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Augmenting graphs for independent sets

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Abstract

We consider a general approach to solve the maximum independent set problem based on finding augmenting graphs. For some special classes of graphs, the approach provides polynomial solutions. In this paper we survey classical and recent results on the topic, and describe a new one that generalizes two previously known algorithms. We also discuss some open questions related to the notion of augmenting graphs.

Keywords: Independent set; Augmenting graph; Polynomial algorithm

1. Introduction

In a simple graph a subset of vertices S is called *independent* if no two vertices in S are linked by an edge. The cardinality of a maximum independent set in a graph G is called the *independence number* of G and is denoted $\alpha(G)$. The *independent set problem* is to find in G an independent set with $\alpha(G)$ vertices. The problem is known to be NP-hard in general graphs. Moreover, it remains NP-hard in many special classes of graphs. To provide some examples, let us call a graph G *H-free* if G does not contain graph H as an induced subgraph. It has been proved in [1] that if H contains either a cycle or a vertex of degree more than 3 or two vertices of degree 3 in the same connected component, then the independent set problem is NP-hard in the class of H-free graphs. On the other hand, there are known classes of graphs for which the problem can be solved in polynomial time. Consider, for example, a line graph L, i.e. a graph whose vertices are in one-to-one correspondence with the edges of an underlying graph G, two vertices being adjacent in L iff the corresponding edges of G have a common vertex. Obviously, the independent set problem in L is equivalent to the problem of finding a maximum matching (independent set of edges) in graph G. A polynomial algorithm for the maximum matching problem has been found by Edmonds in 1965 [6]. The Edmonds' algorithm exploited the idea of Berge that a matching in a graph is maximum if and only if it does not admit an augmenting path [3]. Minty [12] and Sbihi [15] applied the idea of augmenting paths to solve the independent set problem in the class of claw-free graphs that properly contains the line graphs. During almost 20 years the result of Minty and Sbihi remained unimproved and the idea of augmenting graphs remained unused. However, recently, the idea has been applied again to solve the independent set problem efficiently in several new classes of graphs, including an extension of claw-free graphs. In the present paper we survey all those results and describe a new one that generalizes two previously known algorithms. The paper is organized as follows. In the rest of the section we introduce some basic notations. In Section 2 we describe the idea of augmenting graphs and survey the results that use this idea to solve the independent set problem efficiently in certain graph classes. Section 3 is devoted to the characterization of augmenting graphs in some particular classes. In Section 4 we consider the problem of finding augmenting graphs. Together, the results of Sections 3 and 4 provide a polynomial time algorithm

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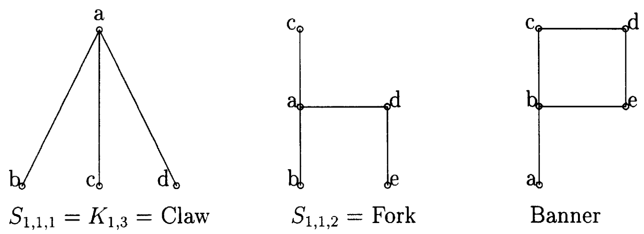

Fig. 1.

to solve the independent set problem in the class of (P_7, banner)-free graphs presented in Section 5. Finally, in Section 6 we discuss some open problems related to the topic.

All graphs considered are undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are denoted VG and EG, respectively. For a vertex $x \in VG$, we denote by $N(x)$ the neighborhood of x, i.e. the set of vertices adjacent to x. If $U \subseteq VG$, then $N_U(x) = N(x) \cap U$, and $N(U) = \bigcup_{x \in U} N_{VG-U}(x)$, and $N_W(U) = N(U) \cap W$, where $W \subseteq VG$. By $G[U]$ we denote the subgraph of G induced by set U.

A bipartite graph $H = (W, B, E)$ consists of two parts of vertices W and B, and set of edges $E \subseteq W \times B$. We allow W or B to be empty. As usual, K_n, P_n, C_n, and $K_{n,m}$ denote, respectively, the complete graph, the chordless path, the chordless cycle on n vertices and the complete bipartite graph with parts of size n and m.

