Quasi-Armendariz rings relative to a monoid

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Abstract

For a monoid $M$, we introduce $M$-quasi-Armendariz rings which are a generalization of quasi-Armendariz rings, and investigate their properties. The $M$-quasi-Armendariz condition is a Morita invariant property. The class of $M$-quasi-Armendariz rings is closed under some kinds of upper triangular matrix rings. Every semiprime ring is $M$-quasi-Armendariz for any unique product monoid and any strictly totally ordered monoid $M$. Moreover, we study the relationship between the quasi-Baer property of a ring $R$ and those of the monoid ring $R[M]$. Every quasi-Baer ring is $M$-quasi-Armendariz for any unique product monoid and any strictly totally ordered monoid $M$.

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0. Introduction

Throughout this paper $R$ and $M$ denote an associative ring with identity and a monoid respectively. Rege and Chhawchharia [13] introduced the notion of an Armendariz ring. A ring $R$ is called Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $i$, $j$. The name “Armendariz ring” was chosen because Armendariz [2, Lemma 1] had shown that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings have been studied in Rege and Chhawchharia [13], Armendariz [2], Anderson and Camillo [1], and Kim and Lee [9]. In [17], Zhongkui studied a generalization of Armendariz rings, which are called $M$-Armendariz rings, where $M$ is a monoid. A ring $R$ is called $M$-Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[M]$, with $g_i, h_j \in M$, satisfy $a \beta = 0$, then $a_i b_j = 0$ for each $i$, $j$. According to Hirano [7], a ring $R$ is called quasi-Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_i Rb_j = 0$ for each $i$, $j$. In [7], Hirano studied some properties of quasi-Armendariz rings.

Motivated by results in Armendariz [2], Anderson and Camillo [1], Kim and Lee [9], Hirano [7], Hong et al. [8], Lee and Wong [10] and Zhongkui [17], we investigate a generalization of a quasi-Armendariz ring which we call an $M$-quasi-Armendariz ring.

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1. Quasi-Armendariz rings relative to a monoid

Definition 1.1. Let $M$ be a monoid. We say that $R$ is $M$-quasi-Armendariz if whenever $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + bmh_m \in R[M]$ satisfy $\alpha R[M] \beta = 0$, then $a_i R b_j = 0$ for each $i, j$.

If $M = (\mathbb{N} \cup \{0\})$, then $R$ is $M$-quasi-Armendariz if and only if $R$ is quasi-Armendariz. If $R$ is reduced and $M$-Armendariz, then $R$ is $M$-quasi-Armendariz.

Let $R$ be a ring and define $T_n(R) = \{ \left( \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right) \mid a, a_{ij} \in R \}$ with $n$ a positive integer $\geq 2$. In [17, Proposition 1.7], Zhongkui proved that if $R$ is $M$-Armendariz and reduced then $T_3(R)$ is $M$-Armendariz. Also he showed that $T_3(R)$ is not $M$-Armendariz. The following results will give more examples of $M$-quasi-Armendariz rings which are not $M$-Armendariz.

Proposition 1.2. Let $R$ be an $M$-Armendariz and reduced ring. Then $T_n(R)$ is $M$-quasi-Armendariz for each $n \geq 2$.

Proof. It is easy to see that there exists an isomorphism of rings $T_n(R)[M] \rightarrow T_n(R[M])$ defined by

$$
\sum_{k=1}^{s} \left( \begin{array}{cccc} a^{k} & a^{k}_{12} & \cdots & a^{k}_{1n} \\ 0 & a^{k} & \cdots & a^{k}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{k} \end{array} \right) g_k \rightarrow \left( \begin{array}{cccc} \sum_{k=1}^{s} a^{g_k} & \sum_{k=1}^{s} a^{g_{12}} & \cdots & \sum_{k=1}^{s} a^{g_{1n}} \\ 0 & \sum_{k=1}^{s} a^{g_k} & \cdots & \sum_{k=1}^{s} a^{g_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=1}^{s} a^{g_k} \end{array} \right) = \left( \begin{array}{cccc} \alpha & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & \alpha & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha \end{array} \right).
$$

We proceed by induction on $n$. First, we claim that $T_2(R)$ is $M$-quasi-Armendariz. Suppose that $X = A_1 g_1 + \cdots + A_s g_s$ and $Y = B_1 h_1 + \cdots + B_m h_m \in T_2(R)[M]$ are such that $XT_2(R)[M]Y = 0$. We claim that $A_i T_2(R) B_j = 0$ for each $i, j$. Assume that

$$
A_i = \left( \begin{array}{cc} a^i & a^i_{12} \\ 0 & a^i \end{array} \right) \quad \text{and} \quad B_j = \left( \begin{array}{cc} b^j & b^j_{12} \\ 0 & b^j \end{array} \right).
$$

