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A nonlinear fourth order diffusion problem: Convergence to the steady state and non-negativity of solutions

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Abstract

The paper deals with nonlinear diffusion, both time-dependent and time-independent. The spatial terms in the partial differential equation (p.d.e.) contain a second order nonlinear part (where the non-negative diffusivity depends on the dependent variable) and a fourth order linear part. In the context of non-null, time-independent boundary conditions, convergence of the unsteady to the steady state is established. The second part of the paper discusses criteria on data ensuring non-negativity of the solutions. This is done for the steady state irrespective of the spatial dimension; and it is done for the unsteady state for the one-dimensional rectilinear case only, using a result from the first part of the paper.

Keywords: Nonlinear fourth order diffusion problem; Convergence; Steady state

1. Introduction

This paper is concerned with nonlinear diffusion, both time-dependent and time-independent. The spatial terms in the relevant partial differential equation contain a second order nonlinear part (where the (non-negative) diffusivity depends on the dependent variable) and a fourth order linear part. Time independent, non-null, Dirichlet type boundary conditions are envisaged, and both the corresponding initial boundary value problem (for the unsteady state) and the boundary value problem (for the steady state) are considered.

Using an extension of a versatile Liapunov functional (e.g. [1,2] and the references quoted therein) as applied to the perturbation (i.e. unsteady–steady state), a lower estimate is obtained for the rate of convergence, with respect to time, of the unsteady to the steady state. In the one-dimensional, rectilinear case, it is possible to deduce a decay estimate for the absolute value of the perturbation. The foregoing may be referred to as the first part of the paper.

We now turn to the second part of this paper: In the context of the initial boundary value problem and the boundary value problem considered, an important question arises: What restrictions on the data ensure non-negative solutions? This question arises as equations of the type considered are often used to model intrinsically non-negative, or positive, quantities, e.g. cell density in biology. In the case of the steady state problem, restrictions are obtained on the data that

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ensure non-negative solutions: this is done by means of a comparison theorem based on maximum principles. In the case of the unsteady state problem, restrictions are obtained on the data guaranteeing non-negative solutions in the one-dimensional rectilineal case only. These restrictions are based on the conditions arising in the case of the steady state problem together with a restriction on the initial data. The latter restriction is obtained using the estimate for the perturbation obtained in the first part of the paper.

2. Steady and unsteady states: Convergence of the latter to the former

A fixed spatial region V is considered, its smooth boundary being denoted by ∂V. The spatial and time variables are denoted by $\mathbf{x}, t$, respectively. It is supposed that the *unsteady state* $T(\mathbf{x},t)$ is a smooth function, satisfying the initial boundary value problem

$$\partial T/\partial t = \nabla\cdot\{k(T)\nabla T\} - \alpha\nabla^4 T \quad \text{in } V, \tag{1}$$

the diffusivity $k(\cdot)$ being a continuously differentiable, positive or non-negative, function of its argument, α being a positive constant, subject to the boundary conditions

$$T = r(\mathbf{x}), \quad \nabla^2 T = s(\mathbf{x}) \quad \text{on } \partial V, \tag{2}$$

and to the initial condition

$$T(\mathbf{x},0) = T_0(\mathbf{x}) \quad \text{in } V, \tag{3}$$

$r(\mathbf{x})$, $s(\mathbf{x})$, $T_0(\mathbf{x})$ being assigned functions of position.

The initial boundary value problem considered above is one of the simpler such problems for the p.d.e. (1). Murray [3] provides motivation for (1) in a biological context: Diffusion with $a = 0$ corresponds to a flux proportional to the concentration gradient ∇T and is appropriate to dilute systems. For relatively high cell concentrations—as may occur in embryological development, for example—it is appropriate to augment the flux by the inclusion of a term involving $\nabla(\nabla^2 T)$. Murray [3] points out that the p.d.e. (1) with

$$k(T) = a + bT^2,$$

a, b being non-negative constants, follows from a Landau–Ginzberg free energy form.

Let $U(\mathbf{x})$ be the corresponding *steady state*, satisfying the same time-independent boundary conditions: it is supposed that $U(\mathbf{x})$ is a smooth function satisfying the boundary value problem

$$\nabla\cdot\{k(U)\nabla U\} - \alpha\nabla^4 U = 0 \quad \text{in } V, \tag{4}$$

subject to the boundary conditions

$$U = r(\mathbf{x}), \quad \nabla^2 U = s(\mathbf{x}) \quad \text{in } V. \tag{5}$$

The analysis subsequently carried out for these is envisaged in the context of a two or three-dimensional region, but the corresponding results in a one-dimensional, rectilineal context will be immediately apparent.

