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The impact of Stieltjes' work on continued fractions and orthogonal polynomials: additional material

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Abstract

In the recent new edition of the collected works of T.J. Stieltjes, one of us gave an impression of the impact of Stieltjes' work a century after his death [43]. In this paper we give an update and mention some observations which were missing from [43] and some results which appeared during the last two years and which are directly related to Stieltjes' work. In particular, there is a large section on special polynomials which were already considered by Stieltjes but which were rediscovered later, often without the knowledge of Stieltjes' work.

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1. Continued fractions and moment problems

In [43] an attempt was made to give an updated review of the impact of Stieltjes' work on continued fractions, orthogonal polynomials, and related topics. Here we will mention some extra information which is directly related to Stieltjes' work and which was missing from [43]. First of all we would like to draw the attention to Kjeldsen's nice historic view on the early history of the moment problem [24], which is recommended reading.

In [43] the remark was made (last paragraph of Section 2.1 and the beginning of Section 2.2) that *"not much work on the Stieltjes moment problem was done after Stieltjes' death"* and *"nothing new happened until 1920"*. What was meant is that the work of Stieltjes was so thorough that he single-handedly almost completed the theory of the Stieltjes moment problem. Of course, there has

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been work after Stieltjes on the Stieltjes moment problem, e.g., Karlin and McGregor on the one hand and Lederman and Reuter on the other hand pointed out the equivalence of Stieltjes' work and the solution of the Chapman–Kolmogorov equations of birth and death processes. Also, the work of Van Vleck, which consists in trying to extend the Stieltjes moment problem to the real line, is quite relevant. Recall that the main object of Stieltjes' work *Sur les fractions continues* is the S-fraction

$$\cfrac{1}{c_1 z + \cfrac{1}{c_2 + \cfrac{1}{c_3 z + \cdots + \cfrac{1}{c_{2n} + \cfrac{1}{c_{2n+1} z + \cdots}}}}}, \tag{1}$$

in which Stieltjes assumes that $c_k > 0$ for $k = 1, 2, 3, \ldots$. This S-fraction corresponds to a Stieltjes moment problem and Stieltjes showed that this moment problem is determinate if and only if $\sum_{k=1}^{\infty} c_k$ diverges. Van Vleck attempted to extend this result to S-fractions for which the coefficients c_k are not all positive. When the positivity of the c_k is dropped, then the analysis becomes more complicated, it is even not clear what the proper condition on the coefficients should be for this more general situation, since several extensions are possible. Van Vleck made relevant progress for this extension, but the full extension to moment problems on the full real line is due to Hamburger [17] in 1920–1921. One of the results of Van Vleck [42] from this period between Stieltjes and Hamburger is that the S-fraction

$$\cfrac{a_0}{1 - \cfrac{a_1 z}{1 - \cfrac{a_2 z}{1 - \cdots}}},$$

for which $a_n \to 0$ (but $a_n \neq 0$ for every n) converges to a meromorphic function of z and the convergence is uniform on every compact set not containing any of the poles of this function. When $a_n \to a \neq 0$ then the S-fraction converges to a function which is meromorphic in the complex plane with a cut along the interval $[\frac{1}{4}a, \infty)$. This S-fraction is related to the S-fraction in (1) by $a_n = -1/(c_n c_{n+1})$ ($c_0 = 1$) with z replaced by $1/z$. When all a_n are negative, then Stieltjes had already shown that the S-fraction is a meromorphic function if and only if $a_n \to 0$ [40, Note, pp. 564–570]. When the a_n are allowed to be real (or complex) then $a_n \to 0$ is only a sufficient condition which is no longer necessary (see also [45, Theorem 54.2, in Section 54]). In modern terminology the situation $a_n \to 0$ corresponds to orthogonal polynomials with a compact Jacobi matrix.

For indeterminate moment problems there has been much work on obtaining and describing the solutions of the moment problem, in particular the so-called Nevanlinna extremal solutions. This is well described in the paper by Berg in these proceedings [6].

