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Von Neumann–Jordan Constant of Absolute Normalized Norms on $\mathbb{C}^2$

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We determine and estimate the von Neumann–Jordan constant of absolute normalized norms on $\mathbb{C}^2$ by means of their corresponding continuous convex functions on [0, 1]. This provides many new interesting examples including those of non-ℓ_p-type as well as some previous ones. It is also shown that all such norms are uniformly non-square except ℓ_1- and ℓ_∞-norms. © 2000 Academic Press

Key Words: absolute normalized norm; convex function; non-ℓ_p-type norm; von Neumann–Jordan constant; uniform non-squareness.

1. INTRODUCTION

The notion of the von Neumann–Jordan constant of Banach spaces (hereafter referred to as NJ constant) was introduced by Clarkson in [5]

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and recently it has been studied by several authors (cf. [5, 6, 8–11, 13], etc.). The NJ constant $C_{\mathrm{NJ}}(X)$ of a Banach space X is the smallest constant C for which

$$\frac{1}{C} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2 + \|y\|^2\right)} \le C$$

holds for all $x, y \in X$, not both 0. From Jordan and von Neumann [6], we have $1 \le C_{\mathrm{NJ}}(X) \le 2$ for any Banach space X, and X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$. Clarkson [5] calculated the NJ constant of L_p by using Clarkson's inequalities. Recently, following his way, it was determined for a sequence of other Banach spaces such as $L_p(L_q)$ (L_q-valued L_p-space), $W_p^k(\Omega)$ (Sobolev space), c_p (Schatten p-class operators), and so on ([9, 11], etc.). On the other hand, thanks to the NJ constant we can describe some geometrical and topological structures of Banach spaces. For example, the second and third authors proved that $C_{\mathrm{NJ}}(X) < 2$ if and only if X is uniformly non-square, whence X is super-reflexive if and only if X admits an equivalent norm with NJ constant less than 2 ([10, 13]). We also have that, if $C_{\mathrm{NJ}}(X) < 5/4$, then the Banach space X has the fixed point property for nonexpansive mappings (cf. [8]). For some other results concerning Rademacher type and cotype we refer the reader to [10].

A norm $\|\cdot\|$ on $\mathbb{C}^2$ is said to be *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z, w \in \mathbb{C}$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. Let N_a denote the family of all absolute normalized norms on $\mathbb{C}^2$, and let Ψ denote the family of all continuous convex functions on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. Then as in Bonsall and Duncan [3, Section 21, Lemma 3], N_a and Ψ are in one-to-one correspondence under the equation $\psi(t) = \|(1-t, t)\|$ $(0 \le t \le 1)$. In particular, owing to this we can consider many non-l_p-type norms easily.

In this paper, we shall determine and estimate the NJ constant of absolute normalized norms on $\mathbb{C}^2$ by means of the corresponding convex functions. In particular, this provides a new way to calculate it with no use of Clarkson's inequalities. The main results are stated as follows. Let $\|\cdot\|_\psi$ be an absolute normalized norm associated with a convex function $\psi \in \Psi$. Let $M_1 = \max_{0 \le t \le 1} \psi(t)/\psi_2(t)$ and $M_2 = \max_{0 \le t \le 1} \psi_2(t)/\psi(t)$, respectively, where $\psi_2(t) := \{(1-t)^2 + t^2\}^{1/2}$ corresponds to the ℓ_2-norm. First we show that, if $\psi \ge \psi_2$ (resp. $\psi \le \psi_2$), then $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = M_1^2$ (resp. M_2^2) (Theorem 1). In general, we prove that $\max\{M_1^2, M_2^2\} \le C_{\mathrm{NJ}}(\|\cdot\|_\psi) \le M_1^2 M_2^2$ (Theorem 2). Theorem 1 gives a class of convex functions for which $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = \max\{M_1^2, M_2^2\}$. We further present a sufficient condition that $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = M_1^2 M_2^2$ $(\max\{M_1^2, M_2^2\} < C_{\mathrm{NJ}}(\|\cdot\|_\psi))$ (Theorem 3) and that $C_{\mathrm{NJ}}(\|\cdot\|_\psi) < M_1^2 M_2^2$ (Theorem 4), respectively. These results enable us to present many new interesting examples, especially those of non-ℓ_p-type, for instance, $\ell_{p,2}$-norm, $p \geq 2$ (Lorentz norm), etc. As a corollary we show that all absolute normalized norms are uniformly non-square except ℓ_1- and ℓ_∞-norms.

2. ABSOLUTE NORMALIZED NORMS ON $\mathbb{C}^2$

A norm $\|\cdot\|$ on $\mathbb{C}^2$ is said to be *absolute* if

$$\|(z,w)\| = \|(|z|,|w|)\| \qquad \text{for all } z,w \in \mathbb{C}$$

and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ $(1 \leq p \leq \infty)$ are basic examples,

$$\|(z,w)\|_p = \begin{cases} (|z|^p + |w|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max(|z|,|w|) & \text{if } p = \infty. \end{cases}$$

Let N_a denote the family of absolute normalized norms on $\mathbb{C}^2$. We recall some basic facts about these norms; for the convenience of the reader we give their proofs following Bonsall and Duncan [3].

