J. Math. Anal. Appl. 393 (2012) 377–388

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On uniqueness class for a heat equation on graphs☆

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ARTICLE INFO

Article history:
Received 5 February 2012
Available online 21 April 2012
Submitted by Stephen Fulling

Keywords:
Uniqueness class
Graphs
Heat equation
Cauchy problem
Stochastic completeness

ABSTRACT

We study the uniqueness class problem for a class of heat equations defined on weighted graphs. As an application, we recover the volume growth criterion for stochastic completeness of Grigor'yan, Huang and Masamune in the case of weighted graphs.

0. Introduction

In the setting of heat equation on Euclidean spaces, it is a classical topic to determine the uniqueness of solutions to the Cauchy problem with zero initial condition

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = 0, \\ u|_{t=0+} = 0 \end{cases} \tag{0.1}$$

on $\mathbb{R}^N \times (0, T]$ for some $T \in (0, +\infty)$. Here Δ is the classical Laplace operator. We briefly review several important results in this direction. Tichonov [1] proved that if

$$|u(x,t)| \le C \exp\left(C|x|^2\right)$$

on $\mathbb{R}^N \times (0, T]$ for some constant $C > 0$, then $u \equiv 0$. (Later on we will preserve C for positive constants.) However if we look for the solutions in a "slightly" larger class where

$$|u(x,t)| \le C \exp\left(C|x|^{2+\epsilon}\right)$$

for some $\epsilon > 0$, Tichonov constructed nonzero solutions to the Cauchy problem with zero initial condition. Then Täcklind [2] gave a more general criterion by asserting that if the solution u to (0.1) satisfies

$$|u(x,t)| \le C \exp\left(|x|h(|x|)\right) \tag{0.2}$$

on $\mathbb{R}^N \times (0, T]$ with a nondecreasing, positive function h on $(0, +\infty)$ such that

$$\int_1^\infty \frac{dr}{h(r)} = \infty, \tag{0.3}$$

☆ Research supported by Project CRC701.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.04.026

then $u \equiv 0$. For example, the following function satisfies (0.3):

$$h(r) = \text{const}\, r \log r\, \log\log r. \tag{0.4}$$

Täcklind also showed that the condition (0.3) is sharp in the sense that for any nondecreasing, positive function h not satisfying (0.3), there exists a non-zero solution of (0.1) that satisfies (0.2). Oleĭnik and Radkevič [3] and Gushchin [4] proposed a new uniqueness condition in the form of an integral, that is, if the solution u to (0.1) satisfies

$$\int_0^T \int_{B(x_0,r)} u^2(x,t)dxdt \le C \exp(rh(r)), \tag{0.5}$$

where x_0 is some fix point in $\mathbb{R}^N$ and h is a nondecreasing, positive function that satisfies (0.3), then $u \equiv 0$.

On a geodesically complete Riemannian manifold M, the Cauchy problem (0.1) with Δ to be the Laplace–Beltrami operator is even more interesting as it is related to the geometry in the large of M as well as to the stochastic completeness of Brownian motion on M. Brownian motion on M is stochastically incomplete if and only if for some/any $T \in (0, +\infty)$, the Cauchy problem (0.1) has a nonzero bounded solution on $M \times (0, T]$ (for details, see Grigor'yan [5]). Many authors contributed to the problem of deciding when a given geodesically complete manifold is stochastically complete: Azencott [6], Gaffney [7], Yau [8], Karp and Li [9], Grigor'yan [10], Hsu [11], and Davies [12] just to name a few. In particular, Grigor'yan [10,5] proved the uniqueness of the Cauchy problem in the class of solutions satisfying (0.5) on an arbitrary geodesically complete Riemannian manifold, where now dx is the Riemannian volume element, $B(x_0, r)$ is a geodesic ball, and $h(r)$ satisfies as before (0.3).

In the setting of weighted graphs, one can introduce a discrete analogue of the Laplace operator. It is then natural to study the uniqueness of the corresponding Cauchy problem. In this paper, we will explore an integrated type uniqueness class for the Cauchy problem with zero initial condition, which is similar to those of Oleĭnik and Radkevič, Gushchin and Grigor'yan. As we will see below, the uniqueness class for the Cauchy problem on graphs is drastically different from that on manifolds. Namely, the optimal function $h(r)$ in (0.5) is

$$h(r) = C \log r,$$

where the borderline between uniqueness and non-uniqueness is determined by the value of the constant C, which is in striking contrast to the conditions (0.3), (0.4) in the case of manifolds. The optimal value C is between $1/2$ and $2\sqrt{2}$, as can be shown by our main result Theorem 0.8, and the example of physical Laplacian on $\mathbb{Z}$. It would be interesting to determine the optimal value of C.

We generally follow the framework of weighted graphs of Keller and Lenz [13] despite that we need some more assumptions to avoid some subtle topological problems. In this paper, we always make the following assumption:

Assumption 0.1. All graphs are assumed to be locally finite, connected, infinite, undirected and without loops and multi-edges.

Let (V, E) be such a graph. Here V is the set of vertices and E is the set of edges that can be viewed as a symmetric subset of $V \times V$. For $(x, y) \in E$, we write $x \sim y$ for short. We call a sequence of points $x_0, \ldots, x_n$ in V a chain connecting x and y if $x_0 = x, x_n = y$ and $x_i \sim x_{i+1}$ for all $i = 0, 1, \ldots, n-1$. The number n is called the length of this chain. A natural graph metric ρ can be defined on V as the minimal length of chains connecting two distinct points. Let $\mu(x)$ be a positive function on V. Then μ can be viewed as a Radon measure on (V, ρ) with full support. Let $w(x, y)$ be a function on $V \times V$ that satisfies:

(1) $w(x, y) \ge 0$;
(2) $w(x, y) = w(y, x)$;
(3) $w(x, y) > 0 \Leftrightarrow (x, y) \in E$.