Then we have

$$
\left( \begin{array}{c} \sum_{i=1}^{s} a^i g_i \\ \vdots \\ \sum_{i=1}^{s} a^i g_i \end{array} \right) A \left( \begin{array}{c} \sum_{j=1}^{m} b^j h_j \\ \vdots \\ \sum_{j=1}^{m} b^j h_j \end{array} \right) = \left( \begin{array}{cc} \alpha & \alpha_{12} \\ 0 & \alpha \end{array} \right) \left( \begin{array}{cc} \beta & \beta_{12} \\ 0 & \beta \end{array} \right) = 0
$$

for each $A \in T_2(R)$. Thus $\alpha \beta = 0$ and $\alpha \beta_{12} + \alpha_{12} \beta = 0$ for each $\alpha \in R$. If we multiply the second equation on the left by $\alpha$, then $\alpha \beta_{12} = 0$ for each $\alpha \in R$, since $R[M]$ is reduced. It follows that $\alpha \beta_{12} = 0$ and so $\alpha_{12} \beta = 0$ for each $\alpha \in R$. Hence $\alpha R \beta_{12} = \alpha_{12} \beta = \alpha \beta = 0$. Therefore $a^i R b^j_{12} = a_{12} R b^j = a^i R b^j = 0$ for all $i = 1, \ldots, s, j = 1, \ldots, m$, since $R$ is Armendariz. Consequently, $A_i T_2(R) B_j = 0$ for all $i = 1, \ldots, s, j = 1, \ldots, m$.

Next, let $X = A_1 g_1 + \cdots + A_s g_s$ and $Y = B_1 h_1 + \cdots + B_m h_m \in T_n(R)[M]$ be such that $XT_n(R)[M]Y = 0$. We claim that $A_i T_n(R) B_j = 0$ for all $i = 1, \ldots, s, j = 1, \ldots, m$. Assume that

$$
A_i = \left( \begin{array}{cccc} a^{i}_{11} & a^{i}_{12} & \cdots & a^{i}_{1n} \\ 0 & a^{i}_{22} & \cdots & a^{i}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{i}_{nn} \end{array} \right) \quad \text{and} \quad B_j = \left( \begin{array}{cccc} b^{j}_{11} & b^{j}_{12} & \cdots & b^{j}_{1n} \\ 0 & b^{j}_{22} & \cdots & b^{j}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^{j}_{nn} \end{array} \right) \quad \text{with} \quad d^{i}_{tt} = a^{i}_{kk}
$$
and $b^j_{it} = b^j_{kk}$ for each $i, j, k, t$. Let

$$\alpha = \begin{pmatrix}
\sum_{i=1}^{s} a_{11}^j g_i & \sum_{i=1}^{s} a_{12}^j g_i & \cdots & \sum_{i=1}^{s} a_{1n}^j g_i \\
0 & \sum_{i=1}^{s} a_{22}^j g_i & \cdots & \sum_{i=1}^{s} a_{2n}^j g_i \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_{i=1}^{s} a_{nn}^j g_i
\end{pmatrix} = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
0 & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_{nn}
\end{pmatrix}$$

and

$$\beta = \begin{pmatrix}
\sum_{j=1}^{m} b_{11}^j h_j & \sum_{j=1}^{m} b_{12}^j h_j & \cdots & \sum_{j=1}^{m} b_{1n}^j h_j \\
0 & \sum_{j=1}^{m} b_{22}^j h_j & \cdots & \sum_{j=1}^{m} b_{2n}^j h_j \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sum_{j=1}^{m} b_{nn}^j h_j
\end{pmatrix} = \begin{pmatrix}
\beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\
0 & \beta_{22} & \cdots & \beta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta_{nn}
\end{pmatrix}.$$
where \( x = \alpha_1 \alpha_{12} \beta_{3n} + \cdots + \alpha_{(n-1)} \alpha_{12} \beta_{(n-1)n} \). Hence

\[
\alpha_1 \alpha_{12} \beta_{3n} + \cdots + \alpha_{(n-1)} \alpha_{12} \beta_{(n-1)n} = 0.
\] (2)