LEMMA 1 ([3, p. 36]). *For any norm* $\|\cdot\| \in N_a$

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1. \tag{1}$$

Indeed, for any $z,w \in \mathbb{C}$

$$\begin{aligned} \|(z,w)\|_\infty &= \max\{\|(z,0)\|, \|(0,w)\|\} \\ &= \tfrac{1}{2}\max\{\|(z,w) + (z,-w)\|, \|(z,w) + (-z,w)\|\} \\ &\leq \tfrac{1}{2}\max\{\|(z,w)\| + \|(z,-w)\|, \|(z,w)\| + \|(-z,w)\|\} \\ &= \|(z,w)\| \\ &\leq \|(z,0)\| + \|(0,w)\| \\ &= \|(z,w)\|_1. \end{aligned}$$

Now let Ψ denote the family of all continuous convex functions ψ on $[0,1]$ with $\psi(0) = \psi(1) = 1$ satisfying

$$\max\{1-t, t\} \leq \psi(t) \leq 1 \qquad (0 \leq t \leq 1). \tag{2}$$

Then N_a and Ψ are in one-to-one correspondence as follows.

LEMMA 2 ([3, p. 37]). (i) *Let* $\|\cdot\| \in N_a$ *and let*

$$\psi(t) = \|(1-t, t)\| \qquad (0 \leq t \leq 1). \tag{3}$$

Then $\psi \in \Psi$: *Conversely*,

(ii) *For a given* $\psi \in \Psi$ *define*

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\,\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{4}$$

Then $\|\cdot\|_\psi \in N_a$, *and* $\|\cdot\|_\psi$ *satisfies* (3).

Proof. (i) This is easy to see (ψ satisfies (2) by Lemma 1).

(ii) Let $\psi \in \Psi$. We only show the triangle inequality. Let us first see that

$$\|(p,q)\|_\psi \leq \|(r,s)\|_\psi \qquad \text{if } 0 \leq p \leq r,\ 0 \leq q \leq s. \tag{5}$$

When $p = 0$ or $q = 0$, (5) is clear (recall Lemma 1). Thus it is enough to show that

$$(p+q)\,\psi\left(\frac{q}{p+q}\right) \leq (r+s)\,\psi\left(\frac{s}{r+s}\right) \qquad \text{if } 0 < p \leq r,\ 0 < q \leq s. \tag{6}$$

Since ψ is convex and $\psi(t) \geq t$, the function $\psi(t)/t$ is non-increasing. Indeed, let $0 < s < t \leq 1$. Then

$$\psi(t) = \psi\left(\frac{1-t}{1-s}s + \frac{t-s}{1-s}1\right) \leq \frac{1-t}{1-s}\psi(s) + \frac{t-s}{1-s}\psi(1).$$

Hence

$$\begin{aligned} \frac{\psi(s)}{s} - \frac{\psi(t)}{t} &\geq \frac{\psi(s)}{s} - \frac{1}{t}\left\{\frac{1-t}{1-s}\psi(s) + \frac{t-s}{1-s}\right\} \\ &\geq \psi(s)\left\{\frac{1}{s} - \frac{1-t}{t(1-s)}\right\} - \frac{t-s}{t(1-s)} \\ &\geq s\left\{\frac{1}{s} - \frac{1-t}{t(1-s)}\right\} - \frac{t-s}{t(1-s)} = 0. \end{aligned}$$

Therefore we have

$$(p+q)\,\psi\left(\frac{q}{p+q}\right) \leq (r+q)\,\psi\left(\frac{q}{r+q}\right). \tag{7}$$

In the same way the function $\psi(t)/(1-t)$ is non-decreasing (use $\psi(t) \geq 1-t$ in this case), which implies

$$(r+q)\psi\left(\frac{q}{r+q}\right) \leq (r+s)\psi\left(\frac{s}{r+s}\right). \tag{8}$$

Combining (7) and (8), we have (6). Now let $(u,v),(z,w) \in \mathbb{C}^2$. Then by (5)

$$\begin{aligned}\|(u,v)+(z,w)\|_\psi &= \|(|u+z|,|v+w|)\|_\psi \\ &\leq \|(|u|+|z|,|v|+|w|)\|_\psi \\ &= (|u|+|z|+|v|+|w|)\,\psi\left(\frac{|v|+|w|}{|u|+|z|+|v|+|w|}\right).\end{aligned}$$

Noting here that

$$\begin{aligned}&\frac{|v|+|w|}{|u|+|v|+|z|+|w|} \\ &\quad= \frac{|u|+|v|}{|u|+|v|+|z|+|w|}\cdot\frac{|v|}{|u|+|v|} + \frac{|z|+|w|}{|u|+|v|+|z|+|w|}\cdot\frac{|w|}{|z|+|w|},\end{aligned}$$

we have by the convexity of ψ

$$\begin{aligned}\|(u,v)+(z,w)\|_\psi &\leq (|u|+|v|)\,\psi\left(\frac{|v|}{|u|+|v|}\right) + (|z|+|w|)\,\psi\left(\frac{|w|}{|z|+|w|}\right) \\ &= \|(u,v)\|_\psi = \|(z,w)\|_\psi,\end{aligned}$$

as desired.

Now let $\psi_p(t) = \{(1-t)^p + t^p\}^{1/p} \in \Psi$. Then, as is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with $\psi_p \in N_a$. In what follows we write $\varphi \leq \psi$ if $\varphi(t) \leq \psi(t)$ for all $0 \leq t \leq 1$. We shall need the following simple facts later.