The triple (V, w, μ) will be called a weighted graph. We call the quantity

$$\deg(x) := \frac{1}{\mu(x)} \sum_{y \in X} w(x, y) \tag{0.6}$$

the weighted degree of $x \in V$ to be distinct from the usual degree of locally finite graphs. Note that we do not require the degree function to be bounded.

Following Keller and Lenz [13], we introduce the following formal Laplacian on (V, w, μ):

$$\Delta v(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(v(x) - v(y)). \tag{0.7}$$

For any real valued function v on V, the expression Δv makes sense since the sum in (0.7) is a finite sum for every $x \in V$. A natural example is the case that $\mu(x) \equiv 1$ and $w(x, y) \in \{0, 1\}$. The formal Laplacian then has a simple form:

$$\Delta v(x) = \sum_{y \in V, y \sim x} (v(x) - v(y)).$$

It is called the physical Laplacian for a graph by Weber [14].

We are interested in the uniqueness class of the Cauchy problem with zero initial condition:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) + \Delta u(x,t) = 0, \\ u|_{t=0+} = 0 \end{cases} \tag{0.8}$$

on $V \times (0, T]$ for some $T \in (0, +\infty)$. Note that because of convention, the sign before Δ here is different from that in (0.1). For each $x \in V$, $u(x, t)$ is a real valued function that is differentiable in t on $(0, T]$ and has zero right limit at 0. Like in the continuous setting, the uniqueness class problem is related to stochastic completeness of the weighted graph. Roughly speaking, stochastic completeness is equivalent to that bounded functions form a uniqueness class. In this context, it is first independently studied by Weber [14] and Wojciechowski [15–17] for the physical Laplacian case. Keller and Lenz [13] set up the general framework of weighted graphs. See also [18–21] for further progresses.

Remark 0.2. For each $x \in V$, the function $u(x, t)$ can be naturally extended to be a continuous function on $[0, T]$. As observed in [13],

$$\frac{\partial}{\partial t}u(x,t) = -\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)(u(x,t) - u(y,t))$$

can be extended continuously on $[0, T]$. The existence of the right derivative of $u(x, t)$ at 0 follows from the mean value theorem.

To investigate the uniqueness class problem for weighted graphs, the graph metric is not adequate. Roughly speaking, we need a family of distances which can "see" the weights.

Definition 0.3. We call a distance d on the weighted graph (V, w, μ) adapted if

$$\frac{1}{\mu(x)}\sum_{y\in V} w(x,y)d^2(x,y) \le 1 \tag{0.9}$$

for every $x \in V$ and $d(x, y) \le 1$ whenever $x \sim y$.

A distance d on a weighted graph satisfying (0.9) is analogous to the geodesic distance on a Riemannian manifold which satisfies $|\nabla d(x, \cdot)| \le 1$. There in fact exist such type distances for our weighted graphs.

Proposition 0.4. *Define a function $\sigma(x, y)$ for all $x \sim y$ by*

$$\sigma(x,y) = \min\left\{\frac{1}{\sqrt{\deg(x)}}, \frac{1}{\sqrt{\deg(y)}}, 1\right\}. \tag{0.10}$$

It naturally induces a distance d on X as follows: $d(x, x) = 0$, and for all distinct points x, y,

$$d(x,y) := \inf\left\{\sum_{i=0}^{n-1}\sigma(x_i, x_{i+1}) : x_0, x_1, \ldots, x_n \text{ is a chain connecting } x \text{ and } y\right\}. \tag{0.11}$$

Proof. It is easy to see that d is nonnegative, symmetric and satisfies the triangle inequality. By locally finiteness, for each $x \neq y \in V$,

$$d(x,y) \ge \min_{z\sim x}\sigma(x,z) > 0.$$

Hence d is a distance on V. □

Remark 0.5. The notion of adapted distance can also be viewed as an analogue of the integrability condition for a Lévy measure. See the work of Masamune and Uemura [22] in the setting of stochastic completeness for jump processes. We are inspired by a general notion of intrinsic metric introduced by Frank et al. [23] where they focus on applications to spectral properties. Folz [24] also came up with similar ideas independently in the context of heat kernel estimates. In [21], Grigor'yan et al. applied adapted distances to the stochastic completeness problem of weighted graphs using a general volume growth criterion for jump processes.

For a weighted graph (V, w, μ) and an adapted distance d on it, we denote the closed balls by $B_d(x, r) = \{y \in V : d(x, y) \le r\}$. We will frequently make use of the following assumption:

Assumption 0.6. There exists an adapted distance d on (V, w, μ) such that the balls $B_d(x, r)$ are finite sets for any $x \in V$, $r > 0$.

Remark 0.7. This assumption is not so restrictive as it appears. Any locally finite, connected weighted graph of bounded weighted degree satisfies it with respect to the graph metric (up to a multiple of some positive constant to make it adapted). For weighted graphs with unbounded weighted degree, under the mild assumption that there is a positive lower bound of the measure μ, upper bounds on volume growth will naturally imply this assumption. So it fits quite well for the purpose of obtaining criterion of stochastic completeness in terms of volume growth with respect to the adapted metric.

Note that since in Assumption 0.1 we assumed that the graphs are infinite, a weighted graph that satisfies Assumption 0.6 is necessarily of infinite radius in the adapted distance there.

We are now ready to formulate our main theorem:

Theorem 0.8. *Let (V, w, μ) be a weighted graph such that its underlying graph satisfies Assumption 0.1. We also assume that d is an adapted distance on V such that Assumption 0.6 holds. Let $u(x, t)$ be a solution to the Cauchy problem (0.8) with zero initial condition. If there are an increasing sequence of positive numbers $\{R_n\}_{n\in\mathbb{N}}$ with*

$$\lim_{n\to\infty} R_n = +\infty,$$

and two constants $C > 0$, $\frac{1}{2} > c > 0$ such that for some $x_0 \in V$ and for all n large enough,

$$\int_0^T \sum_{x\in B_d(x_0,R_n)} u^2(x, t)\mu(x)dt \le C \exp\left(cR_n \log R_n\right), \tag{0.12}$$

then $u(x, t) \equiv 0$ on $V \times [0, T]$.

