J. Math. Anal. Appl. 389 (2012) 1259–1267

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The group of invariants of an inner function with finite spectrum

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ARTICLE INFO

Article history:
Received 29 May 2011
Available online 16 January 2012
Submitted by Richard M. Aron

Keywords:
Blaschke product
Inner function
Group of invariants

ABSTRACT

This paper determines the group of continuous invariants corresponding to an inner function Θ with finitely many singularities on the unit circle $\mathbb{T}$; that is, the continuous mappings $g : \mathbb{T} \to \mathbb{T}$ such that $\Theta \circ g = \Theta$ on $\mathbb{T}$. These mappings form a group under composition.

1. Introduction

In this paper, we study the group $\mathcal{G}$ of continuous invariants corresponding to an inner function Θ with finitely many singularities on the unit circle $\mathbb{T} = \partial\mathbb{D}$; that is, the continuous mappings $g : \mathbb{T} \to \mathbb{T}$ such that $\Theta \circ g = \Theta$ on $\mathbb{T}$. Since Θ is undefined at its singular points on $\mathbb{T}$, we interpret this as meaning that Θ maps singular points to singular points and regular points to regular points. These mappings form a group under composition.

Our motivation for this problem comes from a beautiful geometric result known as Poncelet's theorem [10]. In an article in the American Mathematical Monthly [13], King describes three problems that are connected to a measure: one is Tarski's plank problem, one is a set of questions known as Gelfand's questions, and the third is closely connected to the results in this paper.

To understand the problem King describes, we imagine an ellipse contained inside the open unit disk. Starting with a point z_0 on the unit circle, draw the tangent to the ellipse. The second point of intersection of the tangent line with the unit circle is denoted $z_1 = R(z_0)$. From z_1 draw a tangent to the ellipse, and let z_2 denote the point of intersection of this tangent with the unit circle; that is, $z_2 = R(z_1)$. Now, either this process continues forever or it returns to its starting point in n steps. Poncelet's theorem says that if we return to our starting point in n steps, then starting at any other point of the unit circle and repeating the process produces the same result – the polygon closes in n steps, regardless of the starting point. In [8], a connection was made between this result and finite Blaschke products: if we connect the points of the unit circle identified by a finite Blaschke product, we obtain a closed convex n-gon that circumscribes a closed curve. This is true regardless of which point on the unit circle we choose as our vertex. Thus, these closed curves all have the Poncelet property. If we look at the map R in this context, we see that it is mapping a point z_0 on the unit circle to another point on the unit circle for which $B(z_1) = B(z_0)$. Thus, the map R makes an appearance in this context as well: $B \circ R = B$. In recent work [5] the authors showed that this theorem can be extended to infinite Blaschke products with finitely many

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.01.005

singularities. In this paper, we look at this result from a group theoretic point of view, providing a new angle on an old problem.

In [3], it was shown that the group $\mathcal{G}$ of invariants of a finite Blaschke product B of degree n is isomorphic to the cyclic group $\mathbb{Z}_n$. In this case the circle may be divided into n sub-arcs on each of which B takes all values of $\mathbb{T}$ precisely once; then the mappings $g \in \mathcal{G}$ permute the arcs cyclically. The study of $\mathcal{G}$ was useful in a problem of dual algebra theory [4, Thm. 3.1]. When we come to study general inner functions, the situation is considerably more difficult, but it is still possible to give a complete solution in the case that the spectrum is finite; we shall see that various complicated groups can arise, in particular infinite and non-Abelian groups. We should mention the recent paper [2], which discusses the case of a single singular point: however, Theorem 7 in that paper is incorrect, because it overlooks one possible case, as we shall see later.

Note that if $g \in \mathcal{G}$, then at a regular point of $\mathbb{T}$ (that is, a point that is not a singularity of Θ), we may use the equality $\Theta \circ g = \Theta$ and a simple calculation to conclude that g is analytic on a neighborhood of each regular point with $\arg g(z)$ strictly increasing with $\arg z$.

For general results on the cluster sets of analytic functions we refer to the classic books [7] and [16], although they do not consider the questions addressed in this paper.

2. Notation

We refer to [12] for standard results on Hardy spaces and inner functions. Recall that an inner function Θ is a bounded analytic function in $\mathbb{D}$ whose boundary values satisfy $|\Theta(z)| = 1$ a.e., for z on the unit circle $\mathbb{T}$; such a function may be factorized into the product of a (finite or infinite) Blaschke product and a singular inner function.

An infinite Blaschke product defined on the disc $\mathbb{D}$ is an analytic function B of the form

$$B(z) = \lambda z^p \prod_{j=1}^{\infty} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \bar{a}_j z},$$

where $a_j \in \mathbb{D} \setminus \{0\}$, $|\lambda| = 1$, p is a non-negative integer and $\sum_{j=1}^{\infty}(1 - |a_j|) < \infty$ (a finite Blaschke product is defined analogously).

A singular inner function may be written as

$$S(z) = \exp\left[-\int \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right],$$

where μ is a singular positive measure on $\mathbb{T}$.