Therefore, from Eqs. (1) and (2), we obtain \( \alpha_1 \alpha_{12} \beta_{2n} = 0 \) and that \( \alpha_{12} \beta_{2n} = 0 \). Hence

\[
\begin{pmatrix}
\alpha_1 & \alpha_{13} & \alpha_{14} & \cdots & \alpha_{1(n-1)} & \alpha_{1n} \\
0 & \alpha_{22} & \alpha_{24} & \cdots & \alpha_{2(n-1)} & \alpha_{2n} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \alpha_{(n-3)(n-3)} & \alpha_{(n-3)(n-1)} & \alpha_{(n-3)n} \\
0 & 0 & \cdots & 0 & \alpha_{(n-2)(n-2)} & \alpha_{(n-2)n} \\
0 & 0 & \cdots & 0 & \alpha_{(n-1)(n-1)} & \alpha_{(n-1)n}
\end{pmatrix}
\begin{pmatrix}
\beta_{22} & \beta_{23} & \beta_{24} & \cdots & \beta_{2(n-1)} & \beta_{2n} \\
0 & \beta_{33} & \beta_{34} & \cdots & \beta_{3(n-1)} & \beta_{3n} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \beta_{(n-2)(n-2)} & \beta_{(n-2)(n-1)} & \beta_{(n-2)n} \\
0 & 0 & \cdots & 0 & \beta_{(n-1)(n-1)} & \beta_{(n-1)n} \\
0 & 0 & \cdots & 0 & 0 & \beta_{nn}
\end{pmatrix} = 0.
\]

Then by the induction hypothesis, \( \alpha_1 \beta_{3n} = \cdots = \alpha_{(n-1)} \beta_{(n-1)n} = 0 \). Therefore \( \alpha_{ij} R \beta_{jk} = 0 \) for each \( i, j, k \), since \( R [ M ] \) is reduced. Consequently, \( a_{pq}^i R b_{pq}^j = 0 \) for each \( i, j, p, q, r \), which implies that \( A_i CB_j = 0 \) for each \( i, j \) and \( C \in T_n(R) \). Therefore \( T_n(R) \) is \( M \)-quasi-Armendariz.

Let \( R \) be a ring and let \( n \) be a positive integer. Let \( M_n(R) \) denote the ring of \( n \times n \) matrices over \( R \) and \( e_{ij} \) denote the \((i, j)\)-matrix unit.

\[\square\]

Theorem 1.3. If \( R \) is an \( M \)-quasi-Armendariz ring, let \( S \) be a subring of \( M_n(R) \) such that \( e_{ii} S e_{jj} \subseteq S \) for all \( i, j \in \{1, \ldots, n\} \). Then \( S \) is an \( M \)-quasi-Armendariz ring.

Proof. It is easy to see that there exists an isomorphism of rings \( M_n(R)[M] \rightarrow M_n(R[M]) \) defined by

\[
\sum_{k=1}^{s} \begin{pmatrix}
\alpha_{11}^k & \alpha_{12}^k & \cdots & \alpha_{1n}^k \\
\alpha_{21}^k & \alpha_{22}^k & \cdots & \alpha_{2n}^k \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1}^k & \alpha_{n2}^k & \cdots & \alpha_{nn}^k
\end{pmatrix} g_k \rightarrow \begin{pmatrix}
\sum_{k=1}^{s} \alpha_{11}^k g_k & \sum_{k=1}^{s} \alpha_{12}^k g_k & \cdots & \sum_{k=1}^{s} \alpha_{1n}^k g_k \\
\sum_{k=1}^{s} \alpha_{21}^k g_k & \sum_{k=1}^{s} \alpha_{22}^k g_k & \cdots & \sum_{k=1}^{s} \alpha_{2n}^k g_k \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=1}^{s} \alpha_{n1}^k g_k & \sum_{k=1}^{s} \alpha_{n2}^k g_k & \cdots & \sum_{k=1}^{s} \alpha_{nn}^k g_k
\end{pmatrix} = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{pmatrix}.
\]

Let \( X = A_1 g_1 + \cdots + A_k g_k \) and \( Y = B_1 h_1 + \cdots + B_m h_m \in S[M] \) be such that \( XS[M]Y = 0 \). Assume that \( A_i = \begin{pmatrix}
\alpha_{11}^i & \alpha_{12}^i & \cdots & \alpha_{1n}^i \\
\alpha_{21}^i & \alpha_{22}^i & \cdots & \alpha_{2n}^i \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1}^i & \alpha_{n2}^i & \cdots & \alpha_{nn}^i
\end{pmatrix} \) and \( B_j = \begin{pmatrix}
b_{11}^j & b_{12}^j & \cdots & b_{1n}^j \\
b_{21}^j & b_{22}^j & \cdots & b_{2n}^j \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1}^j & b_{n2}^j & \cdots & b_{nn}^j
\end{pmatrix} \) for each \( i, j \).
Then for each $c \in R$ such that $e_{pp}ce_{qq} \in S$, we have $Xe_{pp}ce_{qq}gY = 0$ with $g \in M$. Thus