Another important contribution after Stieltjes' work and before the work of Hamburger was made by H. Weyl, who wrote his Habilitationsschrift in 1910 [46]. Weyl studied uniqueness or nonuniqueness of spectral measures for second-order ordinary differential operators. Consider the self-adjoint differential operator

$$(py')' - (q - \lambda)y = 0, \qquad 0 \leqslant x \leqslant \ell,$$

where p is a positive function and q is a real function, with boundary conditions $y'(0) - wy(0) = 0$ and $y'(\ell) - hy(\ell) = 0$. Then Weyl showed that there exists a unique solution for exactly one $w = w(h, \lambda)$ which depends on λ and h. If h takes all real values, then the $w(h, \lambda)$ form a circle $C_\ell(\lambda)$ and these circles form a nested family of circles. If ℓ tends to infinity, then there are two possibilities: either $C_\ell(\lambda)$ converges to a circle or $C_\ell(\lambda)$ converges to a point. These two cases are known as the limit-circle and the limit-point case. It later turned out, through work of Hellinger [18], Nevanlinna [33] and Riesz [39] that the same theory can be set up for second-order symmetric difference operators and Weyl's criteria then turn out to be equivalent to Hamburger's criteria, but from a different perspective. The limit-point case corresponds to a determinate moment problem, whereas the limit-circle case gives an indeterminate moment problem.

Another generalization of the moment problem was given by Krein, who considered matrix moment problems. In particular, in [26, 27], he considers matrix moments

$$S_n = \int_{-\infty}^{\infty} \lambda^n \, \mathrm{d}T(\lambda), \quad n = 0, 1, 2, \ldots,$$

where T is a $p \times p$ matrix valued measure. These moments correspond to orthogonal matrix polynomials, i.e., polynomials of a real (complex) variable but with $p \times p$ matrix coefficients. These orthogonal matrix polynomials satisfy a three-term recurrence relation with matrix coefficients. Putting these matrix coefficients into a block tridiagonal matrix gives an operator, and Krein gives a relation between the deficiency indices ν_+ and ν_- of this operator and the nature of the moment problem. In particular, he shows that the matrix moment problem is determinate if and only if both deficiency indices are zero. Furthermore, when both deficiency indices are equal to p then

$$T(\xi+) - T(\xi-) \leqslant \left(\sum_{k=0}^{\infty} P_k^*(\xi) P_k(\xi) \right)^{-1},$$

where P_k ($k = 0, 1, 2, \ldots$) are orthonormal polynomials for T and for fixed ξ equality is attained for only one measure T_ξ. This relation corresponds to a similar relation in the scalar case, where it is known that $\sum_{k=0}^{\infty} p_k^2(x)$ is the maximal possible mass at x for any solution of the moment problem, and there is an N-extremal solution with precisely this mass at x.

A necessary and sufficient condition for the Hamburger moment problem to be determined is that $\sum_{n=0}^{\infty} |p_n(z)|^2 = \infty$ for every $z \in \mathbb{C} \backslash \mathbb{R}$. Quite often this condition can be checked, especially when the orthonormal polynomials are known in closed form. However, when the orthonormal polynomials are not known explicitly or even approximately, then a more convenient sufficient condition is the Carleman condition, which can also be expressed in terms of the recurrence coefficients: if the orthonormal polynomials satisfy

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0,$$

then Carleman's sufficient condition for a determinate moment problem is

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty.$$

This condition is very convenient if the orthogonal polynomials are given by means of their recurrence relation.

As mentioned in [43], recent extensions of the moment problem are the strong moment problem, investigated by Jones, Thron, Waadeland, and Njåstad, and the extended moment problem as given by Njåstad. The strong moment problem is related to doubly infinite Jacobi matrices just as the Hamburger moment problem is related to semi-infinite Jacobi matrices. Berezanskiĭ [5, Ch. VII, Section 3], Nikishin [34] and Ismail et al. [21, Section 6] have shown how a doubly infinite Jacobi matrix can be analyzed by means of two semi-infinite J-matrices. This corresponds to writing a doubly infinite S-fraction in terms of two semi-infinite S-fractions; see also [29, 44]. The extended moment problem is related to (bi)orthogonal rational functions and these have recently also been studied in [23]. See also a recent paper [2] where a q-beta integral on the unit circle leads to some biorthogonal rational functions.