In the simple case of physical Laplacian on $\mathbb{Z}$, it is possible to explicitly write down some solutions to the Cauchy problem with zero initial condition. Then we can make estimates of the solutions to see that they just lie in a slightly larger class of functions. As we will see, Theorem 0.8 is close to being sharp up to the constant in the exponent.

The following result is taken from the author's thesis [25, Theorem 1.5.1], which explicitly states the relation between uniqueness class and stochastic completeness for weighted graphs. We omit the proof here since it is essentially the same as the case of manifolds (see Grigor'yan [5]).

Theorem 0.9. *A weighted graph (V, w, μ) is stochastically incomplete if and only if for any/some $T > 0$, there exists a bounded, nonzero solution to the Cauchy problem with zero initial condition (0.8).*

Remark 0.10. It is also implicitly contained in Keller et al. [19]. The crucial step is a minimum principle for the heat equation (Lemma 3.4 in [19], see also Theorem 1.3.2 in [25]).

Due to Theorem 0.9, we can apply easily Theorem 0.8 to recover the following volume growth criterion for stochastic completeness in [21] in the setting of weighted graphs.

Theorem 0.11. *Let (V, w, μ) be a locally finite, connected weighted graph with an adapted distance d. Furthermore assume that there is a constant $C_0 > 0$ such that $\mu(x) \ge C_0$ for all $x \in V$. If for some point $x_0 \in V$, the volume of balls $\mu(B_d(x_0, r))$ satisfies*

$$\mu(B_d(x_0, r)) \le C \exp\left(cr \ln r\right), \tag{0.13}$$

for some constants $C > 0$, and $0 < c < \frac{1}{2}$ and for all $r > 0$ large enough, then (V, w, μ) is stochastically complete.

Proof. First we note that (V, w, μ) with d satisfies Assumption 0.6 since

$$\#B_d(x_0, r) \le \frac{\mu(B_d(x_0, r))}{C_0} < \infty,$$

where $\#A$ denotes the cardinality of a subset A of V.

Fix some $T > 0$. Let $u(x, t)$ be a solution to the Cauchy problem with zero initial condition (0.8) with

$$\sup_{x\in V} u^2(x, t) = M < \infty.$$

So we have that

$$\int_0^T \sum_{x\in B_d(x_0,R)} u^2(x, t)\mu(x)dt \le MT\mu(B_d(x_0, R)) \le MTC \exp\left(cr \ln r\right).$$

By Theorem 0.8, $u \equiv 0$ on $V \times [0, T]$. Hence by Theorem 0.9, (V, w, μ) is stochastically complete. □

Remark 0.12. In [21], Theorem 0.11 was proved for general jump processes on locally compact metric spaces. When preparing this paper, we are informed that Masamune et al. [26] improved the volume growth criterion for jump processes in [21] by removing the restriction $c < 1/2$ in (0.13). Comparing their new result with our Theorem 0.8, it is interesting to see a big difference between the setting of weighted graphs with manifolds. Namely, the sharp uniqueness class criterion no longer leads to the sharp volume growth criterion for stochastic completeness in the generality of weighted graphs. It should be interesting to study further the relation between these two for more specific classes of weighted graphs.

We organize the paper as follows. In the first section we establish a key estimate, Lemma 1.1. We prove the main theorem in Section 2. Section 3 contains a simple example to show the sharpness of our uniqueness class.

1. Discrete integrated maximum principle

The following estimate can be viewed as a discrete version of Grigor'yan's "integrated maximum principle" [27].

Lemma 1.1. *Let (V, w, μ) be a weighted graph that satisfies Assumption 0.1. Let $u(x,t)$ be a solution to the Cauchy problem (0.8) with zero initial condition. Take two auxiliary functions $\eta(x)$ on V and $\xi(x,t)$ on $V \times [0,T]$ such that*

(1) *the function $\eta(x) \geq 0$ is finitely supported and $\xi(x,t)$ is continuously differentiable in t on $[0,T]$ for each $x \in V$;*
(2) *the inequality $(\eta^2(x) - \eta^2(y))(e^{\xi(x,t)} - e^{\xi(y,t)}) \geq 0$ holds for all $x \sim y$ and $t \in [0,T]$;*
(3) *the inequality $\mu(x)\frac{\partial}{\partial t}\xi(x,t) + \frac{1}{2}\sum_{y\in V} w(x,y)(1 - e^{\xi(y,t)-\xi(x,t)})^2 \leq 0$ holds for any $x \in V$ and $t \in [0,T]$.*

Then for any $\tau \in (0,T]$, we have the following estimate:

$$\sum_{x\in V} u^2(x,\tau)\eta^2(x)e^{\xi(x,\tau)}\mu(x) \leq 2\int_0^\tau \sum_{x\in V}\sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)}dt. \tag{1.14}$$

Remark 1.2. Folz [24] develops a different version of the discrete "integrated maximum principle" independently of us. Note that in [28], Coulhon et al. developed a discrete integrated maximum principle to study discrete time Markov chains.

Before giving a proof of Lemma 1.1, we present an elementary fact that we will use frequently.

Lemma 1.3. *Let $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ be two sequences of real numbers such that*

$$\sum_{i=1}^\infty a_i^2 < \infty, \qquad \sum_{i=1}^\infty b_i^2 < \infty.$$

The inequality

$$\sum_{i=1}^\infty a_i b_i \leq \frac{\delta}{2}\sum_{i=1}^\infty a_i^2 + \frac{1}{2\delta}\sum_{i=1}^\infty b_i^2 \tag{1.15}$$

holds for all $\delta > 0$.