By $S_{i,j,k}$ we denote a tree with exactly three vertices of degree one being on distance i, j, k from the only vertex of degree three. In particular, $S_{1,1,1} = K_{1,3}$ is a *claw*, and $S_{1,1,2}$ is a *fork*. A *banner* is the graph with vertices a, b, c, d, e and edges (a, b), (b, c), (c, d), (d, e) and (e, b). Fig. 1 represents the three graphs mentioned above.

Along the paper, we use term "maximal" with respect to set inclusion, and term "maximum" with respect to size.

2. Augmenting graphs

The idea of augmenting graphs to find a maximum independent set can be described as follows.

Let S be an independent set in a graph G. Throughout the paper we call vertices in S *white* and vertices in $VG - S$ *black*. A bipartite graph $H = (W, B, E)$ with set of white vertices W and set of black vertices B is called *augmenting* for S (and we say that S *admits* the augmenting graph) if $N(B) \cap (S - W) = \emptyset$ and $|B| > |W|$.

Clearly, if $H = (W, B, E)$ is an augmenting graph for S, then S is not maximum since set $S' = (S - W) \cup B$ is independent with $|S'| > |S|$. We shall say that set S' is obtained from S by H-augmentation and call number $|B| - |W| = |S'| - |S|$ the *increment* of H.

Now assume S is not a maximum independent set, and let S' denote an independent set such that $|S'| > |S|$. Then the subgraph of G induced by set $(S - S') \cup (S' - S)$ is augmenting for S. Thus we have

Theorem. (augmenting graphs). *An independent set S in a graph G is maximum if and only if there are no augmenting graphs for S.*

Originally, the augmenting graph approach was formulated by Berge in connection with the maximum matching problem, i.e. the problem of finding a maximum subset of edges no two of which have a vertex in common. As mentioned in the introduction, the maximum matching problem is equivalent to the independent set problem in the class of line graphs. Every line graph is known to be claw-free, and every connected claw-free bipartite graph is obviously either a path or a cycle. It is easy to see that a bipartite cycle cannot be an augmenting graph, because it has equal number of vertices in both parts. Thus, in the class of line graphs any connected augmenting graph is a path. Moreover, if a line graph L is a path, then the corresponding underlying graph G is a path as well, and vice versa. So, the above arguments transform the theorem of augmenting graphs, restricted to the maximum matching problem, to the theorem of augmenting paths proved by Berge in 1957 [3]. In 1965, Edmonds [6] used the idea of Berge to develop an efficient algorithm for the maximum matching problem in general graphs, implying a solution for the independent set problem in the class of line graphs. In 1980, independently Minty [12] and Sbihi [15] extended the approach of Edmonds to a polynomial time procedure to solve the independent set problem in the class of claw-free graphs. Recently, the approach based on finding augmenting graphs has been applied again to solve the problem in some special classes of graphs. Mosca [13] proposed an efficient way to find augmenting graphs in particular subclasses of P_5-free graphs including $(P_5, K_{1,m})$-free graphs, $(P_5, K_{2,3})$-free graphs, and (P_5,cricket)-free graphs, where a cricket is the graph with vertices a,b,c,d,e and edges (a,b), (a,c), (a,d), (a,e) and (b,c). Next Mosca applied the approach to solve the problem in (P_6,C_4)-free graphs [14]. Alekseev [2] discovered an efficient way to find augmenting graphs in fork-free graphs generalizing the result of Minty–Sbihi, and a number of other particular results [5,7,9]. The same approach has been applied to solve the problem in (P_5, banner)-free graphs [10].

In the present paper, we apply the augmenting graph approach to solve the problem in polynomial time in the class of (P_7, banner)-free graphs generalizing both (P_6,C_4)-free and (P_5, banner)-free graphs. We also discuss some open questions related to the topic.