2. Electrostatic interpretation of zeros

Let us recall that Stieltjes showed that the expression

$$L(x_1,x_2,\ldots,x_n) = \sum_{1\leqslant i<j\leqslant n} \log\frac{1}{|x_i-x_j|} + p\sum_{i=1}^{n}\log\frac{1}{|1-x_i|} + q\sum_{i=1}^{n}\log\frac{1}{|1+x_i|}$$

attains a minimum when $x_1,\ldots,x_n$ are the zeros of the Jacobi polynomial $P_n^{(2p-1,2q-1)}(x)$. This is done by setting the partial derivatives $\partial L/\partial x_i$ equal to zero for $i = 1,2,\ldots,n$, and then Stieltjes shows that the corresponding x_i are such that the polynomial $(x-x_1)(x-x_2)\cdots(x-x_n)$ satisfies the second-order differential equation for the Jacobi polynomials. The quantity $L(x_1,\ldots,x_n)$ can be interpreted as the electrostatic energy for a system of $n+2$ charges, n of which are free unit charges on $[-1,1]$, at $+1$ there is a charge $p>0$ and at -1 a charge $q>0$. The vector $(\partial L/\partial x_1,\ldots,\partial L/\partial x_n)$ contains the forces acting on the charges, and thus setting all these forces equal to zero gives an equilibrium for this electrostatic configuration. However, it is not a priori evident that this equilibrium corresponds to a minimum of the electrostatic energy, which is necessary in order to have a stable equilibrium (if the extremum is not a minimum, then small perturbations will quickly result in moving away from the equilibrium position). As far as I could observe, Stieltjes does not explicitly show that the equilibrium position is a minimum (even though he explicitly mentions that it is a minimum), but he seems to deduce this from the physical interpretation that such an electrostatic configuration has a stable equilibrium. Szegő [41, Section 6.7] shows that there is a minimum since $\exp(-L(x_1,\ldots,x_n))$ is a continuous function on the hypercube $-1\leqslant x_i\leqslant 1$ $(i=1,2,\ldots,n)$ so that it attains a maximum. One can also investigate the Hessian matrix $H = (\partial^2 L/\partial x_i\partial x_j)_{1\leqslant i,j\leqslant n}$. Clearly,

$$\frac{\partial L}{\partial x_i} = \sum_{\substack{k=1\\k\neq i}}^{n}\frac{1}{x_k-x_i} - \frac{p}{x_i-1} - \frac{q}{x_i+1}$$

and thus

$$\frac{\partial^2 L}{\partial x_i \partial x_j} = \begin{cases} \dfrac{-1}{(x_j - x_i)^2} & \text{if } i \neq j, \\ \displaystyle\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(x_k - x_i)^2} + \frac{p}{(x_i - 1)^2} + \frac{q}{(x_i + 1)^2} & \text{if } i = j. \end{cases}$$

The Hessian matrix H is thus a sum of the diagonal matrix

$$H_1 = \operatorname{diag}\left(\frac{p}{(x_1 - 1)^2} + \frac{q}{(x_1 + 1)^2}, \frac{p}{(x_2 - 1)^2} + \frac{q}{(x_2 + 1)^2} + \cdots + \frac{p}{(x_n - 1)^2} + \frac{q}{(x_n + 1)^2} \right)$$

and the matrix H_2 which contains on the diagonal the entries

$$(H_2)_{i,i} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(x_k - x_i)^2}$$

and off the diagonal it contains

$$(H_2)_{i,j} = -\frac{1}{(x_i - x_j)^2}.$$

Obviously, H_1 is a positive-definite matrix since $p > 0$ and $q > 0$, and also H_2 is positive definite since by Gerschgorin's theorem the eigenvalues are in

$$\bigcup_{i=1}^{n} \left[0, 2 \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{1}{(x_k - x_i)^2} \right],$$

which implies that all the eigenvalues are nonnegative. Observe that $(1, 1, \ldots, 1)^t$ is an eigenvector of H_2 with eigenvalue 0. We thus conclude that the Hessian matrix H is positive definite, which means that the equilibrium corresponds to a minimum of L and is therefore a stable equilibrium.