$$
\left(\begin{array}{cccc}
0 & \cdots & \sum_{i=1}^{k} a_{1p}cg_{i} & 0 & \cdots \\
0 & \cdots & \sum_{i=1}^{k} a_{2p}cg_{i} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \sum_{i=1}^{k} a_{(n-1)p}cg_{i} & 0 & \cdots \\
0 & \cdots & \sum_{i=1}^{k} a_{np}cg_{i} & 0 & \cdots \\
\end{array}\right)
\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{array}\right)
\left(\begin{array}{ccccc}
b_{q1}gh & \sum_{j=1}^{m} b_{q2}gh & \cdots & \sum_{j=1}^{m} b_{q(n-1)}gh & \sum_{j=1}^{m} b_{qn}gh \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\end{array}\right) = 0.
$$

Hence $(\sum_{i=1}^{k} a_{i}^tg_{i})cg(\sum_{j=1}^{m} b_{j}h_{j}) = 0$ for each $s, p, q, t \geq 1$ and $g \in M$. Since $\{c \in R|e_{pp}ce_{qq} \in S\}$ forms an ideal of $R$, $(\sum_{j=1}^{k} a_{i}^tg_{i})cR[M](\sum_{j=1}^{m} b_{j}h_{j}) = 0$ for each $s, p, q, t$. Since $R$ is $M$-quasi-Armendariz, $a_{i}^pcb_{j}^q = 0$ for all $i, j$. By hypothesis on $S$, every element of $S$ is a sum of such $e_{pp}ce_{qq}$; we conclude that $A_{i}SB_{j} = 0$ for all $i, j$. Therefore $S$ is $M$-quasi-Armendariz.

**Proposition 1.4.** If $R$ is $M$-quasi-Armendariz, then for any nonzero idempotent $e \in R$, $eRe$ is $M$-quasi-Armendariz.

**Proof.** Let $\alpha = a_{1}g_{1} + \cdots + a_{n}g_{n}$ and $\beta = b_{1}h_{1} + \cdots + b_{m}h_{m} \in eRe[M]$ be such that $\alpha eRe[M]\beta = 0$. Since $ae = \alpha$ and $eb = \beta$, we have $\alpha R[M]\beta = 0$. Hence $a_{i}Rb_{j} = 0$, since $R$ is $M$-quasi-Armendariz. Since $a_{i}e = a_{i}$ and $eb_{j} = b_{j}$ for each $i, j$, we conclude that $a_{i}eReb_{j} = 0$ for each $i, j$.

The following is a result of Theorem 1.3 and Proposition 1.4.

**Theorem 1.5.** The endomorphism ring of a finitely generated projective module over an $M$-quasi-Armendariz ring $R$ is $M$-quasi-Armendariz. In particular, the condition of being $M$-quasi-Armendariz is a Morita invariant property.

**Corollary 1.6.** If $R$ is an $M$-quasi-Armendariz ring, then the ring of all $n \times n$ upper triangular matrices over $R$ is also $M$-quasi-Armendariz.

Recall that a monoid $M$ is called a u.p.-monoid (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form $ab$ where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see [5,11,12]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid $M$ has no non-unity element of finite order.

**Proposition 1.7.** Let $M$ be a u.p.-monoid and $R$ be a semiprime ring. Then $R$ is $M$-quasi-Armendariz.

**Proof.** Let $\alpha = a_{1}g_{1} + \cdots + a_{n}g_{n}$ and $\beta = b_{1}h_{1} + \cdots + b_{m}h_{m} \in R[M]$ be such that $\alpha R[M]\beta = 0$. We show that $a_{i}Rb_{j} = 0$ for each $i, j$. We proceed by induction on $n + m$.

Step 1. $n + m = 2$. Then $\alpha = a_{1}g_{1}$ and $\beta = b_{1}h_{1}$. Hence $a_{1}Rb_{1} = 0$.

Step 2. Let $n + m \geq 3$. Since $M$ is a u.p.-monoid, there exists $i, j$ with $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $g_{i}h_{j}$ is uniquely presented by considering two subsets $A = \{g_{1}, \ldots, g_{n}\}$ and $B = \{h_{1}, \ldots, h_{m}\}$ of $M$. Hence $a_{i}Rb_{j}g_{k}h_{l} = 0$ so that $a_{i}Rb_{j} = 0$. Thus $0 = (a_{1}g_{1} + \cdots + a_{n}g_{n})R[M]a_{i}c(b_{1}h_{1} + \cdots + b_{m}h_{m}) = (a_{1}g_{1} + \cdots + a_{n}g_{n})R[M](a_{i}c b_{1}h_{1} + \cdots + a_{i}c b_{j-1}h_{j-1} + a_{i}c b_{j+1}h_{j+1} + \cdots + a_{i}c b_{m}h_{m})$. Continuing this procedure yields $a_{i}Rb_{j} = 0$ for each $1 \leq i \leq n, 1 \leq j \leq m$. Therefore $R$ is $M$-quasi-Armendariz.