LEMMA 3. *Let $\varphi, \psi \in \Psi$ and let $\varphi \leq \psi$. Put*

$$M = \max_{0 \leq t \leq 1} \frac{\psi(t)}{\varphi(t)}.$$

Then

$$\|\cdot\|_\varphi \leq \|\cdot\|_\psi \leq M\|\cdot\|_\varphi.$$

Proof. For any $z, w \in \mathbb{C}$

$$\begin{aligned}\|(z,w)\|_{\varphi} &= (|z|+|w|)\,\varphi\left(\frac{|w|}{|z|+|w|}\right)\\ &\le (|z|+|w|)\,\psi\left(\frac{|w|}{|z|+|w|}\right)\\ &= \|(z,w)\|_{\psi}\\ &\le M(|z|+|w|)\,\varphi\left(\frac{|w|}{|z|+|w|}\right)\\ &= M\|(z,w)\|_{\varphi}.\end{aligned}$$

LEMMA 4. *Let* $\varphi, \psi \in \Psi$ *and let* $1/2 \le \lambda \le 1$. *Then*

$$\max\{\|\cdot\|_{\varphi}, \lambda\|\cdot\|_{\psi}\} = \|\cdot\|_{\max\{\varphi, \lambda\psi\}}.$$

Proof. Note first that $\max\{\varphi, \lambda\psi\} \in \Psi$. Then for any $(z,w) \in \mathbb{C}^2$

$$\begin{aligned}\|(z,w)\|_{\max\{\varphi,\lambda\psi\}} &= (|z|+|w|)\max\left\{\varphi\left(\frac{|w|}{|z|+|w|}\right), \lambda\psi\left(\frac{|w|}{|z|+|w|}\right)\right\}\\ &= \max\{\|(z,w)\|_{\varphi}, \lambda\|(z,w)\|_{\psi}\}.\end{aligned}$$

3. NJ CONSTANT OF ABSOLUTE NORMALIZED NORMS— THE COMPARABLE CASE WITH ψ_2

The *von Neumann–Jordan constant* of a Banach (or normed) space X ([5]; cf. [12, p. 550]), we denote by $C_{\mathrm{NJ}}(X)$, is the smallest constant C for which

$$\frac{1}{C} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \le C$$

holds for all $x, y \in X$, not both 0.

Let us recall some geometrical notions of a Banach space X (cf. [1]). X or its norm $\|\cdot\|$ is called *uniformly convex* if for any $\varepsilon > 0$ $(0 < \varepsilon < 2)$ there exists a $\delta > 0$ such that $\|x-y\| \ge \varepsilon$, $\|x\| \le 1$, $\|y\| \le 1$ implies $\|(x+y)/2\| \le 1-\delta$. X is called *uniformly non-square* provided there exists a $\delta > 0$ such that if $\|(x-y)/2\| \ge 1-\delta$, $\|x\| \le 1$ and $\|y\| \le 1$, then $\|(x+y)/2\| \le 1-\delta$. Clearly uniformly convex spaces are uniformly non-square, for the converse uniform non-squareness does not even imply strict convexity, whereas X admits an equivalent uniformly non-square norm if and only if X is uniformly convexifiable (such a Banach space is precisely super-reflexive).

We summarize some basic facts about the NJ constant.

PROPOSITION A. (i) $1 \le C_{\mathrm{NJ}}(X) \le 2$ *for any Banach space* X; $C_{\mathrm{NJ}}(X) = 1$ *if and only if* X *is a Hilbert space* (*Jordan and von Neumann* [6]).

(ii) $C_{\mathrm{NJ}}(X) < 2$ *if and only if* X *is uniformly non-square* (*Takahashi and Kato* [13]; *see also* [10]).

(iii) $C_{\mathrm{NJ}}(L_p) = C_{\mathrm{NJ}}(\ell_p) = 2^{(2/t)-1}$, *where* $1 \le p \le \infty$, $1/p + 1/p' = 1$, *and* $t = \min\{p, p'\}$ (*Clarkson* [5]).

For a norm $\|\cdot\|$ on $\mathbb{C}^2$ we write $C_{\mathrm{NJ}}(\|\cdot\|)$ for $C_{\mathrm{NJ}}((\mathbb{C}^2, \|\cdot\|))$. We first see that the NJ constant is stable under the symmetric transformation of ψ with respect to the line $t = 1/2$.

PROPOSITION 1. *Let* $\psi \in \Psi$ *and let* $\tilde{\psi}(t) = \psi(1-t)$. *Then* $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = C_{\mathrm{NJ}}(\|\cdot\|_{\tilde{\psi}})$.

Proof. For $x = (z, w) \in \mathbb{C}^2$ put $\tilde{x} = (w, z)$. Then

$$\|x\|_\psi = (|z| + |w|)\,\psi\left(\frac{|w|}{|z| + |w|}\right) = (|w| + |z|)\,\tilde{\psi}\left(\frac{|z|}{|w| + |z|}\right) = \|\tilde{x}\|_{\tilde{\psi}}.$$

Therefore we have

$$\begin{aligned} C_{\mathrm{NJ}}\big(\|\cdot\|_\psi\big) &= \sup_{\|x\|_\psi^2 + \|y\|_\psi^2 \neq 0} \frac{\|x+y\|_\psi^2 + \|x-y\|_\psi^2}{2\big(\|x\|_\psi^2 + \|y\|_\psi^2\big)} \\ &= \sup_{\|\tilde{x}\|_{\tilde{\psi}}^2 + \|\tilde{y}\|_{\tilde{\psi}}^2 \neq 0} \frac{\|\tilde{x}+\tilde{y}\|_{\tilde{\psi}}^2 + \|\tilde{x}-\tilde{y}\|_{\tilde{\psi}}^2}{2\big(\|\tilde{x}\|_{\tilde{\psi}}^2 + \|\tilde{y}\|_{\tilde{\psi}}^2\big)} \\ &= C_{\mathrm{NJ}}\big(\|\cdot\|_{\tilde{\psi}}\big). \end{aligned}$$

THEOREM 1. *Let* $\psi \in \Psi$.