The electrostatic configuration and the equilibrium can easily be simulated on a computer. Fix two charges at ± 1 and in between one puts n free unit charges. At each step one computes the electrostatic energy L. Pick at random one of the n free charges and move it slighly to the left or to the right (at random) and compute the electrostatic equilibrium of the new configuration. If this energy is greater than the previously computed energy, then put the charge back where it came from, otherwise leave the charge. This process can be repeated and by construction it results in a simulation where the electrostatic energy is a decreasing function. The equilibrium position is attained when the charges are no longer moving, and then one can check Stieltjes' theorem by plotting the corresponding Jacobi polynomial, which indeed passes through all the charges.

As mentioned in [43], Stieltjes also obtained similar interpretations of the zeros of Laguerre and Hermite polynomials, and more recently charges in the complex plane were also considered by Forrester and Rogers [12] (on the unit circle) and Hendriksen and van Rossum [19]. The latter considered a configuration with a dipole at the origin and showed that an equilibrium is attained at the zeros of a Bessel polynomial. The dipole is obtained by taking a charge $\frac{1}{2}(a+1)$ at the origin and a charge $\frac{1}{2}(c-a)$ at the point $1/a$, where $a > 0$, and then taking $a \to \infty$. The corresponding electrostatic energy is

$$L(z_1, z_2, \ldots, z_n) = \sum_{1 \leqslant k<j \leqslant n} \log \frac{1}{|z_k - z_j|} + \frac{c+1}{2} \sum_{k=1}^{n} \log \frac{1}{|z_k|} - \frac{1}{2} \sum_{k=1}^{n} \frac{\operatorname{Re} z_k}{|z_k|^2}.$$

Writing the complex numbers as $z_k = x_k + \mathrm{i} y_k$, the extrema are then found by setting the partial derivatives $\partial L/\partial x_k$ and $\partial L/\partial y_k$ to zero ($k = 1, 2, \ldots$), which amounts to setting the forces acting on the charges equal to zero. Hendriksen and van Rossum then showed that the resulting complex numbers z_k are such that the polynomial $(z-z_1)(z-z_2)\cdots(z-z_n)$ satisfies the second-order differential equation for the Bessel polynomial ${}_2F_0(-n, c+n; z)$. In this case it is not at all obvious that the obtained extremum is a minimum. We cannot use Szegő's argument since the charges z_k are not constrained to a compact set. So let us compute the Hessian matrix. First of all we have

$$\frac{\partial L}{\partial x_i} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{x_k - x_i}{(x_k - x_i)^2 + (y_k - y_i)^2} - \frac{c+1}{2} \frac{x_i}{x_i^2 + y_i^2} + \frac{1}{2} \frac{x_i^2 - y_i^2}{(x_i^2 + y_i^2)^2},$$

$$\frac{\partial L}{\partial y_i} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{y_k - y_i}{(x_k - x_i)^2 + (y_k - y_i)^2} - \frac{c+1}{2} \frac{y_i}{x_i^2 + y_i^2} + \frac{1}{2} \frac{2x_i y_i}{(x_i^2 + y_i^2)^2}.$$

Then

$$\frac{\partial^2 L}{\partial x_i \partial x_j} = \begin{cases} \dfrac{-(x_j - x_i)^2 + (y_j - y_i)^2}{[(x_j - x_i)^2 + (y_j - y_i)^2]^2} & \text{if } i \neq j, \\ \displaystyle\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{(x_k - x_i)^2 - (y_k - y_i)^2}{([(x_j - x_i)^2 + (y_j - y_i)^2]^2} + \frac{c+1}{2} \frac{x_i^2 - y_i^2}{(x_i^2 + y_i^2)^2} - \frac{x_i(x_i^2 - 3y_i^2)}{(x_i^2 + y_i^2)^3} & \text{if } i = j, \end{cases}$$