The spectrum $\sigma(\Theta)$ is the complement of the set of points $p \in \mathbb{T}$ such that Θ has an analytic extension into a neighborhood of p. Indeed, writing $Z(\Theta)$ for the zero set of Θ in $\mathbb{D}$ we have

$$\sigma(\Theta) = \mathbb{T} \cap \left\{\overline{Z(\Theta)} \cup \operatorname{supp}(\mu)\right\}.$$

Remark 2.1. The following observation will be crucial in describing the behavior of an inner function at a point $e^{i\theta}$ on the circle at which it is analytic: namely, the continuous branch of the argument at that point is strictly increasing. By Frostman's theorem [11, p. 79], it is enough to consider the case of a Blaschke product B with zeros (a_n). We then have

$$\frac{B'(e^{i\theta})}{B(e^{i\theta})} = \sum_n \frac{1 - |a_n|^2}{(e^{i\theta} - a_n)(1 - \bar{a}_n e^{i\theta})}. \tag{1}$$

Writing $B(e^{i\theta}) = e^{i\phi}$ and using the chain rule we see that

$$\frac{d\phi}{d\theta} = e^{i\theta} \frac{B'(e^{i\theta})}{B(e^{i\theta})} > 0,$$

as claimed.

Finally, for the reader's convenience, we summarize results that will be used frequently: In Lemma 3.3 we show that if an inner function Θ has finitely many singularities and the zeros of the inner function lie in a Stolz angle, then for every point λ of T there are infinitely many solutions to $\Theta = \lambda$ in arbitrarily small intervals on each side of the singularity. In Lemma 4.1, we will show that a map x in the group of invariants is a bijective mapping of $\mathbb{T}$ to $\mathbb{T}$. In Lemmas 4.2, 4.3, 4.4 and 4.5, we will study the behavior of such a map x on an interval I between successive singularities of an inner function.

3. Background

In this section, we collect the relevant background information necessary to read this paper. New results follow in the remaining sections.

Consider an inner function Θ with finitely many points in the spectrum, $\sigma(\Theta)$. It is well known that the essential range of Θ at a singularity ξ_0 is the unit circle, $\mathbb{T}$ (i.e., the inverse image of an arbitrarily small disc centered at any point of the circle meets every neighborhood of ξ_0 in a set of positive measure). For the purposes of this paper, it will be necessary to classify the possible types of limiting behavior of Θ at the point ξ_0. We do so in the next definition.

Definition 3.1. Let Θ be an inner function with finite spectrum. Let $\xi_0 \in \sigma(\Theta) \cap \mathbb{T}$. We say that

1. $\xi_0 = e^{i\theta_0}$ *is of type* $1_{a,L}$ if for $\varepsilon > 0$ sufficiently small, there are infinitely many solutions of $\Theta(\xi) = 1$ in the open interval (i.e., arc of the circle) $(e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})$, finitely many solutions in $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$, and $\lim_{\alpha \to \theta_0^-} \Theta(e^{i\alpha}) = L$;
2. $\xi_0 = e^{i\theta_0}$ *is of type* $1_{b,L}$ if for $\varepsilon > 0$ sufficiently small, there are infinitely many solutions of $\Theta(\xi) = 1$ in the open interval $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$, finitely many solutions in $(e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})$, and $\lim_{\alpha \to \theta_0^+} \Theta(e^{i\alpha}) = L$;
3. $\xi_0 = e^{i\theta_0}$ *is of type* 2 if for all $\varepsilon > 0$ there are infinitely many solutions to $\Theta(\xi) = 1$ in both of the intervals $(e^{i(\theta_0-\varepsilon)}, e^{i\theta_0})$ and $(e^{i\theta_0}, e^{i(\theta_0+\varepsilon)})$.

In view of Remark 2.1 showing that the argument of the Blaschke product is strictly increasing at a point where B is analytic, we see that there is nothing special about the value 1 above; that is, if B has only finitely many solutions to $B(e^{i\theta}) = 1$ in an interval, it has only finitely many solutions to $B(e^{i\theta}) = \lambda$ for every $\lambda \in \mathbb{T}$.

We will soon show that these three cases exhaust all possibilities and we will give examples of Blaschke products satisfying each possible situation. It will, however, be convenient to have a way to refer to the corresponding intervals of each type. Thus, we have the following.

Definition 3.2. Given an inner function Θ, an interval $(e^{i\theta_0}, e^{i\theta_1})$, whose end points are consecutive singularities of Θ,

1. *is of type* 0 (*for* Θ) if there are only finitely many solutions to $\Theta(\xi) = 1$ in the interval;
2. *is of type* 1_a (*for* Θ) if there exist infinitely many solutions to $\Theta(\xi) = 1$ in the interval accumulating precisely at the point $e^{i\theta_0}$;
3. *is of type* 1_b (*for* Θ) if there exist infinitely many solutions to $\Theta(\xi) = 1$ in the interval accumulating precisely at the point $e^{i\theta_1}$;
4. *is of type* 2 (*for* Θ) if there are infinitely many solutions to $\Theta(\xi) = 1$ in the interval accumulating precisely at both $e^{i\theta_0}$ and $e^{i\theta_1}$.