Let $(M, \leq)$ be an ordered monoid. If for any $g_{1}, g_{2}, h \in M$, $g_{1} < g_{2}$ implies that $g_{1}h < g_{2}h$ and $hg_{1} < hg_{2}$, then $(M, \leq)$ is called a strictly ordered monoid.
Proposition 1.8. Let $M$ be a strictly totally ordered monoid and $R$ be a semiprime ring. Then $R$ is $M$-quasi-Armendariz.

Proof. Let $\alpha = a_1g_1 + \cdots + a_ng_n$ and $\beta = b_1h_1 + \cdots + b_mh_m \in R[M]$ be such that $\alpha R[M] \beta = 0$ and $g_1 < \cdots < g_n, h_1 < \cdots < h_m$. We use transfinite induction on the strictly totally ordered set $(M, \leq)$ to show that $a_iRb_j = 0$ for each $i, j$. Clearly, $g_1h_1 < g_ih_j$ if $i \neq 1$ or $j \neq 1$. Hence $a_1Rb_1 = 0$. Now suppose that $\omega \in M$ is such that for any $g_i$ and $h_j$ with $g_ih_j < \omega$, $a_iRb_j = 0$. We will show that $a_iRb_j = 0$ for any $g_i$ and $h_j$ with $g_ih_j = \omega$. Set $X = \{(g_i, h_j) | g_ih_j = \omega \}$. Then $X$ is a finite set. We write $X$ as $\{(g_i, h_j) | i = 1, \ldots, k \}$ such that $g_1 < \cdots < g_k$. Since $M$ is cancellative, $g_{i_1} = g_{i_2}$ and $g_{i_1}h_{j_1} = g_{i_2}h_{j_2} = \omega$ imply $h_{j_1} = h_{j_2}$. Since $\leq$ is a strict order, $g_1 < g_2$ and $g_ih_{j_i} = g_ih_{j_2} = \omega$ imply $h_{j_2} < h_{j_i}$. Thus we have $h_{j_2} < \cdots < h_{j_i}$. Now

$$\sum_{(g_i, h_j) \in X} a_i cb_j = \sum_{t=1}^k a_i cb_j = 0 \quad \text{for each } c \in R. \quad (3)$$

For any $t \geq 2$, $g_{i_2}h_{j_2} < g_{i_1}h_{j_1} = \omega$, and thus, by the induction hypothesis, we have $a_{i_2}Rb_{j_2} = 0$ for each $t = 2, \ldots, k$. Let $d$ be an arbitrary element of $R$. By multiplying Eq. (3) by $a_{i_2}d$, from the left, we have $a_{i_2}Ra_{i_2}Rb_{j_1} = 0$. Since $R$ is semiprime, we have $a_{i_2}Rb_{j_1} = 0$. Now Eq. (3) becomes

$$\sum_{t=2}^k a_{i_2} cb_j = 0 \quad \text{for each } c \in R. \quad (4)$$

By multiplying Eq. (4) by $a_{i_2}d$, from the left, we obtain $a_{i_2}Rb_{j_2} = 0$ in the same way as above. Continuing this process, we can prove $a_iRb_j = 0$ for any $i, j$ with $g_ih_j = \omega$. Therefore, by transfinite induction, $a_iRb_j = 0$ for any $i, j$. Thus $R$ is $M$-quasi-Armendariz.

Corollary 1.9. Let $R$ be a semiprime ring. Then $R$ is $M$-quasi-Armendariz, that is for any $\alpha = a_{-m}x^{-m} + \cdots + b_qx^q, \beta = b_{-n}x^{-n} + \cdots + b_qx^q \in R[x, x^{-1}]$, if $\alpha R[x, x^{-1}] \beta = 0$, then $a_iRb_j = 0$ for each $i, j$.

It was shown in [17, Proposition 1.4] that if $I$ is a reduced ideal of $R$ such that $R/I$ is $M$-Armendariz then $R$ is $M$-Armendariz, where $M$ is a strictly totally ordered monoid. Here we have the following result:

Proposition 1.10. Let $M$ be a strictly totally ordered monoid and $I$ an ideal of $R$. If $I$ is a semiprime ring and $R/I$ is $M$-quasi-Armendariz, then $R$ is $M$-quasi-Armendariz.