(i) *Assume that* $\psi \ge \psi_2$. *Then*

$$C_{\mathrm{NJ}}\big(\|\cdot\|_\psi\big) = \max_{0 \le t \le 1} \frac{\psi(t)^2}{\psi_2(t)^2}. \tag{9}$$

(ii) *Assume that* $\psi \le \psi_2$. *Then*

$$C_{\mathrm{NJ}}\big(\|\cdot\|_\psi\big) = \max_{0 \le t \le 1} \frac{\psi_2(t)^2}{\psi(t)^2}. \tag{10}$$

Proof. (i) Put $M_1 = \max_{0 \le t \le 1} \psi(t)/\psi_2(t)$. Then by Lemma 3

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\le M_1^2\left(\|x+y\|_2^2 + \|x-y\|_2^2\right)\\ &= 2M_1^2\left(\|x\|_2^2 + \|y\|_2^2\right)\\ &\le 2M_1^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right).\end{aligned}$$

Now let ψ/ψ_2 attain the maximum at $t = t_1$ $(0 \le t_1 \le 1)$. Put $x_1 = (1 - t_1, 0)$, $y_1 = (0, t_1)$. Then

$$\begin{aligned}\|x_1+y_1\|_\psi^2 + \|x_1-y_1\|_\psi^2 &= \|(1-t_1, t_1)\|_\psi^2 + \|(1-t_1, -t_1)\|_\psi^2\\ &= 2\psi(t_1)^2\\ &= 2M_1^2\left\{(1-t_1)^2 + t_1^2\right\}\\ &= 2M_1^2\left(\|x_1\|_\psi^2 + \|y_1\|_\psi^2\right), \end{aligned} \tag{11}$$

which implies (9).

(ii) Put $M_2 = \max_{0 \le t \le 1} \psi_2(t)/\psi(t)$. Then

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\le \|x+y\|_2^2 + \|x-y\|_2^2\\ &= 2\left(\|x\|_2^2 + \|y\|_2^2\right)\\ &\le 2M_2^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right).\end{aligned}$$

Assume $M_2 = \psi_2(t_2)/\psi(t_2)$ with some t_2 $(0 \le t_2 \le 1)$. Put $x_2 = (1 - t_2, t_2)$, $y_2 = (1 - t_2, -t_2)$. Then

$$\begin{aligned}\|x_2+y_2\|_\psi^2 + \|x_2-y_2\|_\psi^2 &= 4\left\{(1-t_2)^2 + t_2^2\right\}\\ &= 4M_2^2\psi(t_2)^2\\ &= 2M_2^2\left\{\|(1-t_2, t_2)\|_\psi^2 + \|(1-t_2, -t_2)\|_\psi^2\right\}\\ &= 2M_2^2\left(\|x_2\|_\psi^2 + \|y_2\|_\psi^2\right), \end{aligned} \tag{12}$$

whence we have (10). This completes the proof.

Theorem 1 indicates that the NJ constant of $\|\cdot\|_\psi$ does not depend on the shape of ψ. This is stated in a little more general form:

COROLLARY 1. *Let $\varphi, \psi \in \Psi$ be comparable with ψ_2.*

(i) *Let $\varphi \ge \psi_2$ and $\psi \ge \psi_2$. Then*

$$C_{NJ}\left(\|\cdot\|_\varphi\right) = C_{NJ}\left(\|\cdot\|_\psi\right) \tag{13}$$

if and only if

$$\max_{0 \le t \le 1} \frac{\varphi(t)}{\psi_2(t)} = \max_{0 \le t \le 1} \frac{\psi(t)}{\psi_2(t)}.$$

(ii) *Let* $\varphi \ge \psi_2$ *and* $\psi \le \psi_2$. *Then* (13) *holds if and only if*

$$\max_{0 \le t \le 1} \frac{\varphi(t)}{\psi_2(t)} = \max_{0 \le t \le 1} \frac{\psi_2(t)}{\psi(t)}.$$

The same is true for the other cases.

COROLLARY 2 (Clarkson [5]). *Let* $1 \le p \le \infty$ *and* $1/p + 1/p' = 1$. *Let* $t = \min\{p, p'\}$. *Then*

$$C_{\mathrm{NJ}}(\|\cdot\|_p) = 2^{(2/t)-1}. \tag{14}$$

In particular, $C_{\mathrm{NJ}}(\|\cdot\|_1) = C_{\mathrm{NJ}}(\|\cdot\|_\infty) = 2$.

Indeed, if $1 \le p \le 2$,

$$\psi_2(t) \le \psi_p(t) \le 2^{(1/p)-(1/2)}\psi_2(t) \qquad (0 \le \forall t \le 1),$$

where the constant $2^{(1/p)-(1/2)}$ is the best possible. Hence we have (14) by Theorem 1. For the case $2 \le p \le \infty$ a parallel argument works.

Remark 1. The only known way to calculate NJ constants needs Clarkson's inequalities (cf. [5, 9, 11]), whereas the above discussion to derive (14) does not require them.

Further, Theorem 1 enables us to obtain many examples easily. Let us present some. The following easy lemma is helpful for applying Theorem 1.

LEMMA 5. *Let* $\varphi(t) \ge \psi(t) > 0$ *on* $[a, b]$. *Assume that* $\varphi - \psi$ *has the maximum, resp.* ψ *has the minimum, at* $t = c$ *in* $[a, b]$. *Then* φ/ψ *attains the maximum at* $t = c$.