3. Characterization of augmenting graphs

In this section, we characterize augmenting graphs in some special classes. For the sake of completeness, we first recall some known results and then present several new ones.

1. *Claw-free graphs*: As mentioned above, any connected claw-free bipartite graph is either a cycle or a path. Consequently, any augmenting connected claw-free graph is a path.
2. P_4*-free graphs*: It is an easy task to verify that any connected P_4-free bipartite graph is complete bipartite.
3. *Fork-free graphs* generalize both claw-free and P_4-free graphs. It has been proved in [2] that a connected fork-free bipartite graph is either claw-free or almost complete bipartite, i.e. a bipartite graph whose every vertex has at most one non-neighbor in the opposite part.
4. (P_6,C_4)*-free graphs*: Mosca has proved in [14] that every connected (P_6,C_4)-free augmenting graph is a simple augmenting tree, i.e. a graph defined as follows. A simple augmenting tree T_r has r vertices of degree 1, and r vertices of degree 2, and a unique vertex x of degree r, the *root* of the tree, adjacent to each vertex of degree 2 (Fig. 2). Informally, T_r is the graph obtained from a star $K_{1,r}$ by introducing a new vertex in each edge of the star. In particular, T_1 is a P_3, and T_2 is a P_5, and T_3 is a $S_{2,2,2}$.

We now extend this short list with two new classes, both are subclasses of banner-free graphs. To this end, we first prove a general property of banner-free bipartite graphs (Lemma 2), and then extend the notion of a simple augmenting tree T_r to a notion of a *plant* D_r (Definition 1). To simplify the task, we also restrict our attention to augmenting graphs, which are minimal under inclusion. Obviously, any minimal augmenting graph is connected. However, not every connected augmenting graph is minimal. For example, an augmenting fork contains an augmenting P_3 as an induced subgraph. Similarly, an augmenting $S_{1,2,3}$ contains an augmenting P_5. The following property of minimal augmenting graphs will be helpful in the sequel.

Lemma 1. *Let $H=(W,B,E)$ be an augmenting graph with set of white vertices W and set of black vertices B. If H is minimal, then for every subset $A \subseteq W$, $|A| < |N_B(A)|$.*

Proof. Assume $|A| \geqslant |N_B(A)|$ for some subset A of W. Clearly $A \neq W$, since H is augmenting. Consequently $N_B(A) \neq B$. But then $H-(A \cup N_B(A))$ is a proper induced subgraph of H which is augmenting as well. This contradiction proves the lemma. □

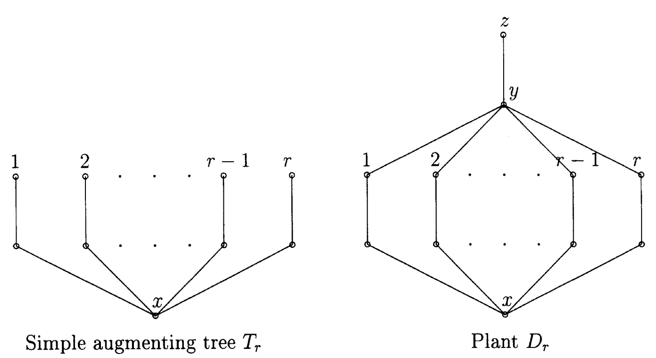

Fig. 2.

Lemma 2. *Let H be a connected bipartite banner-free graph. If H contains a C_4, then it is complete bipartite.*

Proof. Denote by H' a maximal induced complete bipartite subgraph of H containing the C_4. Let x be a vertex outside of H' adjacent to a vertex in the subgraph. Then x must be adjacent to all the vertices in the opposite part of the subgraph, otherwise H contains an induced banner. But then H' is not maximal. This contradiction proves that $H = H'$ is complete bipartite. □

Definition 1. A *plant* D_r is the graph obtained from a simple augmenting tree T_r by introducing two additional vertices y and z, and the edges linking vertex y to z and to each vertex of degree one in T_r (Fig. 2). We call y the *cutpoint* and z the *pendant* of D_r, and keep naming x the root. Moreover, for convenience, we consider a simple augmenting tree as a particular case of a plant.