It should be clear that the definition describes all possibilities for the behavior of an inner function with isolated singularities, but it may not be clear that each situation actually occurs. Before we discuss this further, we note that a theorem of R. Berman [1, Theorem 4.8] shows that given a nonempty set $E \subset \mathbb{T}$ with measure zero and of type F_σ and G_δ, there exists a Blaschke product B such that $B^{-1}(1) = E$. Thus, there are Blaschke products with singularities of each type. Some related results are given by Choike [6]. For discrete singular inner functions, E. Decker has an interesting related result [9, Lemma 6].

Writing our inner function Θ as $\Theta = BS$, where B is a Blaschke product and S is a singular inner function, we discuss the situation in which the zeros of the Blaschke product approach an isolated singularity through a Stolz region at the point $e^{i\theta_0}$; that is, the points are in a region of the form

$$\Gamma_\alpha = \{z \in \mathbb{D}\colon\ |e^{i\theta_0} - z| < \alpha(1 - |z|)\},$$

where $\alpha > 1$, as well as when the singular inner factor has an isolated singularity at $e^{i\theta_0}$.

Lemma 3.3. *Let $\Theta = BS$ be an inner function with finitely many singularities and let $e^{i\theta_0} \in \mathbb{T}$ be a point for which there exists a sequence of zeros of Θ converging to $e^{i\theta_0}$ in a Stolz angle or $e^{i\theta_0} \in \operatorname{supp}(\mu)$, where μ is the singular measure corresponding to S. Then $e^{i\theta_0}$ is of type* 2.

Proof. Let (a_n) denote the zero sequence of the Blaschke product B. Our argument will apply to points lying on either side of the singularity we consider, and therefore we consider only one side of the point. The other will follow in exactly the same way. Under our hypotheses, we know that either B has a singularity at the point $e^{i\theta_0}$, which we assume without loss of generality to be the point 1, or S has a singularity at the point 1, or both.

Thus, we may (without loss of generality) consider two cases: First, the case in which there are infinitely many a_n in a Stolz angle, Γ_α, at the point 1, and second, the case in which the measure corresponding to S has a singularity at the

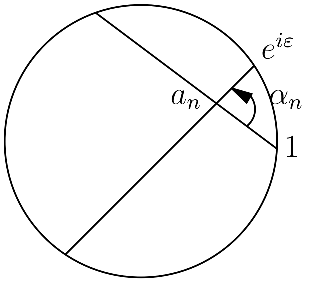

Fig. 1. The angle α_n.

point 1. Note that since the singularity of S is isolated, we may assume that the measure corresponding to S has an atom at the point 1.

So suppose that B has a singularity at the point 1 and infinitely many a_n lie in the Stolz angle. We may consider only those a_n that also lie in the right half-plane.

Recall that on the unit circle at a point $e^{i\theta}$ at which B is analytic, we have the identity (1). Thus, if we let ε be a small positive constant and γ denote the arc of the circle from 1 to $e^{i\varepsilon}$, we have

$$\left|(1/2\pi)\int_{\gamma}\frac{B'(z)}{B(z)}\,dz\right| = \left|\int_0^{\varepsilon}\frac{B'(e^{i\theta})}{e^{-i\theta}B(e^{i\theta})}\,d\theta/2\pi\right| = \int_0^{\varepsilon}\sum_n\frac{1-|a_n|^2}{|e^{i\theta}-a_n|^2}\,d\theta/2\pi. \tag{2}$$

Now [11, Exercise 3, p. 41],

$$\int_0^{\varepsilon}P_{a_n}(\theta)\,d\theta/2\pi = \alpha_n/\pi - \varepsilon/2\pi,$$

where $P_{a_n}(\theta)=\frac{1-|a_n|^2}{|e^{i\theta}-a_n|^2}$ is the Poisson kernel and

$$\alpha_n = \arg\bigl(\bigl(e^{i\varepsilon}-a_n\bigr)/(1-a_n)\bigr).$$

This is illustrated in Fig. 1.

If we choose a_n sufficiently close to 1, the fact that a_n are all in a Stolz angle implies that there exists $\beta>0$ such that $\alpha_n \geqslant (\beta+\varepsilon)/2$ for all such n. Therefore, the change in the argument of B is infinite, and there are infinitely many intervals approaching the point 1 from above that B wraps around the unit circle.

Now consider the case in which the singular inner function has an isolated singularity at the point 1. As noted above, if S has finitely many singularities including a singularity at the point 1, then the measure μ corresponding to S has an atom at the point 1; that is, $\mu=\sum_{k=1}^m \alpha_k\delta_k$ where δ_k is point mass at the singularity $e^{i\theta_k}$, and δ_1 is point mass at the point 1.

In this case, we may write $S=S_1S_2$, where S_2 is a singular inner function continuous at the point 1 and therefore has limit L at the point 1, where $|L|=1$. Thus, the behavior of S is determined by S_1 where $S_1=(S_{\delta_1})^{\alpha_1}=e^{-\alpha_1\frac{1+z}{1-z}}$, where $\alpha_1>0$. Now it is well known (see [17, Thm. VII.7.48]) that every (nonconstant) singular inner function S_μ has the property that for each $\lambda\in\mathbb{T}$ there exist infinitely many points $e^{i\theta_\lambda}$ for which the radial limit of S_μ satisfies $S_\mu^\star(e^{i\theta_\lambda})=\lambda$. Since (as follows from Remark 2.1) S will assume every value on the unit circle between these points, we see that they must cluster at the singularity. Thus, S assumes the value λ on a sequence converging to the point 1. Symmetry considerations show that, in fact, the point 1 must be of type 2.