A perturbation of the steady state is envisaged (at $t = 0$) and we define the perturbation

$$u(\mathbf{x},t) = T(\mathbf{x},t) - U(\mathbf{x}). \tag{6}$$

We introduce a function $\Phi(u,U)$ which is fundamental to the subsequent analysis

$$\Phi(u,U) = \int_0^u d\bar{u}\int_0^{\bar{u}} k(U+\tau)\,d\tau. \tag{7}$$

This function was used in previous papers [1,2], and most of the principal properties, under certain assumptions, are summarized in Appendix A. One may verify that the initial boundary value problem characterizing u may be written

$$\partial u/\partial t = \nabla^2\Phi_u - \alpha\nabla^4 u \quad \text{in } V, \tag{8}$$

subject to the boundary conditions

$$u = \nabla^2 u = 0 \quad \text{on } \partial V, \tag{9}$$

and the initial condition

$$u(\mathbf{x},0)=T_0(\mathbf{x})-U(\mathbf{x}) \quad \text{in } V. \tag{10}$$

Referring to $\Phi(u,U)$, defined by (7), we summarize hereunder some of its salient properties. It is evident that

$$\Phi(0,U)=\Phi_u(0,U)=0. \tag{11}$$

Further, supposing that there exists a non-negative constant k_0 such that

$$k(\cdot)\geqslant k_0, \tag{12}$$

then

$$\Phi(u,U)\geqslant \frac{1}{2}k_0u^2. \tag{13}$$

It follows that $\Phi(u,U)$ is positive-definite in u when $k_0>0$, and non-negative if $k_0=0$. Further relevant properties of Φ are given in Appendix A.

We define

$$E(t)=\int_V \left[\Phi(u,U)+(\alpha/2)(\nabla u)^2\right]dV. \tag{14}$$

In view of the foregoing properties of Φ it is evident that the integrand in (14) is positive-definite in u provided $k_0\geqslant 0$. $E(t)$ is thus a suitable global measure of the perturbation u at time t.

On using (8), (9), etc., one may prove that

$$\frac{dE}{dt}=-\int_V \left[\nabla\Phi_u-\alpha\nabla\left(\nabla^2u\right)\right]^2dV. \tag{15}$$

Remark 1. This result ($dE/dt\leqslant 0$) is sometimes verbalized as follows: the steady state U is stable with respect to the measure (or Liapunov functional) E.

Remark 2. It does not appear to be possible to replace the constant $\alpha/2$ in (14) by another one to obtain a result of the type (15) wherein $dE/dt\leqslant 0$.

Remark 3. The result (15) continues to be valid if the boundary conditions (5) are modified to read

$$\nabla T\cdot\mathbf{n}=0,\quad \nabla\left(\nabla^2T\right)\cdot\mathbf{n}=h(\mathbf{x}) \quad \text{on } \partial V,$$

where $h(\mathbf{x})$ is an assigned function and where $\mathbf{n}$ denotes the unit outward normal to ∂V.

We now obtain a differential inequality for dE/dt, representing an improvement of $dE/dt\leqslant 0$. Consider the following chain of inequalities, etc.:

$$\begin{aligned}\left\{E(t)\right\}^2&=\left[\int_V\left\{\Phi+(\alpha/2)(\nabla u)^2\right\}dV\right]^2\leqslant\left[\int_V\left\{u\Phi_u+\alpha(\nabla u)^2\right\}dV\right]^2\\&=\left[\int_V\left\{u\left(\Phi_u-\alpha\nabla^2u\right)\right\}dV\right]^2\leqslant\int_V u^2\,dV\int_V\left(\Phi_u-\alpha\nabla^2u\right)^2dV,\end{aligned} \tag{16}$$

the four steps, taken in order, being based on

(i) the definition (14),
(ii) $u\Phi_u\geqslant\Phi$ (see Appendix A),
(iii) integration by parts, using (9_1),
(iv) Schwarz's inequality.

Let λ_1 denote the lowest (positive) "fixed-membrane" eigenvalue of V (corresponding to the eigenvalue problem defined by

$$\nabla^2\chi + \lambda\chi = 0 \quad \text{in } V, \qquad \chi = 0 \quad \text{on } \partial V,$$

etc.) Applying the standard variational characterization of λ_1 to (16) we obtain

$$\{E(t)\}^2 \leqslant \int_V u^2\, dV \cdot \lambda_1^{-1} \int_V \{\nabla(\Phi_u - \alpha\nabla^2 u)\}^2\, dV = -\int_V u^2\, dV \cdot \lambda_1^{-1}\, dE/dt \tag{17}$$

where (15) has been used in the last step.