Proof. Let $\alpha = a_1g_1 + \cdots + a_ng_n$ and $\beta = b_1h_1 + \cdots + b_mh_m \in R[M]$ be such that $\alpha R[M] \beta = 0$ and $g_1 < \cdots < g_n, h_1 < \cdots < h_m$. We use transfinite induction on the strictly totally ordered set $(M, \leq)$ to show that $a_iRb_j = 0$ for each $i, j$. Note that in $(R/I)[M], (\alpha g_1 + \cdots + \alpha g_n)R/I(\beta h_1 + \cdots + \beta h_m) = 0$. Thus we have $a_iRb_j \subseteq I$ for each $i, j$, since $R/I$ is $M$-quasi-Armendariz. By analogy with the proof of Proposition 1.8, we can show that $a_iRb_1 = 0$. Take the same induction hypothesis as in the proof of Proposition 1.8. For any $t \geq 2$, $g_{i_1}h_{j_1} < g_{i_2}h_{j_2} = \omega$, and thus, by the induction hypothesis, we have $a_{i_1}Rb_{j_1} = 0$. Now

$$\sum_{(g_i, h_j) \in X} a_i cb_j = \sum_{t=1}^k a_i cb_j = 0 \quad \text{for each } c \in R. \quad (5)$$

Let $d$ be an arbitrary element of $R$. By multiplying Eq. (5) by $a_{i_2}d$, from the left, we have $a_{i_2}Ra_{i_2}Rb_{j_1} = 0$. Hence $(Ra_{i_2}Rb_{j_1}R)^2 = 0$. Since $Ra_{i_2}Rb_{j_1}R \subseteq I$, and $I$ is a semiprime ring, we have $a_{i_2}Rb_{j_1} = 0$. Now Eq. (5), becomes

$$\sum_{t=2}^k a_{i_2} cb_j = 0 \quad \text{for each } c \in R. \quad (6)$$

By multiplying Eq. (6) by $a_{i_2}d$, from the left, we obtain $a_{i_2}Rb_{j_2} = 0$ in the same way as above. Continuing this process, we can prove $a_iRb_j = 0$ for any $i, j$ with $g_ih_j = \omega$. Therefore, by transfinite induction, $a_iRb_j = 0$ for any $i, j$. Thus $R$ is $M$-quasi-Armendariz.
**Proposition 1.11.** Let $M$ be an u.p.-monoid and $I$ an ideal of $R$. If $I$ is a semiprime ring and $R/I$ is $M$-quasi-Armendariz, then $R$ is $M$-quasi-Armendariz.

**Proof.** Let $\alpha = a_1 g_1 + \cdots + a_n g_n$ and $\beta = b_1 h_1 + \cdots + b_m h_m \in R[M]$ be such that $\alpha R[M] \beta = 0$. Note that in $(R/I)[M], (\overline{a_1} \overline{g_1} + \cdots + \overline{a_n} \overline{g_n}) R/I (\overline{b_1} h_1 + \cdots + \overline{b_m} h_m) = 0$. Thus we have $a_i R b_j \subseteq I$ for each $i, j$, since $R/I$ is $M$-quasi-Armendariz. By analogy with the proof of Proposition 1.7, one can complete the proof. \(\square\)

Recall that a monoid $M$ is called torsion-free if the following property holds: if $g, h \in M$ and $k \geq 1$ are such that $g^k = h^k$, then $g = h$.

**Corollary 1.12.** Let $M$ be a commutative, cancellative and torsion-free monoid. If one of the following conditions holds, then $R$ is $M$-quasi-Armendariz:

1. $R$ is semiprime.
2. $R/I$ is $M$-quasi-Armendariz for some ideal $I$ of $R$ and $I$ is a semiprime ring.

**Proof.** If $M$ is commutative, cancellative and torsion-free, then by [14] there exists a compatible strict total order $\leq$ on $M$. Now the results follow from Propositions 1.8 and 1.10. \(\square\)

**Lemma 1.13.** Let $M$ be a cyclic group of order $n \geq 2$ and $R$ a ring with $0 \neq 1$. Then $R$ is not $M$-quasi-Armendariz.

**Proof.** Suppose that $M = \{e, g, g^2, \ldots, g^{n-1}\}$. Let $\alpha = 1e + 1g + 1g^2 + \cdots + 1g^{n-1}$ and $\beta = 1e + (-1)g$. Then $\alpha c \beta = 0$ for each $c \in R$ and $\alpha R[M] \beta = 0$. Thus $R$ is not $M$-quasi-Armendariz. \(\square\)

The following is trivial.