The final case is that in which both functions are discontinuous at the point 1. In this case, the argument of Θ is the sum of the arguments of the two functions, and the result follows from the paragraphs above. □

Example 3.4. There exists a Blaschke product with a singularity of type 1_a on the unit circle.

Suppose 1 is a singularity of the Blaschke product B. Choose a sequence of points (a_n) in $\mathbb{D}$ approaching 1 tangentially from above in such a way that the angular derivative at the point 1 remains bounded. Any sequence (a_n) with $\lim_n a_n = 1$ and

$$\sum\frac{1-|a_n|^2}{|1-a_n|^2}<\infty$$

will work. For example, we may take a_n so that $n^2(1-|a_n|^2)<|1-a_n|^2$, and a_n lies in the upper-half disk. Then, for $-\varepsilon<t\leqslant 0$, we see that $|e^{it}-a_n|>|1-a_n|$ so

$$\sum \frac{1-|a_n|^2}{|e^{it}-a_n|^2} < \sum \frac{1-|a_n|^2}{|1-a_n|^2} < \infty.$$

Therefore the angular derivative is finite at 1 and bounded on the interval $(e^{it}, 1)$. From the version of Eq. (2) for the arc below the point 1, we conclude that the change in argument is bounded on this interval and therefore $B(z) = 1$ can have at most finitely many solutions on this interval. Since 1 is assumed to be a singularity, there must be infinitely many solutions converging to 1 from above. Therefore 1 is a point of type 1_a.

Such examples have appeared in other contexts. One can be obtained, for example, from the aforementioned work of Berman, or from the expression following (2.1) in Lemma 1 of Leung and Linden [14], since we can construct b so that the radial limit exists at 1. Then the estimate provided there shows that $\prod_{n=1}^{N} b(z, a_n)$ must be close to this limit for $z = r(\theta)e^{i\theta}$ where $r(\theta)$ is sufficiently large and $0 < \theta < \varphi_0$.

Lemma 3.5. *If Θ is an inner function with finitely many singularities, then there exists a Blaschke product with precisely the same singularities as Θ of precisely the same type.*

Proof. This is a consequence of Frostman's theorem [11, p. 79], for we may choose $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ where φ_a preserves the order of the points such that $\varphi_a \circ \Theta = B$ is Blaschke. The fact that φ_a is a continuous one-to-one mapping of the unit circle onto itself implies that the singularities remain at the same points and of the same type. □

Therefore, it remains to analyze the group of invariants for a Blaschke product. The general inner function will then follow in the same way.

4. Preliminary observations

In this section, we continue to consider a Blaschke product B with finitely many singularities on the unit circle. We will say that $x \in \mathcal{G}$ *shifts the elements* of an interval I if $x(z_{n,\lambda}) = z_{n+1,\lambda}$ for all n in $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Z}^-$ (depending on whether the intervals are of type 1 or type 2) and all $\lambda \in \mathbb{T}$, where $z_{n,\lambda}$ are the solutions to $B(z) = \lambda$ in I, ordered with increasing argument.

Lemma 4.1. *Let B be a Blaschke product with finitely many singularities and let $x \in \mathcal{G}$. Then x is a bijective mapping of the circle onto itself.*

Proof. We have already observed that x maps singular points to singular points and regular points to regular points. On each interval between two singular points the argument of B is strictly increasing (note that B cannot be constant on a set of positive measure), and thus this interval is mapped bijectively under x onto the whole of another such interval. By the continuity of x, successive intervals are mapped to successive intervals and since there are only finitely many, this establishes that x is a bijection of $\mathbb{T}$. □

Lemma 4.2. *Let I denote a type 2 interval and $x \in \mathcal{G}$ such that $x(I) = I$. If there exists $z \in I$ such that $x(z) = z$, then $x(w) = w$ for all $w \in I$. Hence if $x_1, x_2 \in \mathcal{G}$ with $x_j(I) = I$ for each $j = 1, 2$, and if there exists $z \in I$ with $x_1(z) = x_2(z)$, then $x_1(w) = x_2(w)$ for all $w \in I$.*

Proof. Suppose, without loss of generality, that $B(z) = 1$ and consider the two-sided sequence of points $z_{n,1} \in I$ such that $B(z_{n,1}) = 1$, where $z_{0,1} = z$. These divide I into open subarcs, $(z_{n,1}, z_{n+1,1})$. Now, being continuous and with strictly increasing argument on each such subarc, B takes all possible values in $\mathbb{T}$ except 1 exactly once. Moreover, x fixes $z_{0,1}$, and we now see that x, being bijective (see Lemma 4.1), maps each $z_{n,1}$ to itself and fixes the intermediate intervals. Clearly, the relation $B \circ x = B$ now implies that x fixes all points of I. Finally, if $x_1(z) = x_2(z)$, then $x_1^{-1}x_2$ fixes I and z, so fixes each point of I. □

Lemma 4.3. *Let I denote a type 1 interval or type 0 interval. If $x \in \mathcal{G}$ and $x(I) = I$, then $x(z) = z$ for all $z \in I$.*

Proof. We prove the case for type 1_b intervals; the other cases are similar.