Proof. By the Cauchy–Schwarz inequality we have that

$$\sum_{i=1}^\infty a_i b_i \leq \left(\sum_{i=1}^\infty a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^\infty b_i^2\right)^{\frac{1}{2}}.$$

Then the desired inequality follows by the AM–GM inequality. □

Proof of Lemma 1.1. We multiply the heat equation in (0.8) by $u(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)$ and sum over $x \in V$:

$$\sum_{x\in V}\frac{\partial}{\partial t}u(x,t)\cdot u(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x) + \sum_{x\in V}\sum_{y\in V} w(x,y)(u(x,t)-u(y,t))\cdot u(x,t)\eta^2(x)e^{\xi(x,t)} = 0. \tag{1.16}$$

Note that since $\eta(x)$ is finitely supported, the sums in (1.16) are of finite type. By symmetry of $w(x,y)$, we have

$$\sum_{x\in V}\frac{\partial}{\partial t}u^2(x,t)\cdot\eta^2(x)e^{\xi(x,t)}\mu(x) + \sum_{x\in V}\sum_{y\in V} w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)} - u(y,t)\eta^2(y)e^{\xi(y,t)}) = 0. \tag{1.17}$$

Using the fact that

$$\frac{\partial}{\partial t}u^2(x,t)\cdot e^{\xi(x,t)} = \frac{\partial}{\partial t}(u^2(x,t)e^{\xi(x,t)}) - u^2(x,t)\cdot e^{\xi(x,t)}\frac{\partial}{\partial t}\xi(x,t),$$

and the finiteness of the sums, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)\right) &= \sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)\frac{\partial}{\partial t}\xi(x,t)\\ &\quad - \sum_x\sum_y w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)} - u(y,t)\eta^2(y)e^{\xi(y,t)}).\end{aligned} \tag{1.18}$$

We split the sum in the last line:

$$-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)} - u(y,t)\eta^2(y)e^{\xi(y,t)}) \tag{1.19}$$

$$= -\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2\eta^2(x)e^{\xi(x,t)} \tag{1.20}$$

$$- \sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)} \tag{1.21}$$

$$- \sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)\eta^2(y)(e^{\xi(x,t)} - e^{\xi(y,t)}), \tag{1.22}$$

which is a discrete analogue to the Leibniz rule. Then we apply Lemma 1.3 to (1.21) and (1.22) to cancel the term in (1.20).

First, for any $\delta_1 > 0$,

$$\begin{aligned}&-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)}\\ &\leq \frac{\delta_1}{2}\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta(x)+\eta(y))^2e^{\xi(x,t)}\\ &\quad + \frac{1}{2\delta_1}\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(y,t)e^{\xi(x,t)}.\end{aligned}$$

Applying the elementary fact

$$(\eta(x)+\eta(y))^2 \leq 2(\eta^2(x)+\eta^2(y)),$$

we have that by the symmetry of $w(x,y)$,

$$\begin{aligned}&-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)}\\ &\leq \delta_1\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta^2(x)+\eta^2(y))e^{\xi(x,t)}\\ &\quad + \frac{1}{2\delta_1}\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(y,t)e^{\xi(x,t)}\\ &= \frac{\delta_1}{2}\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta^2(x)+\eta^2(y))(e^{\xi(x,t)}+e^{\xi(y,t)})\\ &\quad + \frac{1}{2\delta_1}\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(y,t)e^{\xi(x,t)}.\end{aligned}$$

Using the condition (2) of Lemma 1.1, it follows that

$$\begin{aligned}&-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)(\eta^2(x)-\eta^2(y))e^{\xi(x,t)}\\ &\leq \delta_1\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2(\eta^2(x)e^{\xi(x,t)}+\eta^2(y)e^{\xi(y,t)})\\ &\quad + \frac{1}{2\delta_1}\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(y,t)e^{\xi(x,t)}\\ &= 2\delta_1\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2\eta^2(x)e^{\xi(x,t)}\\ &\quad + \frac{1}{2\delta_1}\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(x,t)e^{\xi(y,t)}.\end{aligned} \tag{1.23}$$

Similarly, for any $\delta_2 > 0$,

$$-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(y,t)\eta^2(y)(e^{\xi(x,t)}-e^{\xi(y,t)})$$
$$=-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))u(x,t)\eta^2(x)(e^{\xi(x,t)}-e^{\xi(y,t)})$$
$$\le \frac{\delta_2}{2}\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))^2\eta^2(x)e^{\xi(x,t)}$$
$$+\frac{1}{2\delta_2}\sum_x\sum_y w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2u^2(x,t)\eta^2(x)e^{\xi(x,t)}.$$

Choose $\delta_1 = 1/4$ and $\delta_2 = 1$ and apply the above estimates of (1.21) and (1.22) to (1.19). It follows that

$$-\sum_x\sum_y w(x,y)(u(x,t)-u(y,t))(u(x,t)\eta^2(x)e^{\xi(x,t)}-u(y,t)\eta^2(y)e^{\xi(y,t)})$$
$$\le 2\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(x,t)e^{\xi(y,t)}+\frac{1}{2}\sum_x\sum_y w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2u^2(x,t)\eta^2(x)e^{\xi(x,t)}.$$

And hence by (1.18),

$$\begin{aligned}\frac{\partial}{\partial t}\left(\sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)\right) &\le \sum_x u^2(x,t)\eta^2(x)e^{\xi(x,t)}\mu(x)\frac{\partial}{\partial t}\xi(x,t)\\ &\quad +2\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(x,t)e^{\xi(y,t)}\\ &\quad +\frac{1}{2}\sum_x\sum_y w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2u^2(x,t)\eta^2(x)e^{\xi(x,t)}\\ &\le 2\sum_x\sum_y w(x,y)(\eta(x)-\eta(y))^2u^2(x,t)e^{\xi(y,t)},\end{aligned}$$

where in the last inequality we used the condition (3) on the auxiliary function. Integrate the above inequality with respect to t on $[0,\tau]$, we get the desired inequality as $u(x,0)\equiv 0$. □

Remark 1.4. In practical applications, $\eta(x)$ is often chosen to be some kind of "cut off" function. The idea is that this estimate together with the "cut off" property of $\eta(x)$, allow us to compare the "energy"

$$\sum u^2(x,t)\mu(x)$$

at different places.

