one point, at which the germ of the germ of $F_\Delta \mid F_\Delta^{-1}\rho_\Delta^{-1}(\epsilon)$ is a ministable germ presenting $E_{2,0}(\lambda)^2$ and $E_{3,0}(\lambda).E_{1,0}$, respectively.

We construct a tubular neighborhood (T, π) about the 1-dimensional stratum which agrees with the fibration by level sets $\rho_\Delta^{-1}(\epsilon)$ near the 0-dimensional strata, as in Fig. 6. We construct these by gluing local sprays together [5, Chapter IV] which give the wanted tubular neighborhoods near the 0-dimensional strata.

The restriction of F_ϵ^1 to the level set is a stable unfolding for $E_{2,0}(*).E_{1,0}^2$. By the product retraction construction on page 459, we *can* construct E-tame retractions $F \to F^+$ for the stable multigerm $F_{(2,0).(1,0)^2}$ representing $E_{2,0}(*).E_{1,0}^2$, formed by combining the retractions for $F_{(1,0)}$ and $F_{(2,0)}$ as in (2). Pulling back using the choice of coordinates in the
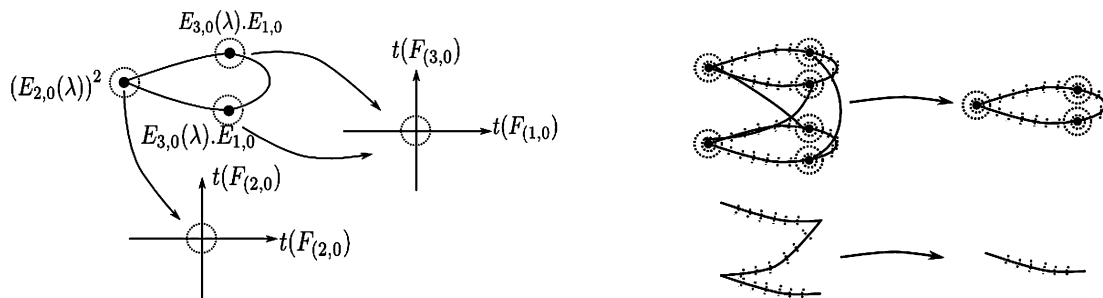
Fig. 6. Left: We choose local coordinates at the 0-dimensional strata in which F_ϵ^1 is represented by the ministable unfoldings $F_{(2,0)^2}$ and $F_{(3,0).(1,0)}$ of $(E_{2,0}(\lambda))^2$ and $E_{3,0}(\lambda).E_{1,0}$. Right: We find a tubular neighborhood of the 1-dimensional stratum which coincides with the trivialization by level sets of the local weighted distance functions at the 0-dimensional strata. We can find E-tame retractions inside all fibers, and we can fit them smoothly together through the tubular neighborhood by using a path in the group $\mathscr{A}_H$ of coordinate changes in the fiber which leave the fiber germ H invariant.

fiber, we thus obtain a retraction in the tubular fiber; however, the target of this retraction is not contained in the target of $t(F_\epsilon^+)$.

It is tempting to try to use the following procedure: Define an E-tame retraction inside the fibers of the tubular neighborhoods of the 1-dimensional strata. Near the 0-dimensional strata, define a retraction (r_ϵ, s_ϵ) in the level sets $(F^{-1}\rho^{-1}(\epsilon), \rho^{-1}(\epsilon))$ by combining the one from the tubular neighborhood fiber with smooth retractions $F_y^1 \to F_y^+$ found off the positive instability locus, and then use the weighted homogeneity at the 0-dimensional stratum $\{y_0\}$ to extend this retraction over a neighborhood of y_0. However, this is impossible!

Let us see why. The singularity presented at y_0 is either $(E_{2,0}(\lambda))^2$ or $E_{3,0}(\lambda).E_{1,0}$, and in order to use the weighted homogeneity we need to work inside the local coordinate system $t(F_{(2,0)}) \times t(F_{(2,0)})$ or $t(F_{(3,0)}) \times t(F_{(1,0)})$, in which the germ $(F_\epsilon^1)_{y_0}$ takes the form F_Δ, where Δ is either $(2,0)^2$ or $(3,0).(1,0)$, and construct a retraction $F_\Delta \to F_\Delta^+$ as in (2) by combining retractions for $F_{(q,0)}$ with $q \leqslant 3$. The retraction onto F_Δ^+ induced by the level set retraction *cannot* be an E-tame retraction $F_\Delta \to F_\Delta^+$. If that was the case, then the instability locus of F_Δ^+ would be the 1-dimensional stratum, because the retraction in the level set is smooth off the 1-dimensional stratum, inducing local smooth stability of F_Δ^+ – but the instability locus of F_Δ^+ is much larger than that. We conclude that the fibers of the retraction (r_ϵ, s_ϵ) cannot be transverse to the target of F_Δ^+, and this strategy does not work.