We may label the solutions in I to $B(z) = 1$ in increasing order of argument as $z_{n,1}$ for $n \geqslant 0$ (only). Now, x can only act as a shift on these solutions, but since x is invertible it must fix the first point $z_{0,1}$ as otherwise either x or x^{-1} would have to map $z_{0,1}$ to a point in I of strictly smaller argument. Now considerations as in the proof of Lemma 4.2 show that x fixes every point of I. □

As a consequence we have:

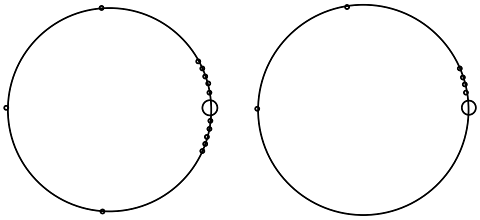

Fig. 2. Singularities of type 2, and type 1_a.

Lemma 4.4. *Let $x \in \mathcal{G}$. Then x maps intervals of types 0, 1_a, 1_b and 2 to intervals of exactly the same type. Moreover, in the case when an endpoint is of type $1_{a,L}$ or $1_{b,L}$, it is mapped under x to another endpoint of exactly the same type.*

Lemma 4.5. *Let I and J be intervals of the same type and suppose that $y_1, y_2 \in \mathcal{G}$ map I to J. If $y_1(z) = y_2(z)$ for some z in I, then $y_1 = y_2$ on I.*

Proof. In this case $y_2^{-1} y_1$ maps I to I and fixes a point of I; hence it fixes all points of I by Lemmas 4.2 and 4.3. □

We note that type 0 intervals can only be mapped to each other if their images under B are identical (including multiplicities of points).

5. 1, 2 and *p* singularities: a detailed description

Before analyzing the general case, we consider the cases when the number of singularities is 1 or 2, and then an arbitrary prime number. Here the number of possible symmetry groups is very limited, and it is possible to give detailed descriptions of the corresponding groups.

5.1. One singularity

Theorem 5.1. *If B is a Blaschke product with a singularity at the point $e^{i\theta_0}$ only, then we have the following*:

1. *If $e^{i\theta_0}$ is of type 2, then $\mathcal{G} = \mathbb{Z}$*;
2. *If $e^{i\theta_0}$ is of type 1_a or 1_b, the group is the trivial group*; *that is, $\mathcal{G} = \{e\}$.*

Proof. We refer to Fig. 2 at this point. Without loss of generality, we assume that $e^{i\theta_0} = 1$.

First suppose that 1 is a point of type 2. Note that B is continuously differentiable with nonvanishing derivative on the interval $I := \mathbb{T} \setminus \{1\}$, and for each $\lambda \in \mathbb{T}$ we may enumerate the points such that $B(z) = \lambda$ as a sequence $\{z_{n,\lambda}\}_{n\in\mathbb{Z}}$, taken with increasing argument, such that all points $z_{0,\lambda}$ lie in the arc between $z_{0,1}$ and $z_{1,1}$. Moreover, for each $\lambda \in \mathbb{T}$ we have $z_{n,\lambda} \to 1$ as $n \to \pm\infty$.

We may define a mapping $x_0 \in \mathcal{G}$ by $x_0(z_{n,\lambda}) = z_{n+1,\lambda}$ for each $n \in \mathbb{Z}$ and each $\lambda \in \mathbb{T}$, and $x_0(1) = 1$. From our definition, it is clear that the two functions $B \circ x_0$ and B are equal; also x_0 is (in argument) a monotonic increasing bijection from I to itself, and hence is continuous.

Now x_0 generates the entire group $\mathcal{G}$, for if $y \in \mathcal{G}$ then $y : z_{0,1} \mapsto z_{n,1}$ for some $n \in \mathbb{Z}$, and then $y = x_0^n$ by Lemma 4.2.

If 1 is a point of type 1, then necessarily I is an interval of type 1, and the group is trivial, by Lemma 4.3. □

The complications begin with the next case.

5.2. Two singularities

Theorem 5.2. *Suppose that B is a Blaschke product with spectrum consisting of exactly two points of $\mathbb{T}$. Then, depending on the nature of the two intervals comprising $\mathbb{T} \setminus \sigma(B)$, the group $\mathcal{G}$ is one of the following*: $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$. *The last of these cases is a semi-direct product, which may be presented as*

$$\mathcal{G} = \langle x_1, x_2, y\colon\ y^2 = e,\ x_1 x_2 = x_2 x_1,\ x_2 y = y x_1 \rangle.$$

We refer to [15], for example, for more on semi-direct products. Recall that if $\mathcal{G} \cong K \rtimes \mathcal{S}$, then $\mathcal{G}$ has a normal subgroup isomorphic to K with quotient $\mathcal{G}/K \cong \mathcal{S}$. These do not determine $\mathcal{G}$ uniquely, but a presentation in terms of generators and relations can be given.