Lemma 3. *A minimal augmenting $(S_{1,2,3}, banner)$-free graph $H = (W, B, E)$ is either a path or a plant or a complete bipartite graph.*

Proof. We assume that H is C_4-free (otherwise it is complete bipartite by Lemma 2) and has a vertex of degree at least 3 (otherwise it is a path). Assume first that every black vertex has degree at most 2. Then there is a white vertex a of degree at least 3. Let $\{b_1, \dots, b_k\}$ $(k \geqslant 3)$ be the neighborhood of a in graph H. Due to minimality of H, there is at most one black vertex of degree 1 among $b_1, \dots, b_k$, say b_k. For $i < k$, we denote by c_i the second neighbor of b_i in H. Notice that $c_i \neq c_j$ due to C_4-freeness of H. By Lemma 1, each c_i is adjacent to one more black vertex $x \neq b_i$. Obviously x is not a neighbor of a else H contains a C_4. Furthermore, x is adjacent to c_j for each $j \neq i$, otherwise vertices $\{a, b_i, b_j, b_k, c_i, c_j, x\}$ induce $S_{1,2,3}$. But then $k = 3$ (else x has degree more than 2) and hence $H = D_2$.

Now let b be a black vertex of degree at least 3. By minimality of H, $|B| = |W| + 1$. From Lemma 1 and König-Hall theorem it follows that graph $H - b$ has a perfect matching. We denote by $m(x)$ the vertex matched with x relative to the matching. Let $W_1 = N_W(b)$, $W_0 = W - W_1$, $B_1 = m(W_1)$, $B_0 = m(W_0)$. Clearly every vertex in B_1 has exactly one neighbor in W_1 else H contains a C_4. Consequently, if $W_0 = \emptyset$ then H is a simple augmenting tree. If W_0 is not empty, then there is a vertex $z \in W_0$ that has a neighbor x in B_1 due to Lemma 1. If z is not adjacent to a vertex $y \in B_1$, then H contains $S_{1,2,3}$ induced by vertices $\{b, x, y, z, m(x), m(y), u\}$, where $u \in W_1 - \{m(x), m(y)\}$. Therefore, z is adjacent to every vertex in B_1. Moreover, because of C_4-freeness of H, vertex z is the only neighbor in W_0 for each vertex of B_1. Finally, vertex $m(z)$ has no neighbor in W_1 (else H contains a C_4), and $m(z)$ has no neighbor in W_0 different from z (else H contains an $S_{1,2,3}$ induced by vertices $b, m(x), x, y, z, m(z), z'$ with $x, y \in B_1$ and $z' \in W_0 \cap N(m(z)) - \{z\}$). But now $W_1 \cup B_1 \cup \{b, z, m(z)\}$ induces an augmenting graph which is a plant. □

Lemma 4. *A minimal augmenting (P_7, banner)-free graph H is either a complete bipartite graph or a plant.*

Proof. Due to Lemma 2 we assume that H is C_4-free. If in addition H is $S_{1,2,3}$-free, then the proposition follows from Lemma 3. Indeed, there are only two P_7-free augmenting paths, P_3 and P_5, and both of them are simple augmenting trees (plants).

Now let H contain an induced $S_{1,2,3}$ formed by path $P_6 = (1, 2, 3, 4, 5, 6)$ and vertex 7 adjacent to 3. Suppose x is a neighbor of vertex 1 other than 2. Then x is not adjacent to 3, because of C_4-freness of H. But now H contains either an induced $P_7 = (x, 1, 2, 3, 4, 5, 6)$ (if x is not adjacent to 5) or an induced $P_7 = (7, 3, 2, 1, x, 5, 6)$ (if x is adjacent to 5). This contradiction shows that vertex 1 has only one neighbor in graph H. From minimality of H and Lemma 1, we deduce that 1 is a black vertex, and hence 6 and 7 are white. For the same reason, vertex 6 has a neighbor x different from 5 in graph H, and vertex 7 has a neighbor y different from 3. Due to C_4-freness of H, x is not adjacent to 4, and y is not adjacent to 2 and 4. Furthermore, y is not adjacent to 6 else H contains $P_7 = (1, 2, 3, 7, y, 6, 5)$, and x is adjacent to 2 else H contains $P_7 = (1, 2, 3, 4, 5, 6, x)$. Consequently, x is not adjacent to 7 because of C_4-freeness of H. But then H contains an induced $P_7 = (y, 7, 3, 4, 5, 6, x)$. □