2. Uniqueness class

Now we can prove the main Theorem 0.8 by specifying the auxiliary functions in Lemma 1.1. In the following, (V,w,μ) will be a weighted graph satisfying Assumption 0.1 and d is an adapted distance on V to make Assumption 0.6 hold.

Lemma 2.1. *Fix a point $x_0 \in V$. Define*

$$\xi(x,t) = -\frac{1}{2}\alpha^2e^{2\alpha}t-\alpha\,(d(x,x_0)-\delta R)_+$$

where $\alpha > 0$, $R > 0$ and $0 < \delta < \frac{1}{2}$ are to be chosen later. Then $\xi(x,t)$ satisfies the condition (3) in Lemma 1.1, that is, the following inequality holds for any $x \in V$ and $t \in [0,T]$:

$$\mu(x)\frac{\partial}{\partial t}\xi(x,t)+\frac{1}{2}\sum_{y\in V}w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2 \le 0. \tag{2.24}$$

Proof. Note the following elementary inequality for $a \in \mathbb{R}$:

$$(e^a-1)^2 \le e^{2|a|}a^2.$$

Hence

$$\begin{aligned}\frac{1}{2}\sum_{y\in V} w(x,y)(1-e^{\xi(y,t)-\xi(x,t)})^2 &\le \frac{1}{2}\sum_{y\in V} w(x,y)(\xi(y,t)-\xi(x,t))^2 e^{2|\xi(y,t)-\xi(x,t)|}\\ &\le \frac{1}{2}\sum_{y\in V} w(x,y)\alpha^2 d^2(x,y)e^{2\alpha d(x,y)}\\ &\le \frac{1}{2}\alpha^2\mu(x)e^{2\alpha},\end{aligned}$$

where in the last inequality we used the facts that

$$w(x,y)>0 \Leftrightarrow x\sim y$$

and that $d(x,y)\le 1$ when $x\sim y$. Then the inequality (2.24) follows easily. □

Remark 2.2. Note that $e^{\xi(x,t)}$ is decaying quickly in space. This will help cancel the growth of

$$\int_0^T \sum_{x\in B_d(x_0,R_n)} u^2(x,t)\mu(x)dt.$$

Lemma 2.3. *Fix a point $x_0\in V$. Define*

$$\eta(x)=\min\left\{\frac{(R-1-d(x,x_0))_+}{\delta R},1\right\},$$

where $R>0$ and $0<\delta<\frac{1}{2}$ are the same as in Lemma 2.1. Then $\eta(x)$ satisfies the following inequality for any $x\in V$:

$$\sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 \le \frac{1}{\delta^2R^2}\mu(x)\chi_{\{(1-\delta)R-2\le d(x,x_0)\le R\}}, \tag{2.25}$$

where χ is the characteristic function of a set.

Proof. For $x\in V$ such that

$$d(x,x_0)<(1-\delta)R-2,$$

we see that if $y\sim x$,

$$d(y,x_0)\le d(x,x_0)+d(x,y)\le(1-\delta)R-1.$$

Hence $\eta(x)=\eta(y)=1$. Similarly, for x such that

$$d(x,x_0)>R,$$

if $y\sim x$,

$$d(y,x_0)\ge d(x,x_0)-d(x,y)\ge R-1.$$

And then $\eta(x)=\eta(y)=0$.

Finally, for x such that

$$(1-\delta)R-2\le d(x,x_0)\le R,$$

we have

$$\sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 \le \frac{1}{\delta^2R^2}\sum_{y\in V} w(x,y)d^2(x,y)\le\frac{1}{\delta^2R^2}\mu(x). \quad \square$$

Proof of Theorem 0.8. Recall the notations in Theorem 0.8, Lemmas 2.1 and 2.3. Let x_0 be the fixed point there and $\epsilon=1-2c>0$. Choose $0<\delta<\frac{1}{2}$ small enough such that

$$\left(1-\frac{\epsilon}{4}\right)(1-2\delta)\ge 1-\frac{\epsilon}{2}. \tag{2.26}$$

Define $\xi(x,t)$ as in Lemma 2.1 with $R>\frac{2}{1-2\delta}$ and $\alpha>0$ to be chosen later. Define $\eta(x)$ as in Lemma 2.3.

It is easy to check the conditions (1) and (2) in Lemma 1.1 and the condition (3) is proven in Lemma 2.1. So we can apply Lemma 1.1 to assert that for any $\tau \in (0, T]$,

$$\sum_{x\in V} u^2(x,\tau)\eta^2(x)e^{\xi(x,\tau)}\mu(x) \le 2\int_0^\tau \sum_{x\in V}\sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)}dt. \tag{2.27}$$

Note that since $R > \frac{2}{1-2\delta}$, we have $\eta(x) = 1$ when $d(x, x_0) \le \delta R$. For the left side of (2.27), we have

$$\sum_{x\in B_d(x_0,\delta R)} u^2(x,\tau)\mu(x)e^{-\frac{1}{2}\tau\alpha^2 e^{2\alpha}} \le \sum_{x\in V} u^2(x,\tau)\eta^2(x)e^{\xi(x,\tau)}\mu(x).$$