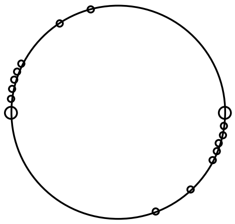

Fig. 3. The case where y interchanges I_1 and I_2.

Proof. We may suppose, without loss of generality, that B is a Blaschke product with singularities at 1 and -1. Call the resulting intervals I_1 and I_2, where I_1 denotes the upper-half of the unit circle.

Note first that $\mathcal{G}$ contains a normal subgroup

$$K := \{x \in \mathcal{G}\colon\ x(I_k) = I_k \text{ for each } k\}.$$

Indeed, we may define a homomorphism $\psi : \mathcal{G} \to \mathbb{Z}_2 = \{[0], [1]\}$ by defining $\psi(x) = [r]$ if $x(I_k) = I_{k+r}$ for each k (here intervals are numbered modulo 2). Clearly any $x \in \mathcal{G}$ either fixes both intervals I_k or it interchanges them. Now $\mathcal{G}/K \cong \psi(\mathcal{G})$, which is either $\{e\}$ or $\mathbb{Z}_2$. We consider three cases.

Case 1. Both intervals are of type 2.
Case 2. One interval is type 1 (or type 0) and one interval is type 2.
Case 3. Both intervals are type 1.

Case 1. (Both intervals type 2.) From the proof of Theorem 5.1, we know that there exists $x_j \in \mathcal{G}$ such that x_j shifts the elements of I_j (from $z_{n,\lambda}$ to $z_{n+1,\lambda}$) for $j = 1, 2$ and each x_j leaves the elements of the remaining interval fixed. Again as in the case of one interval, we may argue that x_j is continuous and it follows that $x_j(1) = 1$ and $x_j(-1) = -1$. As above, we see from this that $B(x_j(e^{i\theta})) = B(e^{i\theta})$ for $\theta \neq 0, \pi$.

Now consider the point i. Since i is not a singularity, the radial limit of B exists at $i \in I_1$ and we may write $B(i) = \lambda$. Choose a point, $z_- \in I_2$ such that $B(z_-) = \lambda$. We claim that there is a unique element $y \in \mathcal{G}$ such that y interchanges i and z_-: for on I_1 we may define $y(x_1^n(i)) = x_2^n(z_-)$, and define the image under y of the remaining subintervals of I_1 in the unique way that guarantees that y is continuous and $B \circ y = B$. Similarly, we define $y(x_2^n(z_-)) = x_1^n(i)$, and fill in the values on I_2 in the same way. Note that $y^2 = e$.

We claim that

$$\mathcal{G} = \langle x_1, x_2, y\colon\ y^2 = e,\ x_1x_2 = x_2x_1,\ x_2y = yx_1\rangle.$$

Our argument above shows that the right-hand side is a subset of $\mathcal{G}$. Now choose an element $x \in \mathcal{G}$.

Note that we have various possibilities:

- $x = e$;
- $x|I_1 = e$, $x|I_2 \neq e$. In this case $x = x_2^n$ for some $n \in \mathbb{Z}$. The case $x|I_1 \neq e$, $x|I_2 = e$ is handled in the same way.
- $x(I_j) = I_j$ and $x|I_j \neq e$ for $j = 1, 2$. Then $x = x_1^n \circ x_2^m$ for $n, m \in \mathbb{Z} \setminus \{0\}$.
- $x(I_1) = I_2$. In this case $x \circ y(I_1) = I_1$ and we obtain the result from the previous cases.

This completes the proof in which we have two intervals of type 2.

Case 2. (One interval type 2, one not.) In this case one interval, say I_1, is type 1 (or type 0) and I_2 is type 2. By Lemma 4.5 we cannot map I_1 to I_2, and therefore by Lemma 4.3 every element of $\mathcal{G}$ must fix the points of I_1. Let x_0 denote the shift of elements in I_2. Then

$$G = \langle x_0 \rangle \cong \mathbb{Z}.$$

Case 3. (Neither interval type 2.) In this case, B has a singularity at 1 and -1. If there were a type 0 interval, one interval would have to be type 2. Thus I_1 and I_2 are type 1 intervals. Further, since there are no type 2 intervals, each singularity is a point of the same type. Thus, the Blaschke product has one-sided limits at each singularity. We have several cases:

Case 3a: The one-sided limits exist and are not equal. Without loss of generality, we assume that $\lim_{\theta \to \pi^+} B(e^{i\theta}) \neq \lim_{\theta \to 0^+} B(e^{i\theta})$. By Lemma 4.3, the only maps $x \in \mathcal{G}$ for which $x(I_j) = I_j$ are the identity. The only other possibility is that there exists $y \in G$ such that y interchanges I_1 and I_2 (see Fig. 3). In this case, since the points where B assumes the value 1 are mapped to each other, we must map -1 to 1. However, for $\theta \neq 0, \pi$ we have $B(y(e^{i\theta})) = B(e^{i\theta})$ with different one-sided limits at 1 and -1, and y is continuous at 1 and -1, so this is impossible and the group must be the trivial group.