Again using the variational characterization of λ_1, as applied to u, together with (13), we obtain from (14)

$$E(t) \geqslant (1/2)(k_0 + \alpha\lambda_1) \int u^2\, dV. \tag{18}$$

Combining (17), (18) we obtain the requisite, improved differential inequality

$$dE/dt + (\lambda_1/2)(k_0 + \alpha\lambda_1)E \leqslant 0. \tag{19}$$

Integration gives the following result:

Theorem 1. *The global measure of the perturbation u (defined by (6)–(9), etc.) given by the Liapunov functional $E(t)$, defined by (14), satisfies*

$$E(t) \leqslant E(0)\exp[-(\lambda_1/2)(k_0 + \alpha\lambda_1)t], \tag{20}$$

λ_1 being the lowest (positive) fixed membrane eigenvalue of the region V, k_0 being defined by (12).

Remark 4. The theorem may be expressed verbally as follows: as $t \to \infty$, the unsteady state converges asymptotically and exponentially to the steady state, in the measure E.

Remark 5. The result (20) continues to hold if the constant k_0 is omitted.

Remark 6. An upper bound for $\int u^2\, dV$ is deducible from (18), (20).

Remark 7. The previous, and subsequent, analysis continues to hold in a one-dimensional rectilineal context, *mutatis mutandis*, e.g. consider the fixed domain $0 \leqslant x \leqslant L$, x being a rectangular cartesian coordinate; ∇, ∇^2, ∇^4 are replaced by d/dx, d^2/dx^2, d^4/dx^4, respectively; the boundary conditions (2), etc. are replaced by

$$T(0) = r(0), \qquad d^2T/dx^2(0) = s(0), \qquad T(L) = r(L), \qquad d^2T/dx^2(L) = s(L), \tag{21}$$

where $r(0)$, $s(0)$, $r(L)$, $s(L)$ are constants, etc. In this context, λ_1 is replaced by π^2/L^2.

In this (one-dimensional) case, it is possible to obtain pointwise bounds for $u(x,t)$: for $0 < x < L$,

$$\begin{aligned} E(t) &= \int_0^L [\Phi(U,u) + (\alpha/2)(du/dx)^2]\, dx \geqslant (\alpha/2)\int_0^L (du/dx)^2\, dx \\ &\geqslant (\alpha/2)\{x(1 - x/L)\}^{-1}\{u(x,t)\}^2, \end{aligned} \tag{22}$$

on using a well-known, optimal bound [4]. The requisite, explicit bound for $u(x,t)$ follows on combining this with (20). It is possible to improve this along the following lines: for $0 < x < L$,

$$E(t) \geqslant (\alpha/2)\int_0^L [(du/dx)^2 + (k_0/\alpha)u^2]\, dx \geqslant (\alpha/2)c(x, L, (k_0/\alpha))\{u(x,t)\}^2,$$

where $c(x, L, k_0/\alpha)$ is a non-negative quantity, the best value of which can be written down explicitly [5].

Remark 8. If a positive constant k_m is known such that

$$k(\cdot) \leqslant k_m,$$

then a slight generalization of the analysis leading to (20) yields

$$E(t) \leqslant E(0)\exp\bigl[-\bigl(p^2\lambda_1/2\bigr)(k_0+\alpha\lambda_1)t\bigr], \tag{23}$$

where

$$p = 1 + k_0/k_m;$$

(A.5) of Appendix A plays a central role in this generalization. The result (23) is optimal in the limit $k_0/k_m \to 1$: when the diffusivity is constant, there are non-trivial circumstances in which the equality sign is realized in (23).

3. Non-negativity of solutions

In this section the following important issue is considered: under what conditions (i.e. for what data) are the solutions of the steady and unsteady state problems non-negative? Whereas it is possible to obtain such conditions for the steady state irrespective of the number of spatial dimensions, in this paper this proves possible for the unsteady state in the one-dimensional rectilineal case only. The importance of the general issue arises as the dependent variable in p.d.e.s of the type (1), often model intrinsically non-negative, or positive, quantities, e.g. cell density in biology.

To address the issue for the steady state, we need the following (simplified version of the) comparison theorems, given in [6], for example. Suppose $z(\mathbf{x})$, $Z(\mathbf{x})$ are smooth functions satisfying

(i) $\nabla^2 z - a(\mathbf{x})z = 0$ in V, $z = F(\mathbf{x})$ on ∂V,
(ii) $\nabla^2 Z - a(\mathbf{x})Z \leqslant 0$ in V, $Z \geqslant F(\mathbf{x})$ on ∂V,

where $a(x) \geqslant 0$, then

$$Z(\mathbf{x}) \geqslant z(\mathbf{x}). \tag{24}$$

If we define

$$W = -\nabla^2 U \tag{25}$$

and if we suppose that the derivative of the diffusivity $k(\cdot)$ with respect to its argument k' satisfies

$$k'(\cdot) \geqslant 0$$

then (4), (25) imply an inequality of the type

$$\nabla^2 W - a(\mathbf{x})W \leqslant 0 \quad \text{in } V, \tag{26}$$

where $a(\mathbf{x}) \geqslant 0$. Suppose, in addition, that

$$W(\mathbf{x}) \geqslant 0 \quad \text{on } \partial V, \tag{27}$$

then, on putting $F \equiv 0$, $z \equiv 0$, $Z \equiv W$ in the comparison theorem (24), it follows that

$$W(\mathbf{x}) \geqslant 0 \quad \text{in } V, \tag{28}$$

i.e.