For the right side of (2.27), the following estimate holds

$$\begin{aligned}
&2\int_0^\tau \sum_{x\in V}\sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(y,t)}dt\\
&\quad\le 2e^\alpha \int_0^\tau \sum_{x\in V}\sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 u^2(x,t)e^{\xi(x,t)}dt\\
&\quad= 2e^\alpha \int_0^\tau \sum_{x\in V} u^2(x,t)e^{\xi(x,t)} \sum_{y\in V} w(x,y)(\eta(x)-\eta(y))^2 dt\\
&\quad\le \frac{2}{\delta^2R^2}e^\alpha \int_0^\tau \sum_{x\in V} \mu(x)\chi_{\{(1-\delta)R-2\le d(x,x_0)\le R\}} u^2(x,t)e^{-\alpha(d(x,x_0)-\delta R)_+}dt\\
&\quad\le \frac{2}{\delta^2R^2}e^{3\alpha-(1-2\delta)\alpha R} \int_0^\tau \sum_{x\in B_d(x_0,R)} \mu(x)u^2(x,t)dt.
\end{aligned}$$

Put them together, we have

$$\sum_{x\in B_d(x_0,\delta R)} u^2(x,\tau)\mu(x)e^{-\frac{1}{2}\tau\alpha^2 e^{2\alpha}} \le \frac{2}{\delta^2R^2}e^{3\alpha-(1-2\delta)\alpha R} \int_0^\tau \sum_{x\in B_d(x_0,R)} \mu(x)u^2(x,t)dt. \tag{2.28}$$

Let R_n be as in Theorem 0.8. As in (0.12),

$$\int_0^T \sum_{x\in B_d(x_0,R_n)} u^2(x,t)\mu(x)dt \le C\exp\left(\frac{1}{2}(1-\epsilon)R_n\log R_n\right).$$

For n big enough such that $R_n > \max\{\frac{2}{1-2\delta}, 1\}$, choose $\alpha = \frac{1}{2}(1-\frac{\epsilon}{4})\log R_n$ and let $R = R_n$ in (2.28). We have

$$\begin{aligned}
\sum_{x\in B_d(x_0,\delta R_n)} u^2(x,\tau)\mu(x) &\le \frac{2}{\delta^2R_n^2}e^{\frac{1}{2}\tau\alpha^2e^{2\alpha}+3\alpha-(1-2\delta)\alpha R_n} \int_0^\tau \sum_{x\in B_d(x_0,R_n)} \mu(x)u^2(x,t)dt\\
&\le \frac{2C}{\delta^2R_n^2}\exp\left(\frac{1}{2}\tau\alpha^2e^{2\alpha}+3\alpha-(1-2\delta)\alpha R_n+\frac{1}{2}(1-\epsilon)R_n\log R_n\right)\\
&\le \frac{2C}{\delta^2R_n^2}\exp\left(\frac{1}{8}\tau\left(1-\frac{\epsilon}{4}\right)^2(\log R_n)^2R_n^{1-\frac{\epsilon}{4}}+\frac{3}{2}\left(1-\frac{\epsilon}{4}\right)\log R_n\right.\\
&\qquad\left.+\frac{1}{2}(1-\epsilon)R_n\log R_n-\frac{1}{2}(1-2\delta)\left(1-\frac{\epsilon}{4}\right)R_n\log R_n\right).
\end{aligned}\tag{2.29}$$

Apply (2.26), we arrive at

$$\sum_{x\in B_d(x_0,\delta R_n)} u^2(x,\tau)\mu(x) \le \frac{2}{\delta^2R_n^2}\exp\left(\frac{1}{8}\tau\left(1-\frac{\epsilon}{4}\right)^2(\log R_n)^2R_n^{1-\frac{\epsilon}{4}}+\frac{3}{2}\left(1-\frac{\epsilon}{4}\right)\log R_n-\frac{\epsilon}{4}R_n\log R_n\right). \tag{2.30}$$

Let n approaches to $+\infty$. As R_n increases to $+\infty$, we can see that the right side of (2.30) tends to zero while the left side is nonnegative and nondecreasing. Hence

$$u(x,\tau) \equiv 0$$

for all $x \in V$. Note that τ is arbitrarily chosen in $(0, T]$, the theorem follows. $\square$

3. A sharpness example

The example here is a discrete analogue of the classical construction of Tichonov [1]. See also the textbook of John [29]. However, the discrete case turns out to have a different behavior.

We equip $\mathbb{Z}$ with a graph structure such that for $m, n \in \mathbb{Z}$,

$$m \sim n \Leftrightarrow |m-n| = 1.$$

Let the weights $\mu(n) \equiv 1$ and $w(m,n) \in \{0,1\}$. The heat equation takes a simple form in this case:

$$\frac{\partial}{\partial t} u(n,t) + 2u(n,t) - u(n-1,t) - u(n+1,t) = 0. \tag{3.31}$$

As before, we have a natural graph distance ρ on $\mathbb{Z}$ which is just given by $\rho(m,n) = |m-n|$. By choosing $\sigma(n, n+1) = \frac{\sqrt{2}}{2}$ in Definition 0.4, we have $d = \frac{\sqrt{2}}{2}\rho$ is an adapted distance on $(\mathbb{Z}, w, \mu)$. It is also direct to see that Assumption 0.6 holds.

Let $g(t)$ be a smooth function on $\mathbb{R}$ such that all orders of derivatives of it goes to zero at zero. For example, we can take

$$g(t) = \begin{cases} \exp(-t^{-\beta}), & t > 0, \\ 0, & t \le 0, \end{cases} \tag{3.32}$$

where $\beta > 1$ is a constant.

For $0 < T < +\infty$, we define a function $u(n,t)$ on $\mathbb{Z} \times [0,T]$ as follows:

$$u(n,t) = \begin{cases} g(t), & n = 0, \\ g(t) + \displaystyle\sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1), & n \ge 1, \\ u(-n-1,t), & n \le -1. \end{cases} \tag{3.33}$$

In the above definition, the function $(n+k)\cdots(n+1)n\cdots(n-k+1)$ plays the role of the power x^{2k} in the continuous setting. However, the main difference is that

$$(n+k)\cdots(n+1)n\cdots(n-k+1)$$

vanishes for all $k > |n|$. Hence the sums in (3.33) are all of finite type. It is an elementary calculation to check that in fact $u(n,t)$ solves the heat equation. By the property of $g(t)$, we see that $u(n,t)$ satisfies the zero initial condition.