Case 3b: The one-sided limits exist and are equal. Again, we assume without loss of generality that $\lim_{\theta\to\pi^+} B(e^{i\theta}) = \lim_{\theta\to 0^+} B(e^{i\theta})$. In this case, there is an element y of order 2 that maps the sequence of points $\{z_{n,\lambda,j}\} \in I_j$ for which $B(z_{n,\lambda,j}) = \lambda$ to the corresponding points in the other interval. Our map is $y(z_{n,\lambda,1}) = z_{n,\lambda,2}$ and our group is $\mathbb{Z}_2$. □

It is clear from the previous discussions that each of the listed subgroups can occur.

5.3. Prime number of singularities

In what follows, we let $K = \{y \in \mathcal{G}\colon\ y(I_k) = I_k \text{ for all } k\}$. Define $\psi : \mathcal{G} \to \mathbb{Z}_n$ as follows: for $y \in \mathcal{G}$ such that $y(I_j) = I_{j+r} \pmod n$, let $\psi(y) = [r]$. Note that $K = \ker\psi$, $K \cong \mathbb{Z}^k$ for some k with $0 \leqslant k \leqslant n$ and that k is the number of intervals of type 2.

Theorem 5.3. *Suppose that B is a Blaschke product with spectrum consisting of exactly p points of $\mathbb{T}$, where $p \geqslant 2$ is a prime number. Then, depending on the nature of the intervals comprising $\mathbb{T} \setminus \sigma(B)$, the group $\mathcal{G}$ is one of the following*: $\{e\}$, $\mathbb{Z}_p$, $\mathbb{Z}^r$ *for some* $1 \leqslant r < p$, $\mathbb{Z}^p \rtimes \mathbb{Z}_p$. *The last of these cases is a semi-direct product, which may be presented as*

$$\mathcal{G} = \langle x_1, x_2, \ldots, x_p, y\colon\ y^p = e,\ x_j x_k = x_k x_j,\ \textit{and}\ y x_j = x_{j+1} y \pmod p \rangle.$$

Proof. We suppose that the p singularities of B divide the circle into intervals $I_1, \ldots, I_p$. Note that if $\psi : \mathcal{G} \to \mathbb{Z}_p$ is the map defined in the paragraph preceding the proof of Theorem 5.3, then $\psi(\mathcal{G})$ can be only one of two groups: the trivial group $\{e\}$ or the group $\mathbb{Z}_p$ and $\mathcal{G}/K \cong \mathcal{S}$ where S is a subgroup of $\mathbb{Z}_p$. Thus, $\mathcal{G}/K \cong \{e\}$ or $\mathcal{G}/K \cong \mathbb{Z}_p$. Since we already know K, this gives us a description of $\mathcal{G}/K$, but we give a different way to visualize the group below.

We suppose that $\mathcal{G}$ is nontrivial. (The trivial case can happen if each interval is of type 1, but the singularities are not all of the same type, or we have intervals of type 1 and type 0.)

Case 1. In case $\psi(\mathcal{G}) = \mathbb{Z}_p$, we have $\mathcal{G}/K \cong \mathbb{Z}_p$. If we choose an element $y \in \mathcal{G}$ that is not mapped to the identity, then y must have order p. Therefore, we may assume that $y(I_j) = I_{j+1}$. Consequently, every interval is of the same type. There are three cases to consider: each interval of type 2, each of type 1 and each of type 0.

Now, we cannot have every interval of type 0, for then there are no singularities.

So suppose every interval is type 2. Then K is nontrivial and we have shifts $x_1, \ldots, x_p$ corresponding to intervals I_j, for $j = 1, \ldots, p$, and

$$\mathcal{G} = \langle x_1, x_2, \ldots, x_p, y\colon\ y^p = e,\ x_j x_k = x_k x_j\ \forall j,k,\ \text{and}\ y x_j = x_{j+1} y \pmod p \rangle.$$

The remaining case is the case in which every interval is type 1 (with B having equal one-sided limits). In this case, K is trivial and

$$\mathcal{G} = \langle y\colon\ y^p = e \rangle \cong \mathbb{Z}_p.$$

Case 2. In case $\psi(\mathcal{G}) = \{e\}$, there is no nontrivial rotation. Note that because there are singularities, not all intervals are of type 0. Thus, we know that at least two intervals are of different types. Consequently, we must leave all intervals of type 0 and type 1 fixed and we may shift intervals of type 2 only. Thus, if there are k intervals of type 2 we have k generators that shift the points of the corresponding intervals of type 2, and we have no rotations. Clearly $k < p$. Therefore $\mathcal{G} \cong \mathbb{Z}^k$. □

6. General case

We use the same notation as above for the case in which we have n singularities: $K = \{y \in \mathcal{G}\colon\ y(I_j) = I_j \text{ for all } j\}$. From the discussion in the preceding sections it is clear that in the group of invariants, there are only certain possibilities. First, an element $x \in \mathcal{G}$ may map every interval to itself. If this is the case, then $x \in K$. If $x \notin K$, then x maps some interval I_j to I_k where $j \neq k$. As we have seen, x can be written in terms of elements that shift the points of a specific interval and leave all other intervals fixed, and elements that move one interval to another.