Indeed, the conclusion is immediate from the identity

$$\frac{\varphi(t)}{\psi(t)} - 1 = \frac{\varphi(t) - \psi(t)}{\psi(t)}.$$

EXAMPLE 1. Let $\|\cdot\| = \max\{\|\cdot\|_2, \lambda\|\cdot\|_1\}$ $(1/\sqrt{2} \le \lambda \le 1)$. Then

$$C_{\mathrm{NJ}}(\|\cdot\|) = 2\lambda^2.$$

In fact by Lemma 4, $\|\cdot\| = \|\cdot\|_{\max\{\psi_2, \lambda\psi_1\}}$. Then by Theorem 1 we have

$$C_{\mathrm{NJ}}(\|\cdot\|) = \max_{0 \le t \le 1} \left[\frac{\max\{\psi_2(t), \lambda\psi_1(t)\}}{\psi_2(t)} \right]^2$$

$$= \max_{0 \le t \le 1} \left\{ \frac{\lambda\psi_1(t)}{\psi_2(t)} \right\}^2 = \frac{\lambda^2}{1/2} = 2\lambda^2.$$

The following example treats a nonnormalized norm.

EXAMPLE 2 ([10, Proposition 1]). Let $\|\cdot\| = \max\{\|\cdot\|_2, \lambda\|\cdot\|_\infty\}$ $(1 \le \lambda \le \sqrt{2})$. Then

$$C_{\mathrm{NJ}}(\|\cdot\|) = \lambda^2.$$

Indeed, put $\|\cdot\|_0 = \max\{\lambda^{-1}\|\cdot\|_2, \|\cdot\|_\infty\}$. Then $\|\cdot\|_0 = \|\cdot\|_{\max\{\lambda^{-1}\psi_2, \psi_\infty\}}$ by Lemma 4 and $\|\cdot\| = \lambda\|\cdot\|_0$. Hence we have

$$\begin{aligned} C_{\mathrm{NJ}}(\|\cdot\|) &= C_{\mathrm{NJ}}(\lambda\|\cdot\|_0) \\ &= C_{\mathrm{NJ}}(\|\cdot\|_0) \\ &= \max_{0 \le t \le 1} \left[\frac{\psi_2(t)}{\max\{\lambda^{-1}\psi_2(t), \psi_\infty(t)\}} \right]^2 \end{aligned}$$

by Theorem 1. Now clearly $\psi_2(t)/[\max\{\lambda^{-1}\psi_2(t), \psi_\infty(t)\}]$ is symmetric with respect to $t = 1/2$. Let t_0 be such that $\lambda^{-1}\psi_2(t_0) = \psi_\infty(t_0)$ $(0 \le t_0 \le 1/2)$. Then we have

$$\max_{0 \le t \le t_0} \left[\frac{\psi_2(t)}{\max\{\lambda^{-1}\psi_2(t), \psi_\infty(t)\}} \right]^2 = \lambda^2$$

by Lemma 5, and clearly

$$\max_{t_0 \le t \le 1/2} \left[\frac{\psi_2(t)}{\max\{\lambda^{-1}\psi_2(t), \psi_\infty(t)\}} \right]^2 = \max_{t_0 \le t \le 1/2} \left[\frac{\psi_2(t)}{\lambda^{-1}\psi_2(t)} \right]^2 = \lambda^2.$$

Therefore we have $C_{\mathrm{NJ}}(\|\cdot\|) = \lambda^2$.

EXAMPLE 3. Let $2 \le p < \infty$. Let $\|\cdot\|_{p,2}$ be the (Lorentz) $\ell_{p,2}$-norm.

$$\|(z, w)\|_{p,2} = \left\{ |z|^{*2} + 2^{(2/p)-1}|w|^{*2} \right\}^{1/2},$$

where $\{|z|^*, |w|^*\}$ is the non-increasing rearrangement of $\{|z|, |w|\}$; that is, $|z|^* \geq |w|^*$. (Note that if $p < 2$, $\|\cdot\|_{p,2}$ is a quasi-norm; cf. [7, Proposition 1; 14, p. 126; 2, p. 8]). Then

$$C_{\mathrm{NJ}}(\|\cdot\|_{p,2}) = \frac{2}{1 + 2^{2/p-1}}.$$

Indeed, $\|\cdot\|_{p,2} \in N_a$, and the corresponding convex function is given by

$$\psi_{p,2}(t) = \begin{cases} \left\{(1-t)^2 + 2^{2/p-1}t^2\right\}^{1/2} & \text{if } 0 \leq t \leq 1/2, \\ \left\{t^2 + 2^{2/p-1}(1-t)^2\right\}^{1/2} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Since $\psi_{p,2} \leq \psi_2$ and $\psi_2/\psi_{p,2}$ is symmetric with respect to $t = 1/2$, we find the maximum of $\psi_2^2/\psi_{p,2}^2$ in the interval $[0, 1/2]$. The difference $\psi_2(t)^2 - \psi_{p,2}(t)^2 = (1 - 2^{2/p-1})t^2$ takes its maximum at $t = 1/2$, and $\psi_{p,2}$ has the minimum at $t = 1/2$. Therefore by Lemma 5 we have

$$\max_{0 \leq t \leq 1} \frac{\psi_2(t)^2}{\psi_{p,2}(t)^2} = \frac{\psi_2(1/2)^2}{\psi_{p,2}(1/2)^2} = \frac{2}{1 + 2^{2/p-1}},$$

which implies the conclusion by Theorem 1.