4. Finding augmenting graphs

4.1. Augmenting plants in ($S_{1,2,4}$, banner)-free graphs

Theorem 1. *Let G be a $S_{1,2,4}$- and banner-free graph with n vertices, and S be a maximal independent set in G. Then one can determine if S admits an augmenting plant in time $O(n^6)$.*

Proof. One can verify if S admits an augmenting P_3 in time $O(n^2)$. In the rest of the proof, we assume there is no augmenting P_3 with respect to S.

Consider a vertex x outside of S with $N_S(x)=\{a_1,\dots,a_k\}$. Our aim is to determine whether set S admits an augmenting plant with root x. To this end, we fix a vertex y as the cutpoint of the plant and a vertex z as its pendant. Clearly y must be a white vertex non-adjacent to x, and z a black vertex non-adjacent to x whose only neighbor in S is y. If such y and z exist for x, we define for each white neighbor a_i of x, set $K(a_i)$ in the following way:

$$K(a_i)=\{b\in VG-S \mid (b,x)\notin EG \text{ and } (b,z)\notin EG \text{ and } N_S(b)=\{a_i,y\}\}.$$

If there is no pair of vertices y,z satisfying the above condition, we try to determine if S admits a simple augmenting tree with root x. In that case, we define $K(a_i)$ as follows:

$$K(a_i)=\{b\in VG-S \mid (b,x)\notin EG \text{ and } N_S(b)=\{a_i\}\}.$$

Notice, in either case subset $K(a_i)$ is a clique in G. Indeed, if c and d are two non-adjacent vertices in $K(a_i)$, then either x,a_i,c,y,d induce a banner (first case) or c,a_i,d form an augmenting P_3 (second case).

Clearly, set S admits an augmenting plant with root x if and only if there is a independent set $P=\{b_1,\dots,b_k\}$ with $b_i\in K(a_i)$. To find such a set (if any), we use the following arguments.

Consider a vertex $b_1\in K(a_1)$, and denote by $K_{b_1}(a_j)$ the set of non-neighbors of vertex b_1 in set $K(a_j)$ $(j>1)$. If $K_{b_1}(a_j)$ is empty for some $j>1$, then obviously the required set P with given vertex b_1 does not exist. Assume now that $K_{b_1}(a_j)\neq\emptyset$ for all $j>1$, and let $k>3$. We claim that for any given pair $i,j\in\{2,\dots,k\}$, either each pair of vertices $c\in K_{b_1}(a_i)$ and $d\in K_{b_1}(a_j)$ forms an edge in G or none of them does. To prove the claim, we assume, without loss of generality, that there is a vertex $c\in K_{b_1}(a_i)$ adjacent to a vertex $d\in K_{b_1}(a_j)$ but non-adjacent to a vertex $e\in K_{b_1}(a_j)$. Then G contains $S_{1,2,4}$ induced by vertices b_1,a_1,x,a_i,c,d,e,a_l, where l is different from $1,i,j$. We thus can determine whether G contains the sought for independent set P in the following way. If there is an edge between two sets $K_{b_1}(a_i)$, $K_{b_1}(a_j)$ $(i,j\geqslant 2, i\neq j)$, then the desired independent set cannot exist. If on the contrary there is no edge between any two such sets, then we can obtain such a P by taking an arbitrary vertex b_i in each $K_{b_1}(a_i)$ $(i=2,\dots,k)$.