We choose $g(t)$ as in (3.32). The following estimate is taken from [29, p. 172]: for all $0 < \theta < 1$ small enough,

$$|g^{(k)}(t)| < \frac{k!}{(\theta t)^k} \exp\left(-\frac{1}{2}t^{-\beta}\right). \tag{3.34}$$

Hence for $n \ge 1$, $t > 0$, we have

$$|u(n,t)| \le \exp(-t^{-\beta}) + \sum_{k=1}^{n} \frac{k!}{(2k)!(\theta t)^k}(n+k)\cdots(n+1)n\cdots(n-k+1)\exp\left(-\frac{1}{2}t^{-\beta}\right). \tag{3.35}$$

We are going to make an estimate of $|u(n,t)|$ independent of t. The following estimate is elementary.

Lemma 3.1. *Let $k > 0$, $\theta > 0$. Then for $t > 0$,*

$$\frac{1}{(\theta t)^k} \exp\left(-\frac{1}{2}t^{-\beta}\right) \le \left(\frac{2k}{e\beta\theta^\beta}\right)^{\frac{k}{\beta}}.$$

Proof. Note that

$$\frac{1}{t^k}\exp\left(-\frac{1}{2}t^{-\beta}\right) = \exp\left(-k\log t - \frac{1}{2}t^{-\beta}\right).$$

Consider

$$\left(-k\log t - \frac{1}{2}t^{-\beta}\right)' = -\frac{k}{t} + \frac{\beta}{2}t^{-\beta-1} \begin{cases} > 0, & 0 < t < \left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}, \\ = 0, & t = \left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}, \\ < 0, & t > \left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}. \end{cases}$$

Then we see that $\frac{1}{t^k}\exp(-\frac{1}{2}t^{-\beta})$ attains its maximum at $\left(\frac{\beta}{2k}\right)^{\frac{1}{\beta}}$. The assertion follows. □

So now we have that

$$|u(n,t)| \le 1 + \sum_{k=1}^{n} \frac{k!}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1)\left(\frac{2k}{e\beta\theta^{\beta}}\right)^{\frac{k}{\beta}}. \tag{3.36}$$

Roughly speaking, the following lemma shows that the last term in the estimate (3.36) of $|u(n,t)|$ is the dominating one.

Lemma 3.2. *Fix some $n \ge 1$. Let $1 \le k \le n-1$. Denote*

$$\frac{k!}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1)\left(\frac{2k}{e\beta\theta^{\beta}}\right)^{\frac{k}{\beta}}$$

by a_k. Then for $0 < \theta < 1$ small enough, we have

$$a_k \le a_{k+1}.$$

Proof. By direct calculations,

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\frac{(k+1)!}{(2k+2)!}(n+k+1)\cdots(n+1)n\cdots(n-k)\left(\frac{2k+2}{e\beta\theta^{\beta}}\right)^{\frac{k+1}{\beta}}}{\frac{k!}{(2k)!}(n+k)\cdots(n+1)n\cdots(n-k+1)\left(\frac{2k}{e\beta\theta^{\beta}}\right)^{\frac{k}{\beta}}} \\
&= \frac{(k+1)(n+k+1)(n-k)}{(2k+2)(2k+1)} \times \left(1+\frac{1}{k}\right)^{\frac{k}{\beta}}\left(\frac{2}{e\beta\theta^{\beta}}\right)^{\frac{1}{\beta}}(k+1)^{\frac{1}{\beta}} \\
&\ge \frac{(k+1)(2k+2)}{(2k+2)(2k+1)}\left(\frac{2}{e\beta\theta^{\beta}}\right)^{\frac{1}{\beta}} \\
&\ge \frac{1}{2\theta}\left(\frac{2}{e\beta}\right)^{\frac{1}{\beta}}.
\end{aligned}$$

Hence for

$$0 < \theta < \frac{1}{2}\left(\frac{2}{e\beta}\right)^{\frac{1}{\beta}},$$

we have

$$\frac{a_{k+1}}{a_k} \ge 1. \quad \square$$

By Lemmas 3.1 and 3.2, we have for $\theta > 0$ small enough, for all $n \ge 1$, $t > 0$,

$$\begin{aligned}
|u(n,t)| &\le 1 + n a_n \\
&= 1 + n \cdot \frac{n!}{(2n)!}(2n)!\left(\frac{2n}{e\beta\theta^{\beta}}\right)^{\frac{n}{\beta}} \\
&= 1 + n \cdot n!\left(\frac{2n}{e\beta\theta^{\beta}}\right)^{\frac{n}{\beta}}.
\end{aligned}$$

For $\theta > 0$ small enough, in a similar way as in the proof of Lemma 3.2, we have that the sequence

$$1 + n \cdot n!\left(\frac{2n}{e\beta\theta^{\beta}}\right)^{\frac{n}{\beta}}$$

is nondecreasing.

Choose $x_0 = 0$, and a sequence $R_m = \frac{\sqrt{2}}{2}m$. For any $T \in (0, +\infty)$, we have

$$\begin{aligned}
\int_0^T \sum_{x \in B_d(x_0,R_m)} u^2(x,t)\mu(x)dt &\le \int_0^T \sum_{n=-m}^{m} u^2(n,t)dt \\
&\le 2\int_0^T \sum_{n=0}^{m} u^2(n,t)dt
\end{aligned}$$

$$\leq 2\int_0^T (1+m)\left(1+m\cdot m!\left(\frac{2m}{e\beta\theta^\beta}\right)^{\frac{m}{\beta}}\right)^2 dt$$

$$\leq 2T(1+m)\left(1+m\cdot m!\left(\frac{2m}{e\beta\theta^\beta}\right)^{\frac{m}{\beta}}\right)^2.$$

Apply Stirling's approximation to the last term in the above inequality, we have that for all big enough m,

$$\int_0^T \sum_{x\in B_d(x_0,R_m)} u^2(x,t)\mu(x)dt \leq CT\exp\left(2\sqrt{2}\left(1+\frac{1}{\beta}\right)(1+\epsilon)R_m\log R_m\right), \tag{3.37}$$

where $\epsilon > 0$ is a constant and $C > 1$ is a large enough constant depending on ϵ.