**Lemma 1.14.** Let $M$ be an abelian monoid and $N$ a submonoid of $M$. If $R$ is $M$-quasi-Armendariz, then $R$ is $N$-quasi-Armendariz.

**Lemma 1.15 ([17, Lemma 1.13]).** Let $M$ and $N$ be u.p.-monoids. Then so is the monoid $M \times N$.

**Theorem 1.16.** Let $M$ be a finitely generated abelian group. Then the following are equivalent:

1. $M$ is torsion-free.
2. There exists a ring $R$ with $|R| \geq 2$ such that $R$ is $M$-quasi-Armendariz.

**Proof.** (1) $\Rightarrow$ (2). Since $M$ is a finitely generated torsion-free abelian group, then $M \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$, a finite product of group $\mathbb{Z}$. By Lemma 1.15, $M$ is a u.p.-monoid. Let $R$ be a semiprime ring. Then $R$ is $M$-quasi-Armendariz, by Proposition 1.7.

(2) $\Rightarrow$ (1). Let $g \in M$ be an element of finite order with $g \neq e$. Then $N = \langle g \rangle$ is a cyclic group of finite order. If a ring $R \neq \{0\}$ is $M$-quasi-Armendariz, then $R$ is $N$-quasi-Armendariz, a contradiction with Lemma 1.14. Thus $M$ is torsion-free. \(\square\)

A ring $R$ is a subdirect sum of a family of rings $\{R_i\}_{i \in I}$ if there is an injective homomorphism $f : R \rightarrow \Pi_{i \in I} R_i$ such that, for each $j \in I$, $\pi_j f : R \rightarrow R_j$ is a surjective homomorphism, where $\pi_j : \Pi_{i \in I} R_i \rightarrow R_j$ is the $j$th projection. In [17], Zhongkui showed that if $R$ is a subdirect sum of $M$-Armendariz rings, then $R$ is an $M$-Armendariz ring. Similarly we have the following.

**Proposition 1.17.** If $R$ is a subdirect sum of $M$-quasi-Armendariz rings, then $R$ is $M$-quasi-Armendariz.

**Proof.** Let $I_k$ $(k \in K)$ be ideals of $R$ such that $R/I_k$ is $M$-quasi-Armendariz and $\cap_{k \in K} I_k = 0$. Suppose that $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m \in R[M]$ satisfy $\alpha R[M] \beta = 0$. Since $R/I_j$ is $M$-quasi-Armendariz, for each $j \in J$, we have $a_i R b_j \subseteq I_k$ for each $i, j$. Hence $a_i R b_j \subseteq \cap_{k \in K} I_k = 0$. \(\square\)
2. Quasi-Baer rings

A ring $R$ is called quasi-Baer if the left annihilator of every left ideal of $R$ is generated by an idempotent. Note that this definition is left–right symmetric. Some results for a quasi-Baer ring can be found in [3,4,6,12]. Following [7], for a ring $R$, put $r\Ann_R(id(R)) = \{ rU \mid U \text{ is an ideal of } R \}$.

**Proposition 2.1.** Let $M$ be a monoid. Then the following statements are equivalent:

1. $R$ is $M$-quasi-Armendariz;
2. $\psi : r\Ann_R(id(R)) \to r\Ann_R(M)(id(R[M])); A \to A[M]$ is bijective.

**Proof.** (1) $\Rightarrow$ (2). Let $A \in r\Ann_R(id(R))$. Then there exists an ideal $I$ of $R$ such that $A = rR(I)$. Clearly $I[M]$ is an ideal of $R[M]$ and $I[M]A[M] = 0$. Let $\alpha = a_1g_1 + \cdots + a_ng_n \in r(I[M])$. Then $I[M]R[M]\alpha = 0$. Hence $Ia_i = 0$ for each $i$, since $R$ is $M$-quasi-Armendariz. Thus $a_i \in I$ and $rR(I[M](I[M]) = A[M]$. Consequently, $\psi$ is a well defined map. Assume that $B \in r\Ann_R(M)(id(R[M]));$ then there exists an ideal $J$ of $R[M]$ such that $B = rR(M)(J)$. Let $B_1$ and $J_1$ denote the set of coefficients of elements of $B$ and $J$ respectively. We claim that $rR(J_1 R) = B_1 R$. Let $\alpha = a_1g_1 + \cdots + a_ng_n \in J$ and $\beta = b_1h_1 + \cdots + b_nb_m \in B$. Then $\alpha R[M]\beta = 0$. Hence $a_iRb_j = 0$ for all $a_i, b_j$, since $R$ is $M$-quasi-Armendariz. Thus $J_1 R \subseteq B_1 R$. Clearly $rR(J_1 R) \subseteq B_1 R$. Thus $rR(J_1 R) = B_1 R$; hence $rR(I[M] = B_1 R[M]$.