4. NJ CONSTANT OF ABSOLUTE NORMALIZED NORMS—THE GENERAL CASE

LEMMA 6. *Let $1/2 \leq \alpha \leq 1$ and let*

$$\psi_\alpha(t) = \begin{cases} \dfrac{\alpha - 1}{\alpha} t + 1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases}$$

Then

$$M_1 = \max_{0 \leq t \leq 1} \frac{\psi_\alpha(t)}{\psi_2(t)} = \left\{\left(2 - \frac{1}{\alpha}\right)^2 + 1\right\}^{1/2}, \tag{15}$$

$$M_2 = \max_{0 \leq t \leq 1} \frac{\psi_2(t)}{\psi_\alpha(t)} = \left\{\left(\frac{1}{\alpha} - 1\right)^2 + 1\right\}^{1/2}. \tag{16}$$

Proof. If $\alpha = 1/2$ or $\alpha = 1$, the conclusion is clear by Lemma 5. Let $1/2 < \alpha < 1$. Easy calculation shows that the function ψ_α/ψ_2 attains the maximum at $t = (2\alpha - 1)/(3\alpha - 1)$, which gives (15). The function ψ_2/ψ_α clearly has the maximum at $t = \alpha$, which implies (16).

Although the notation ψ_α is not consistent with ψ_p corresponding to the ℓ_p-norm, there will be no confusion in the following.

THEOREM 2. *Let $\psi \in \Psi$ and let*

$$M_1 = \max_{0 \le t \le 1} \frac{\psi(t)}{\psi_2(t)} \quad \textit{and} \quad M_2 = \max_{0 \le t \le 1} \frac{\psi_2(t)}{\psi(t)}. \tag{17}$$

Then

$$\max\{M_1^2, M_2^2\} \le C_{\rm NJ}(\|\cdot\|_\psi) \le M_1^2 M_2^2. \tag{18}$$

Further we have

$$1 \le \max\{M_1^2, M_2^2\} \le M_1^2 M_2^2 \le 2. \tag{19}$$

Proof. For all $x, y \in \mathbb{C}^2$ we have

$$\begin{aligned} \|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\le M_1^2\left(\|x+y\|_2^2 + \|x-y\|_2^2\right) \\ &= 2M_1^2\left(\|x\|_2^2 + \|y\|_2^2\right) \\ &\le 2M_1^2 M_2^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right), \end{aligned} \tag{20}$$

which implies that $C_{\rm NJ}(\|\cdot\|_\psi) \le M_1^2 M_2^2$. Next let

$$M_1 = \frac{\psi(t_1)}{\psi_2(t_1)}, \qquad M_2 = \frac{\psi_2(t_2)}{\psi(t_2)} \tag{21}$$

with some $0 \le t_1, t_2 \le 1$. Put $x_1 = (1 - t_1, 0)$, $y_1 = (0, t_1)$. Then by (11) we have $C_{\rm NJ}(\|\cdot\|_\psi) \ge M_1^2$. In the same way, by putting $x_2 = (1 - t_2, t_2)$, $y_2 = (1 - t_2, -t_2)$, we have $C_{\rm NJ}(\|\cdot\|_\psi) \ge M_2^2$ by (12).

Now we prove (19). The first two inequalities are obvious. If $\psi \ge \psi_2$ or $\psi \le \psi_2$, the last inequality in (19) is easy to see (merely note that $\psi_1/\psi_2 \le \sqrt{2}$ and $\psi_2/\psi_\infty \le \sqrt{2}$). So assume this is not the case. Then $M_1, M_2 > 1$, whence we have (21) with $0 < t_1, t_2 < 1$. Owing to Proposition 1 we may assume that $t_1 < t_2$. Let (α, α) be the intersection of the line $s = t$ and the line combining the points $(0, 1)$ and $(t_2, \psi(t_2))$. Then evidently $1/2 < \alpha < 1$ and $t_2 < \alpha$. Hence

$$M_2 = \frac{\psi_2(t_2)}{\psi(t_2)} = \frac{\psi_2(t_2)}{\psi_\alpha(t_2)} \le \frac{\psi_2(\alpha)}{\psi_\alpha(\alpha)},$$

where ψ_α is as in Lemma 6. On the other hand, since $\psi_2(t_1) \le \psi(t_1) \le \psi_\alpha(t_1)$ by the convexity of $\psi(t)$ and (21), we have

$$M_1 = \frac{\psi(t_1)}{\psi_2(t_1)} \le \frac{\psi_\alpha(t_1)}{\psi_2(t_1)}.$$

Therefore by Lemma 6 we have

$$\begin{aligned} M_1 M_2 &= \frac{\psi(t_1)}{\psi_2(t_1)} \frac{\psi_2(t_2)}{\psi(t_2)} \\ &\le \max_{0\le t\le 1} \frac{\psi_\alpha(t)}{\psi_2(t)} \max_{0\le t\le 1} \frac{\psi_2(t)}{\psi_\alpha(t)} \\ &= \left\{\left(2 - \frac{1}{\alpha}\right)^2 + 1\right\}^{1/2} \left\{\left(\frac{1}{\alpha} - 1\right)^2 + 1\right\}^{1/2}. \end{aligned}$$

Put $u = 1/\alpha - 1$. Then, $0 < u < 1$ and

$$\begin{aligned} M_1^2 M_2^2 &\le (u^2 + 1)\{(1-u)^2 + 1\} \\ &= u(u-1)(u^2 - u + 2) + 2 \\ &< 2. \end{aligned} \tag{22}$$

This completes the proof.

Remark 2. (i) In Theorem 2 we have $M_1^2 M_2^2 = 2$ if and only if $\alpha = 1$ or $\alpha = 1/2$; in this case $\psi = \psi_1$ or $\psi = \psi_\infty$. In fact, the "if" part is clear, and the opposite follows directly from (22).

(ii) $\max\{M_1, M_2\} = 1$ if and only if $\psi = \psi_2$.

(iii) $\max\{M_1, M_2\} = M_1 M_2$ if and only if $\psi \ge \psi_2$ or $\psi \le \psi_2$: In particular, Theorem 1 is also a result of this fact.