$$\nabla^2 U \leqslant 0 \quad \text{in } V. \tag{29}$$

Suppose, in addition to (29), that

$$U(\mathbf{x}) \geqslant 0 \quad \text{on } \partial V, \tag{30}$$

then, on putting $a \equiv 0$, $Z \equiv U$, $F \equiv 0$, $z = 0$ in the comparison theorem (24), one finds that

$$U(\mathbf{x}) \geqslant 0 \quad \text{in } V. \tag{31}$$

We have thus obtained conditions on the data under which the steady state—defined by (4), (5)—is non-negative (assuming, of course, the existence of smooth solutions). These are embodied in the following theorem

Theorem 2. *The steady state $U(x)$, defined by* (4), (5), *is non-negative provided that*

(a) *the diffusivity $k(\cdot)$ is a non-decreasing function of its argument,*
(b) *the boundary conditions* (5) *are such that $r(\mathbf{x}) \geqslant 0$, $s(\mathbf{x}) \leqslant 0$.*

It is possible to deduce from the foregoing analysis sufficient conditions on the data to ensure non-negative solutions in the time-dependent case, provided one confines attention to the one-dimensional, rectilineal case. This may be done by using the results of Theorems 1 and 2 (*mutatis mutandis*) together with the pointwise bound (22). Plainly one has

$$T(x,t) \geqslant 0$$

provided that the conditions of Theorem 2 hold *mutatis mutandis* together with

$$\{u(x,t)\}^2 \leqslant \{U(x)\}^2.$$

Recall that Theorem 1 (or, indeed, (15)) implies

$$E(t) \leqslant E(0),$$

and the foregoing considerations lead to:

Theorem 3. *In the one-dimensional rectilinear context $0 \leqslant x \leqslant L$ (see* [1–5] *and Remark* 7*), suppose that*

(a) *the diffusivity $k(\cdot)$ is a non-decreasing function of its argument,*
(b) *the boundary conditions satisfy $r(0) \geqslant 0$, $r(L) \geqslant 0$, $s(0) \leqslant 0$, $s(L) \leqslant 0$,*
(c) $$x(1-x/L)\left[\int_0^L \left\{(2/\alpha)\int_0^{u_0(x)} d\bar{u}\int_0^{\bar{u}} k(U+\tau)\,d\tau + \big(du_0(x)/dx\big)^2\right\} dx\right] \leqslant \{U(x)\}^2,$$
where $u_0(x)$ is the initial value of the perturbation, then

$$T(x,t) \geqslant 0 \quad \textit{for } t \geqslant 0.$$

Remark 9. The issue dealt with in the latter theorem is similar to that dealt with by Bartuccelli et al. [7] and in the references quoted therein, although there are considerable differences in the nonlinear diffusion problems considered in both cases. Among the similarities, is a condition restricting the size of the absolute values of the initial values of the perturbation and that of its spatial derivative. See also [8].

Remark 10. It is, of course, possible to weaken the restriction (c) in Theorem 3 if one merely requires

$$T(x,t) \geqslant 0 \quad \text{for } t \geqslant t_0,$$

where $t_0 > 0$.

Appendix A. Properties of Φ defined by (7)

Suppose that the diffusivity k satisfies (12). Then the remainder form of Taylor's theorem together with (7), (11), (13) imply that

$$\Phi(u,U) \geqslant \frac{1}{2}k_0u^2. \tag{A.1}$$

The positive-definiteness in u of $\Phi(u,U)$ follows.

One easily shows that

$$u\Phi_u - \Phi = \int_0^u \bar{u}\,k(U+\bar{u})\,d\bar{u}. \tag{A.2}$$

Assuming (12), it follows that

$$u\Phi_u \geqslant \Phi, \tag{A.3}$$

for if u is non-negative there is nothing to prove, while if it is negative a change of variable makes the proof transparent.

If one assumes the existence of a positive constant k_m such that

$$k_m \geqslant k(\cdot)$$

one may prove a result similar to (A.1)

$$\frac{1}{2}k_m u^2 \geqslant \Phi(u, U). \tag{A.4}$$

Combining (A.1), (A.2), (A.4) one may obtain

$$u\Phi_u \geqslant (1 + k_0/k_m)\Phi. \tag{A.5}$$

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