The above arguments lead to an obvious polynomial time procedure to determine if S admits an augmenting plant. Trivial observations show that the procedure can be implemented in time $O(n^6)$. □

4.2. P_4-free augmenting graphs in banner-free graphs

Along the section we denote by G a banner-free graph and by S a maximal independent set in G. Our purpose is to find a P_4-free augmenting graph for S with maximum increment. Recall from Section 3 that every connected component of a P_4-free bipartite graph is complete bipartite. So, any two black vertices in the same connected component of a P_4-free augmenting graph have the same neighborhood in set S. Let us call two black vertices x and y with $N_S(x)=N_S(y)$ *similar*. Clearly the similarity is an equivalence relation. We denote the similarity classes by $Q_1,\dots,Q_k$ and assume that

$$|N_S(Q_i)|\geqslant 3 \quad (i=1,\dots,k), \tag{1}$$

$$\text{each vertex in } Q_i \text{ has a similar non-neighbor } (i=1,\dots,k). \tag{2}$$

To meet condition (1), we first find augmenting graphs of form $K_{1,2}$ or $K_{2,3}$. If S does not admit such augmenting graphs, we may delete similarity classes non-satisfying (1) without loss of generality. Under condition (1), any vertex that has no non-neighbor in its own class of similarity is of no interest to us. Hence condition (2).

Assuming (1) and (2), we prove the following lemma.

Lemma 5. *Let $x\in Q_i$ and $y\in Q_j$ $(i\neq j)$. If x is not adjacent to y then $N_S(x)\cap N_S(y)=\emptyset$, and if x is adjacent to y then $N_S(x)\subseteq N_S(y)$ or $N_S(y)\subseteq N_S(x)$.*

Proof. Assume first x is not adjacent to y. Since x and y are not similar, we may consider a vertex a in S adjacent to x but not to y. Then $|N_S(x)\cap N_S(y)|\leqslant 1$, otherwise G would contain an induced Banner (a,x,b,y,c), where b and c are two distinct vertices in $N_S(x)\cap N_S(y)$. Suppose now $N_S(x)\cap N_S(y)=\{b\}$. Due to (1), there is a vertex c in $N_S(x)$ different from a and b, and due to (2), there is a vertex z in Q_i non-adjacent to x. If y is not adjacent to z, then G contains induced Banner(y,b,x,a,z). If y is adjacent to z, then G contains induced Banner (y,z,a,x,c). In both cases we have a contradiction, and therefore $N_S(x)\cap N_S(y)=\emptyset$.

Suppose now that x is adjacent to y and assume, to the contrary, $a\in N_S(x)-N_S(y)$ and $b\in N_S(y)-N_S(x)$. Consider also vertices $c\in N_S(x)$ and $z\in Q_i$ with $(x,z)\notin EG$, like above. If y is adjacent to z, then G contains induced Banner

(b,y,z,a,x). If y is not adjacent to z, then, by the first part of the lemma, y is not adjacent to c and hence G contains induced Banner (y,x,a,z,c). This contradiction proves that either $N_S(x)\subseteq N_S(y)$ or $N_S(y)\subseteq N_S(x)$. $\square$

Let us define a graph Γ as follows:

$$V\Gamma=\{Q_1,\dots,Q_k\},$$

$$E\Gamma=\{(Q_i,Q_j)\mid N_S(Q_i)\cap N_S(Q_j)\neq\emptyset\}.$$

With each vertex Q_i of Γ we associate an integer, the weight of the vertex, equal to $\alpha(G[Q_i])-|N_S(Q_i)|$. The weight of a subset of vertices is the sum of weights of its elements.