As a consequence of Theorem 2 we have

COROLLARY 3. *Let* $\|\cdot\| \in N_a$. *Then* $C_{\mathrm{NJ}}(\|\cdot\|) = 2$ *if and only if* $\|\cdot\|$ *is an* ℓ_1- *or* ℓ_∞-*norm: In other words, all norms in* N_a *except* ℓ_1- *and* ℓ_∞-*norms are uniformly non-square.*

Proof. By Theorem 2, $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = 2$ if and only if $M_1^2 M_2^2 = 2$, which occurs only when $\psi = \psi_1$ or $\psi = \psi_\infty$ by Remark 2 (i).

Now, according to Theorem 1, the identity $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = \max\{M_1^2, M_2^2\}$ holds in the estimate (18) of Theorem 2 if ψ is comparable with ψ_2. The next theorem asserts that for another wide class of convex functions we have $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = M_1^2 M_2^2$ and $C_{\mathrm{NJ}}(\|\cdot\|_\psi) > \max\{M_1^2, M_2^2\}$.

THEOREM 3. *Let $\psi \in \Psi$ and let $\psi(t) = \psi(1-t)$ for all $0 \le t \le 1$. Assume that $M_1 = \max_{0 \le t \le 1} \psi(t)/\psi_2(t)$ or $M_2 = \max_{0 \le t \le 1} \psi_2(t)/\psi(t)$ is taken at $t = 1/2$. Then*

$$C_{\mathrm{NJ}}(\|\cdot\|_\psi) = M_1^2 M_2^2. \tag{23}$$

Moreover, if neither $\psi \ge \psi_2$ nor $\psi \le \psi_2$,

$$\max\{M_1^2, M_2^2\} < C_{\mathrm{NJ}}(\|\cdot\|_\psi). \tag{24}$$

Proof. Suppose first $M_1 = \psi(1/2)/\psi_2(1/2)$. Take an arbitrary t with $0 \le t \le 1$ and put $x = (t, 1-t)$ and $y = (1-t, t)$. Then

$$\|x\|_\psi^2 = \psi(1-t)^2 = \psi(t)^2, \qquad \|y\|_\psi^2 = \psi(t)^2.$$

On the other hand

$$\|x+y\|_\psi^2 = \|(1,1)\|_\psi^2 = 4\psi(1/2)^2,$$
$$\|x-y\|_\psi^2 = \|(2t-1, 1-2t)\|_\psi^2 = 4(2t-1)^2\psi(1/2)^2.$$

Therefore

$$\begin{aligned}\frac{\|x+y\|_\psi^2 + \|x-y\|_\psi^2}{2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right)} &= \frac{4\psi(1/2)^2\{(2t-1)^2+1\}}{4\psi(t)^2} \\ &= \frac{\psi(1/2)^2\{(1-t)^2+t^2\}}{\psi(t)^2/2} \\ &= \frac{\psi(1/2)^2\psi_2(t)^2}{\psi_2(1/2)^2\psi(t)^2} = M_1^2\frac{\psi_2(t)^2}{\psi(t)^2}.\end{aligned}$$

Since t is arbitrary, we have

$$C_{\mathrm{NJ}}(\|\cdot\|_\psi) \ge M_1^2 M_2^2, \tag{25}$$

which, combined with (18), implies (23). In case of $M_2 = \psi_2(1/2)/\psi(1/2)$, let x and y be as above and put $u = x + y$ and $v = x - y$. Then since

$$\begin{aligned}\frac{\|u+v\|_\psi^2 + \|u-v\|_\psi^2}{2\left(\|u\|_\psi^2 + \|v\|_\psi^2\right)} &= \frac{2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right)}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} \\ &= \frac{\psi_2(1/2)^2\psi(t)^2}{\psi(1/2)^2\psi_2(t)^2} = M_2^2\frac{\psi(t)^2}{\psi_2(t)^2},\end{aligned}$$

we have (25) and hence (23). The inequality (24) is a direct consequence of (23) and Remark 2 (iii).

EXAMPLE 4. Let $1/2 \leq \beta \leq 1$. Let $\varphi_\beta(t) = \max\{1-t, t, \beta\}$ (note that neither $\varphi_\beta \geq \psi_2$ nor $\varphi_\beta \leq \psi_2$ if $1/\sqrt{2} < \beta < 1$). Then

$$C_{\mathrm{NJ}}\left(\|\cdot\|_{\varphi_\beta}\right) = \begin{cases} \dfrac{1}{\beta^2}\left\{(1-\beta)^2 + \beta^2\right\} & \text{if } \dfrac{1}{2} \leq \beta \leq \dfrac{1}{\sqrt{2}}, \\ 2\left\{(1-\beta)^2 + \beta^2\right\} & \text{if } \dfrac{1}{\sqrt{2}} \leq \beta \leq 1. \end{cases}$$

Indeed, by Lemma 5

$$M_1 = \begin{cases} 1 & \text{if } \dfrac{1}{2} \leq \beta \leq \dfrac{1}{\sqrt{2}}, \\ \dfrac{\varphi_\beta(1/2)}{\psi_2(1/2)} = \dfrac{\beta}{1/\sqrt{2}} = \sqrt{2}\,\beta & \text{if } \dfrac{1}{\sqrt{2}} \leq \beta \leq 1 \end{cases}$$

and

$$M_2 = \frac{\psi_2(\beta)}{\varphi_\beta(\beta)} = \frac{1}{\beta}\left\{(1-\beta)^2 + \beta^2\right\}^{1/2},$$

whence we have the conclusion by Theorem 3.