(2) $\Rightarrow$ (1). Let $\alpha = a_1g_1 + \cdots + a_ng_n \in J$ and $\beta = b_1h_1 + \cdots + b_nb_m \in R[M]$ satisfy $\alpha R[M]\beta = 0$. Then $\beta \in rR(J_1 R[M] \alpha R[M]) = A[M]$, where $A$ is an ideal of $R$. Hence $b_1, \ldots, b_m \in A$ and so $\alpha Rb_j = 0$ for $j = 1, \ldots, m$. Thus $a_iRb_j = 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Therefore $R$ is $M$-quasi-Armendariz.

**Definition 2.2.** A submodule $N$ of a left $R$-module $M$ is called a pure submodule if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right $R$-module $L$. Following Tominaga [16], an ideal $I$ of $R$ is said to be right $s$-unital if, for each $a \in I$, there is an $x \in I$ such that $ax = a$. If an ideal $I$ of $R$ is right $s$-unital, then for any finite subset $F$ of $I$, there exists an element $e \in I$ such that $xe = x$ for all $x \in F$. By [15, Proposition 11.3.13], for an ideal $I$, the following conditions are equivalent:

1. $I$ is pure as a left ideal in $R$;
2. $R/I$ is flat as a left $R$-module;
3. $I$ is right $s$-unital.

**Theorem 2.3.** Let $M$ be a u.p.-monoid or $(M, \leq)$ be a strictly totally ordered monoid. Then the following are equivalent:

1. $\ell(Ra)$ is pure as a left ideal in $R$ for any element $a \in R$;
2. $\ell(R[M]b)$ is pure as a left ideal in $R[M]$ for any element $b \in R[M]$; in this case $R$ is $M$-quasi-Armendariz.

**Proof.** We prove it for a u.p.-monoid and the other case is similar.

(1) $\Rightarrow$ (2). Assume that condition (1) holds. First we shall prove that $R$ is $M$-quasi-Armendariz. Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n \in R[M]$ are such that $\alpha R[M]b = 0$. Then

$$((a_1g_1 + \cdots + a_ng_n)c(b_1h_1 + \cdots + b_nh_m) = 0 \text{ for each } c \in R. \quad (\dagger)$$

Since $M$ is a u.p.-monoid, there exists $j$, with $1 \leq i \leq n, 1 \leq j \leq m$ such that $g_jh_j$ is uniquely presented by considering two subsets $A = \{g_1, \ldots, g_n\}$ and $B = \{h_1, \ldots, h_m\}$ of $M$. Thus $a_i c_{i/j} h_j = 0$ for each $c \in R$ and $a_i Rb_j = 0$. Hence $a_i \in \ell(Rb_j)$. By hypothesis, $\ell(Rb_j)$ is right $s$-unital, and hence there exists $e_j \in \ell(Rb_j)$ such that $a_i = a_i e_j$. Replacing $c$ by $e_j c$ in Eq. $(\dagger)$, we obtain

$$0 = (a_1g_1 + \cdots + a_ng_n)e_j c(b_1h_1 + \cdots + b_nh_m)$$

$$= (a_1g_1 + \cdots + a_ng_n)c(b_1h_1 + \cdots + b_jh_j + b_{j+1}h_{j+1} + \cdots + b_nh_m)$$

$$= (a_1e_jg_1 + \cdots + a_ne_jg_n)c(b_1h_1 + \cdots + b_jh_j + b_{j+1}h_{j+1} + \cdots + b_nh_m)$$

$$= (a_1e_jg_1 + \cdots + a_ne_jg_n)c(b_1h_1 + \cdots + b_jh_j + b_{j+1}h_{j+1} + \cdots + b_nh_m).$$

Using induction on $n + m$, it follows that $a_i Rb_k = 0$ for each $1 \leq k \leq m$. Thus we have $(a_1g_1 + \cdots + a_{i-1}g_{i-1} + a_{i+1}g_{i+1} + \cdots + a_ng_n)c(b_1h_1 + \cdots + b_nh_m) = 0$, for each $c \in R$. By induction, it follows that $a_i Rb_j = 0$ for each
Let $M$ be a $u.p.$-monoid or ... Hence we have the following:

Corollary 2.4. Let $M$ be a $u.p.$-monoid or $(M, \leq)$ be a strictly totally ordered monoid. Then $R$ is quasi-Baer if and only if the monoid ring $R[M]$ is quasi-Baer.

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References