Let $Q=\{Q_1,\dots,Q_p\}$ be an independent set in Γ. With each vertex $Q_j\in Q$ we associate a complete bipartite graph $H_j=(W_j,B_j,E_j)$, where $W_j=N_S(Q_j)$ and B_j is an independent set of maximum cardinality in $G[Q_j]$. By Lemma 5, subsets $W_1,\dots,W_p$ are pairwise disjoint. Hence the union of graphs $H_1,\dots,H_p$, denoted H_Q, is a P_4-free bipartite graph. The increment of H_Q, equal to $\sum_{j=1}^{p}|B_j|-|W_j|$, coincides with the weight of Q, equal to $\sum_{j=1}^{p}\alpha(G[Q_i])-|N_S(Q_i)|$. If the weight of Q is positive, then H_Q is an augmenting graph for S. Moreover, if Q is a maximum weight independent set in Γ, then the increment of H_Q is maximum over all P_4-free augmenting graphs S. Indeed, if H is a P_4-free augmenting graph for S with greater increment, then the union of similarity classes corresponding to components of H is an independent set in Γ, whose weight is obviously at least the increment of H. We thus have proved

Lemma 6. *If Q is a maximum weight independent set in graph Γ, then the increment of the corresponding graph H_Q is maximum over all possible P_4-free augmenting graphs for S.*

To find a maximum weight independent set in graph Γ, we characterize it as follows.

Lemma 7. *Graph Γ is (P_4,C_4)-free.*

Proof. Suppose Γ contains a P_4 or C_4 on vertices Q_1,Q_2,Q_3,Q_4 with edges (Q_1,Q_2), (Q_2,Q_3), (Q_3,Q_4) and non-edges (Q_1,Q_3), (Q_2,Q_4) ((Q_1,Q_4) can be an edge or not). By Lemma 5 we have, up to the symmetry, $N_S(Q_2)\subseteq N_S(Q_3)$. But now $N_S(Q_1)\cap N_S(Q_3)=\emptyset$ implies $N_S(Q_1)\cap N_S(Q_2)=\emptyset$ that contradicts the fact $(Q_1,Q_2)\in E\Gamma$. Hence neither P_4 nor C_4 can occur in Γ. $\square$

The graphs containing no P_4 and C_4 as induced subgraphs have been studied in the literature extensively under different names, like trivially perfect graphs [8], quasi-threshold graphs [16], etc. The problem of finding an independent set of maximum weight can be solved in that class in linear time using the "co-tree" structure of P_4-free graphs [4].

5. Computing maximum independent sets in (P_7, banner)-free graphs

We now summarize the obtained results to prove polynomial solvability of the independent set problem in the class of (P_7, banner)-free graph.

Theorem 2. *Given a (P_7, banner)-free graph G with n vertices, one can find a maximum independent set in G in time* $O(n^8)$.

Proof. To prove the theorem, we present a recursive procedure ALPHA that finds a maximum independent set in G.

ALPHA (G)
Input: A (P_7, banner)-free graph G.
Output: A maximum independent set S in G.

1. Find an arbitrary maximal independent set S in G.
2. Apply $K_{1,2}$-, $K_{2,3}$- or plant-augmentation for S as long as possible.
3. Partition vertices in $VG-S$ into similarity classes $Q_1,\dots,Q_k$ and delete the classes which do not satisfy conditions (1)–(2).
4. For every $j=1,\dots,k$, find a maximum independent set $B_j=ALPHA(G[Q_j])$.
5. Construct an auxiliary graph Γ and find a maximum weight independent set $Q=\{Q_1,\dots,Q_p\}$ in it.

6. If the weight of Q is positive, make augmentation of S by exchanging $N_S(Q_i)$ by B_i for each $i = 1, \ldots, p$.
7. Return S and STOP.

Correctness of algorithm ALPHA follows from Lemmas 4–7 and the simple observation that an independent set obtained by H-augmentation is maximum if and only if the increment of graph H is maximum over all augmenting graphs.