Finally we see a class of convex functions for which the identity $C_{\mathrm{NJ}}(\|\cdot\|_\psi) = M_1^2 M_2^2$ fails to hold.

THEOREM 4. *Let $\psi \in \Psi$. Let M_1 and M_2 be as in Theorem 2. Assume that*

$$\max\left\{t;\, M_1 = \frac{\psi(t)}{\psi_2(t)}\right\} < \min\left\{s;\, M_2 = \frac{\psi_2(s)}{\psi(s)}\right\} \tag{26}$$

or

$$\min\left\{t;\, M_1 = \frac{\psi(t)}{\psi_2(t)}\right\} > \max\left\{s;\, M_2 = \frac{\psi_2(s)}{\psi(s)}\right\} \tag{27}$$

(*hence ψ is not symmetric with respect to $t = 1/2$*). *Then*

$$C_{\mathrm{NJ}}\left(\|\cdot\|_\psi\right) < M_1^2 M_2^2. \tag{28}$$

Proof. It is enough to show (28) in the case (26) by Proposition 1. Put

$$t_0 = \max\left\{t;\, M_1 = \frac{\psi(t)}{\psi_2(t)}\right\}, \qquad s_0 = \min\left\{s;\, M_2 = \frac{\psi_2(s)}{\psi(s)}\right\}.$$

Then clearly $0 < t_0 < s_0 < 1$. Assume that (28) is not valid. Then there are $x, y \in \mathbb{C}^2$, not both 0, such that

$$\|x + y\|_\psi^2 + \|x - y\|_\psi^2 = 2M_1^2 M_2^2 \left(\|x\|_\psi^2 + \|y\|_\psi^2\right).$$

It should be noted that all of x, y, $x + y$, and $x - y$ are not 0 because $\psi \neq \psi_2$. Then by (20) we have

$$\|x + y\|_\psi = M_1 \|x + y\|_2, \qquad \|x - y\|_\psi = M_1 \|x - y\|_2 \tag{29}$$

and

$$\|x\|_2 = M_2 \|x\|_\psi, \qquad \|y\|_2 = M_2 \|y\|_\psi. \tag{30}$$

Put $x = (a, b)$ and $y = (c, d)$. Then by (29)

$$\psi\left(\frac{|b + d|}{|a + c| + |b + d|}\right) = M_1 \psi_2\left(\frac{|b + d|}{|a + c| + |b + d|}\right),$$
$$\psi\left(\frac{|b - d|}{|a - c| + |b - d|}\right) = M_1 \psi_2\left(\frac{|b - d|}{|a - c| + |b - d|}\right),$$

from which it follows that

$$\frac{|b + d|}{|a + c| + |b + d|} \leq t_0, \qquad \frac{|b - d|}{|a - c| + |b - d|} \leq t_0.$$

In the same way, by (30) we have

$$\frac{|b|}{|a| + |b|} \geq s_0, \qquad \frac{|d|}{|c| + |d|} \geq s_0.$$

Therefore

$$|b + d| \leq \frac{t_0}{1 - t_0} |a + c|,$$
$$|b - d| \leq \frac{t_0}{1 - t_0} |a - c|$$

and

$$|b| \geq \frac{s_0}{1 - s_0} |a|, \qquad |d| \geq \frac{s_0}{1 - s_0} |c|.$$

Hence we have

$$|b+d|^2+|b-d|^2 \le \left(\frac{t_0}{1-t_0}\right)^2 \left(|a+c|^2+|a-c|^2\right) = 2\left(\frac{t_0}{1-t_0}\right)^2 \left(|a|^2+|c|^2\right)$$

and

$$|b+d|^2+|b-d|^2 = 2\left(|b|^2+|d|^2\right) \ge 2\left(\frac{s_0}{1-s_0}\right)^2 \left(|a|^2+|c|^2\right).$$

Consequently we have $t_0/(1-t_0) \ge s_0/(1-s_0)$ because $|a|^2+|c|^2 \ne 0$, and hence $t_0 \ge s_0$, which contradicts our assumption. This completes the proof.

COROLLARY 4. *Let $\psi \in \Psi$. Let M_1 and M_2 be as in Theorem 2. If there exists uniquely one point such that ψ/ψ_2 (resp. ψ_2/ψ) attains M_1 (resp. M_2), then*

$$C_{\mathrm{NJ}}(\|\cdot\|_\psi) < M_1^2 M_2^2.$$

In fact, these points at which M_1 and M_2 are attained are different by the assumption.

Remark 3. (i) In Theorem 3 the condition $\psi(t) = \psi(1-t)$ is essential. Indeed, modify the function φ_β in Example 4 as follows: Let $1/\sqrt{2} < \gamma < 1$. Let t_0 be the smaller solution of the equation, $\psi_2(t) = \gamma$. Define

$$\omega_\gamma(t) = \begin{cases} \psi_2(t) & \text{if } 0 \le t \le t_0, \\ \gamma & \text{if } t_0 \le t \le \gamma, \\ t & \text{if } \gamma \le t \le 1. \end{cases}$$

Then, ω_γ/ψ_2 has the maximum at $t = 1/2$, but ω_γ is not symmetric with respect to $t = 1/2$. On the other hand, ω_γ satisfies the condition in Corollary 4, and hence we have $C_{\mathrm{NJ}}(\|\cdot\|_{\omega_\gamma}) < M_1^2 M_2^2$.

(ii) For ψ_α in Lemma 6 we have $C_{\mathrm{NJ}}(\|\cdot\|_{\psi_\alpha}) < M_1^2 M_2^2$ by Corollary 4, where M_1 and M_2 are as in Lemma 6.

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