Recall that for $y \in \mathcal{G}$ satisfying $y(I_j) = I_{j+r} \pmod n$ we defined $\psi(y) = [r]$. Also recall that $K = \ker\psi$, $K \cong \mathbb{Z}^k$ for some k with $0 \leqslant k \leqslant n$ and that k is the number of intervals of type 2.

We begin with a brief discussion of the case of four singularities, which will help with an understanding of the general (composite n) case.

Theorem 6.1. *Suppose that the Blaschke product B has exactly 4 singularities on $\mathbb{T}$. Then the group $\mathcal{G}$ is one of the following possibilities*: $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}^k$ ($k = 1$, 2 *or* 3), $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$, $\mathbb{Z}^4 \rtimes \mathbb{Z}_4$.

Proof. As above, we note that $\mathcal{G}$ has a normal subgroup K isomorphic to $\mathbb{Z}^k$ for some $0 \leqslant k \leqslant 4$, with quotient $\mathcal{G}/K$ isomorphic to either $\{e\}$, $\mathbb{Z}_2$ or $\mathbb{Z}_4$. Which cases can occur now depends on the types of the intervals $I_1, \ldots, I_4$ comprising $\mathbb{T} \setminus \sigma(B)$.

Case 1. All intervals type 2. Then we have four shifts, x, y, z, w, and we can rotate with r, where $r^4 = e$. In this case, the group is

$$\langle x, y, z, w, r\colon r^4 = e,\ xy = yx,\ \ldots,\ zw = wz;\ rw = xr,\ \ldots\rangle \cong \mathbb{Z}^4 \rtimes \mathbb{Z}_4.$$

Case 2. One interval type 1, three type 2. Then $G = \mathbb{Z}^3$ and there is no rotation.
Case 3. Three intervals type 1, one type 2. Then $G = \mathbb{Z}$ and there is no rotation.
Case 4. Two intervals type 2, two intervals type $1_{a,L}$ (or two intervals type $1_{b,L}$): Then to get a nontrivial quotient, we must have the type 1 intervals alternating with the type 2 intervals, and agreement of the one-sided limits at the points of type 1. In this case, we have G described by

$$\langle x, y, r\colon r^2 = e,\ rx = yr,\ ry = xr,\ xy = yx\rangle \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_2.$$

The other possibilities (two type 2 and two type 1 that are not alternating, and cases including type 0 intervals) can be handled in the same way. □

We can summarize the general result as follows, noting that some of the cases described below may not occur. For example, it is not possible to have $k = 3$ and $d = 2$ in the following theorem.

Theorem 6.2. *Suppose that B has exactly n singularities on $\mathbb{T}$. Then $\mathcal{G} \cong \mathbb{Z}^k \rtimes \mathbb{Z}_d$ for some $0 \leqslant k \leqslant n$ and $1 \leqslant d \leqslant n$ with $d|n$. Here k denotes the number of intervals of type 2, and d the number of cyclic permutations of the n intervals comprising $\mathbb{T} \setminus \sigma(B)$ that map each interval onto another of the same type.*

Proof. Say that the number of intervals of type 2 is $k \in \{0, \ldots, n\}$. So $K \cong \mathbb{Z}^k$. Consider the map $\psi : \mathcal{G} \to \mathbb{Z}_n$. Then ψ maps $\mathcal{G}$ onto a subgroup $\mathcal{S}$ of $\mathbb{Z}_n$. Therefore $\mathcal{G}/K \cong \mathcal{S}$, where $\mathcal{S}$ is a subgroup of $\mathbb{Z}_n$; in other words, $\mathcal{G}/K$ is $\mathbb{Z}_d$ where $d|n$. Then, letting $x_1, \ldots, x_k$ denote the shifts of the corresponding interval of type 2 that leave all other intervals fixed, we have

$$\mathcal{G} = \langle x_j, y\colon\ j = 1, \ldots, k;\ y^d = e,\ yx_j = x_{j+n/d}\, y\ \forall j,\ x_j x_l = x_l x_j\ \forall j, l\rangle \cong \mathbb{Z}^k \rtimes \mathbb{Z}_d. \qquad \square$$

An application. Suppose that B is a Blaschke product with finitely many singularities $\lambda_1, \lambda_2, \ldots, \lambda_m$ and the zeros of B lie in a nontangential region at the point λ_j for each j, then the group of invariants is $\mathbb{Z}^m \rtimes \mathbb{Z}_m$.

Proof. By Lemma 3.3 we know that every interval is a type 2 interval. Therefore, letting $x_1, \ldots, x_m$ be the corresponding shifts of each interval and y the map that moves I_j to I_{j+1} in such a way that $y(z_{n,\lambda,j}) = z_{n,\lambda,j+1}$, then

$$\mathcal{G} = \langle x_j, y\colon\ j = 1, \ldots, m;\ y^m = e, yx_j = x_{j+1} y,\ x_j x_l = x_l x_j\ \forall j\rangle. \qquad \square$$

Acknowledgment

We wish to thank the referee for several helpful suggestions.

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