What we do instead is first find local retractions at the 0-dimensional strata, see that they induce retractions $F_{(2,0).(1,0)^2} \to F_{(2,0).(1,0)^2}^+$ in the fibers, and extend these retractions near the boundaries of the 1-dimensional stratum over the whole stratum by using the local symmetries, as described in Section 2.1.1. Carefully controlling the geometry, we see that we obtain coinciding local retractions whose foliations are transverse to $t(F_\epsilon^+)$ in $t(F_\epsilon^1)$, hence inducing an E-tame retraction $F_\epsilon^1 \to F_\epsilon^+$, which is what we are looking for. The details of the construction, which are rather technical, can be found in [4, Chapter 4.1.6].

3.4. The $Z_{3,0}(*)$-singularity

Again, it is not difficult to construct E-tame retractions $F \to F^+$ for $Z_{q,0}(\lambda)$ when $q = 1$ or 2, using techniques from [2]. We do this in [2, Chapter 4.2] and in fact, in [2, p. 504-507], tame (but not E-tame) retractions *are* explicitly constructed for these singularities. Once more, as q increases, the problem becomes significantly more difficult due to the complexity of the instability locus. We study the case $q = 3$.

As before, we use weighted homogeneity to reduce our problem to one of finding an E-tame retraction $F_\epsilon^1 \to F_\epsilon^+$ of the restrictions to level sets $((F^1)^{-1}\rho^{-1}(\epsilon), \rho^{-1}(\epsilon))$ of the weighted distance function $\rho : t(F) \to \mathbb{R}$.

Also, as in the previous case, the positive instability locus is stratified by singularity sets, and its intersection with $\rho^{-1}(\epsilon)$ has the structure seen in Fig. 4. The presented singularities are marked in Fig. 4, and they are all singularities from the E- and Z-series for which we know how to construct E-tame retractions. The 1-dimensional strata are deformations of *one* of the singularities appearing in 0-dimensional strata. The 0-dimensional strata are multigerm presentations where at most two of the monogerm components are deformable within the positive instability locus; in the deformations giving rise to 1-dimensional strata, one is kept constant while the other is deformed.

Note that this instability locus is even more complicated than the one we had for $E_{4,0}(\lambda)$ – it is not even a topological manifold, due to self-intersections. However, our techniques only use the stratified structure, and hence we are able to use the same strategy as for the E-series – that is: First construct a tubular neighborhood about the 1-dimensional stratum which agrees with the local fibrations at the 0-dimensional strata by certain well-chosen distance functions (see [4, Chapter 4.2] for details). Secondly, find local retractions at the 0-dimensional strata. Then use the contractibility of the quotient $\mathscr{A}_H/G$ by a maximal compact subgroup G of $\mathscr{A}_H$ for the ministable representatives of the multigerms H presented at the 1-dimensional stratum in order to find retractions in the tubular neighborhoods which agree with the chosen retractions at the 0-dimensional strata.

4. Possible generalizations

Our constructions of E-tame retractions above and in [4] have followed the pattern

i) Find the positive instability locus, and show that it is a stratified set with respect to stratification by presented singularity type. Realize that by induction, we already know how to find E-tame retractions onto the positively weighted unfoldings – at least for a reduced part of a problem such as that given by restriction to a level set.

ii) Using this, find local retractions near the instability locus.

iii) Combine the local retractions by controlling the geometry near the positive instability locus, forcing the local retractions to coincide.

We conjecture that it is possible to construct E-tame retractions for $E_{p,0}(*)$ and $Z_{q,0}(*)$ for all $p, q \in \mathbb{N}$, having seen that it *is* possible for $p \leqslant 4$ and $q \leqslant 3$ using this strategy.

We have already completed i) for all p and q. In iii) we suggest constructing a system of trivial tubular neighborhoods about the strata of the instability locus, and using the results on local symmetries of singularities as in Section 2.1.1 in order to combine retractions along the strata.

Step ii) suggests an inductive procedure, but as we have seen in the construction for the $E_{4,0}(*)$ and $Z_{3,0}(*)$ cases, we need our retractions $F_{(m,0)} \rightarrow F_{(m,0)}^{+}$ to be equivariant with respect to $MC(\mathscr{A}_{F_{(m,0)}})$ for $F_{(m,0)}$ occurring on dim > 1 strata of the positive instability locus, in order to use the combination in iii). Thus, in attacking the general problem, we need to construct *equivariant* retractions in each step.

We also conjecture that our methods may be used for other singularities, possibly with more complicated instability loci, as long as we are able to describe the structure of the instability locus and the presented singularities are of types which we know how to handle.

Acknowledgements

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