To estimate the time complexity, we notice that finding augmenting plants applies at most n times, and hence the complexity of Step 2 is $O(n^7)$. Obviously, any other step does not exceed this bound. Therefore, the recursion in Step 4 results in the total time $O(n^8)$. □

6. Concluding remarks and open questions

In this paper we study the augmenting graph approach to solve the maximum independent set problem in special classes of graphs. Specifically, we investigate the following two problems: structural characterization and finding augmenting graphs in special classes. As a particular consequence of the obtained results, we derive a polynomial time algorithm to solve the independent set problem in the class of (P_7, banner)-free graphs. The algorithm generalizes two previously known results dealing with (P_6, C_4)-free and (P_5, banner)-free graphs. However, the time complexity of the new algorithm is worse than that of previously known ones. We leave the problem of finding a better algorithm for the class of (P_7, banner)-free graphs for the future research.

Another open research problem is to apply the obtained results to other classes of graphs. Among those, let us distinguish the class of ($S_{1,2,3}$, banner)-free graphs which seems to be the most perspective case. At first, this class generalizes simultaneously several previously known results, including the class of claw-free graphs (see Fig. 3 for the inclusion relationship between classes under consideration). Second, Section 3 of the present paper provides a complete characterization of minimal augmenting graphs in that class. The only remaining problem is to derive an efficient procedure for finding augmenting paths. A natural way to develop such a procedure is to extend the approach of Minty. We state, without proof, that Minty's approach to find augmenting paths can be extended with no extra work from claw-free to $S_{1,2,2}$-free graphs. As a byproduct, we obtain in this way a polynomial time algorithm to solve the independent set problem in the class of ($S_{1,2,2}$, banner)-free graphs. However, for ($S_{1,2,3}$, banner)-free graphs the problem remains open.

The class of $S_{1,2,2}$-free graphs is also of special interest as an extension of several positive results (see Fig. 3). Moreover, a complete characterization of bipartite graphs in this class has been obtained recently in [11]. However, for the time being the complexity status of the independent set problem is unknown even for P_5-free graphs that form a proper subclass of $S_{1,2,2}$-free graphs.

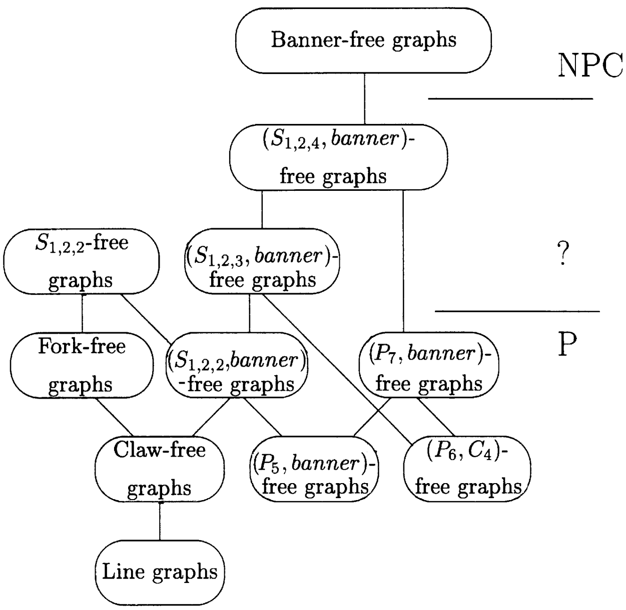

Fig. 3.

The class of ($S_{1,2,4}$, banner)-free graphs seems to be a more distant perspective for obtaining a polynomial algorithm, because we do not yet know the structure of augmenting graphs in this class. Obviously it includes paths, plants and complete bipartite augmenting graphs. However, this list does not exhaust all types of augmenting graphs in the class. Consider, for example, a chordless cycle on 8 vertices with one additional vertex x. Assume x has exactly two neighbors on the cycle being on distance 4 from each other. Clearly the described graph is not of any type studied in the paper. On the other hand, one can easily verify that it is a minimal augmenting ($S_{1,2,4}$, banner)-free graph. So, the problem of characterizing augmenting graphs in the class of ($S_{1,2,4}$, banner)-free graphs is open.

As a conclusion, let us point out that in the class of banner-free graphs the independent set problem is NP-hard due to result in [1].

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