Remark 3.3. Note that the gap between the estimate for a nonzero solution (3.37) and the uniqueness class bound (0.12) only lies in the constant in the exponent. This is very different from the known uniqueness class in the smooth setting where the gap appears as different classes of functions in the exponent.

Acknowledgments

This paper is based on part of the author's Ph.D. Thesis [25] at Bielefeld University. The author is grateful to his supervisor, Prof. Grigor'yan, who introduced to him this topic and continuously supported him. The author also would like to thank R. Wojciechowski, M. Keller, D. Lenz and J. Masamune for their encouragement and inspiring discussions.

References

[1] A.N. Tichonov, Uniqueness theorems for the equation of heat conduction, Matem. Sbornik 42 (1935) 199–215 (in Russian).
[2] S. Täcklind, Sur les classes quasianalytiques des solutions des équations aux dérivées partielles du type parabolique, Nova Acta Soc. Sci. Upsal., IV. Ser. 10 (3) (1936) 1–57.
[3] O.A. Oleĭnik, E.V. Radkevič, The method of introducing a parameter for the investigation of evolution equations, Usp. Mat. Nauk 33 (5(203)) (1978) 7–76. 237.
[4] A.K. Gushchin, On the uniform stabilization of solutions of the second mixed problem for a parabolic equation, Mat. Sb. (NS) 119(161) (4) (1982) 451–508. 590.
[5] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. 36 (1999) 135–249.
[6] R. Azencott, Behavior of diffusion semi-groups at infinity, Bull. Soc. Math. (France) 102 (1974) 193–240.
[7] M.P. Gaffney, The conservation property of the heat equation on Riemannian manifolds, Comm. Pure Appl. Math. 12 (1959) 1–11.
[8] S.T. Yau, On the heat kernel of a complete Riemannian manifold, J. Math. Pures Appl. 57 (1978) 191–201.
[9] L. Karp, P. Li, The heat equation on complete Riemannian manifolds, unpublished manuscript 1983.
[10] A. Grigor'yan, On stochastically complete manifolds, DAN SSSR 290 (1986) 534–537 (in Russian); Engl. transl.: Soviet Math. Dokl. 34 (2) (1987) 310–313.
[11] Elton P. Hsu, Heat semigroup on a complete Riemannian manifold, Ann. Probab. 17 (1989) 1248–1254.
[12] E.B. Davies, Heat kernel bounds, conservation of probability and the Feller property, J. D' Analyse Math. 58 (1992) 99–119.
[13] M. Keller, D. Lenz, Dirichlet forms and stochastic completeness of graphs and subgraphs, J. Reine Angew. Math., http://dx.doi.org/10.1515/CRELLE.2011.122 (in press).
[14] A. Weber, Analysis of the laplacian and the heat flow on a locally finite graph, J. Math. Anal. App. 370 (2010) 146–158.
[15] R.K. Wojciechowski, Stochastic completeness of graphs, ProQuest LLC, Ann Arbor, MI, 2008, Ph.D. Thesis-City University of New York.
[16] R.K. Wojciechowski, Heat kernel and essential spectrum of infinite graphs, Indiana Univ. Math. J. 58 (3) (2009) 1419–1441.
[17] R.K. Wojciechowski, Stochastically incomplete manifolds and graphs, Prog. Probab. 64 (2011) 163–179.
[18] M. Keller, R.K. Wojciechowski, Volume growth, discreteness of the spectrum, and stochastic incompleteness of infinite graphs, 2010. Preprint.
[19] M. Keller, D. Lenz, R.K. Wojciechowski, Volume growth, spectrum and stochastic completeness of infinite graphs, 2011. Preprint arXiv:1105.0395v1 [math.SP].
[20] X. Huang, Stochastic incompleteness for graphs and weak Omori–Yau maximum principle, J. Math. Anal. Appl. 379 (2) (2011) 764–782.
[21] A. Grigor'yan, X. Huang, J. Masamune, On stochastic completeness for nonlocal dirichlet forms, Math. Z., http://dx.doi.org/10.1007/s00209-011-0911-x (in press).
[22] J. Masamune, T. Uemura, Conservation property of symmetric jump processes, Ann. Inst. Henri. Poincare Probab. Statist. 47 (3) (2011) 650–662.
[23] R. Frank, D. Lenz, D. Wingert, Intrinsic metrics for (non-local) symmetric Dirichlet forms and applications to spectral theory. Preprint arXiv:1012.5050v1 [math.FA].
[24] M. Folz, Gaussian upper bounds for heat kernels of continuous time simple random walks, Elec. J. Prob. 16 (2011) 1693–1722.
[25] X. Huang, On stochastic completeness of weighted graphs, Ph.D. Thesis, Universität Bielefeld.
[26] J. Masamune, T. Uemura, J. Wang, On the conservativeness and recurrence of symmetric jump-diffusions, 2011. Preprint.
[27] A. Grigor'yan, Integral maximum principle and its applications, Proc. Roy. Soc. Edinburgh Sect. A 124 (2) (1994) 353–362.
[28] Thierry Coulhon, Alexander Grigor'yan, Fabio Zucca, The discrete integral maximum principle and its applications, Tohoku Math. J. 57 (2) (2005) 559–587. no. 4.
[29] Fritz John, Partial Differential Equations, fourth ed., in: Applied Mathematical Sciences, vol. 1, Springer-Verlag, New York, 1991.