and

$$\frac{\partial^2 L}{\partial y_i \partial y_j} = \begin{cases} \dfrac{(x_j - x_i)^2 - (y_j - y_i)^2}{[(x_j - x_i)^2 + (y_j - y_i)^2]^2} & \text{if } i \neq j, \\ \displaystyle\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{-(x_k - x_i)^2 + (y_k - y_i)^2}{([(x_j - x_i)^2 + (y_j - y_i)^2]^2} - \frac{c+1}{2} \frac{x_i^2 - y_i^2}{(x_i^2 + y_i^2)^2} + \frac{x_i(x_i^2 - 3y_i^2)}{(x_i^2 + y_i^2)^3} & \text{if } i = j, \end{cases}$$

and similar formulas for the mixed derivatives $\partial^2 L/\partial x_i \partial y_j$, which we will not reproduce here. The Hessian matrix H is not positive definite, since for

$$\boldsymbol{u} = (\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}, \underbrace{0, 0, \ldots, 0}_{n \text{ zeros}})^{\mathrm{t}}$$

we have

$$\boldsymbol{u}^{\mathrm{t}} H \boldsymbol{u} = \frac{c+1}{2} \sum_{i=1}^{n} \frac{x_i^2 - y_i^2}{(x_i^2 + y_i^2)^2} - \sum_{i=1}^{n} \frac{x_i(x_i^2 - 3y_i^2)}{(x_i^2 + y_i^2)^3},$$

and for

$$\boldsymbol{v} = (\underbrace{0,0,\ldots,0}_{n \text{ zeros}},\underbrace{1,1,\ldots,1}_{n \text{ ones}})^{\mathrm{t}}$$

we have

$$\boldsymbol{v}^{\mathrm{t}} H \boldsymbol{v} = -\frac{c+1}{2} \sum_{i=1}^{n} \frac{x_i^2 - y_i^2}{(x_i^2+y_i^2)^2} + \sum_{i=1}^{n} \frac{x_i(x_i^2 - 3y_i^2)}{(x_i^2+y_i^2)^3} = -\boldsymbol{u}^{\mathrm{t}} H \boldsymbol{u},$$

and both quantities cannot be positive. This means that the extremum is not a minimum, or a maximum and consequently the equilibrium is not a stable equilibrium. This became clear when simulating the electrostatic configuration in the presence of a dipole: even when the initial location of the charges is at the zeros of a the corresponding Bessel polynomial, the n free charges very quickly move away from this (unstable) equilibrium and move towards infinity, which is a global minimum. If one wants the equilibrium position to be a stable equilibrium, then one needs to find an appropriate constraint for the charges. For Jacobi polynomials the constraints are that all x_i are on the interval $[-1,1]$, for Laguerre polynomials the x_i need to be positive and such that $\sum_{i=1}^{n} x_i \leqslant nK$ for some positive constant K, for Hermite polynomials the x_i need to be real with $\sum_{i=1}^{n} x_i^2 \leqslant nK$ for some constant K. For Bessel polynomials it is not clear what the appropriate constraint should be. In general, the normalized zeros of a Bessel polynomial approach a smooth curve Γ in the complex plane, which is symmetric with respect to the horizontal axis [14, Theorem 10, Ch. 10]. The analytic representation of this curve is known [36, 7], hence a possible constraint would be that the charges need to be located at points z_i which, after normalization, are in the neighborhood of this curve Γ.

In [43] it was pointed out that the electrostatic interpretation of zeros of orthogonal polynomials indirectly leads to Fekete's transfinite diameter and logarithmic capacity, and nowadays logarithmic potential theory is again a powerful tool for investigating zeros of orthogonal polynomials (in general extremal polynomials) and zeros and poles of rational approximants. An extension of the equilibrium problem in logarithmic potential theory to vector potentials was made in [13] (see also [35, p. 177]). Let Δ_i $(i = 1,2,\ldots,k)$ be n compact sets in the complex plane and $A = (a_{i,j})$ a real symmetric positive-definite $k \times k$ matrix such that $a_{i,j} \geqslant 0$ when $\Delta_i \cap \Delta_j \neq \emptyset$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector measure such that μ_j is a positive measure on Δ_j $(j = 1,2,\ldots,k)$. The energy of μ is then given by

$$J(\mu) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{i,j} I(\mu_i, \mu_j),$$

where

$$I(\mu_i, \mu_j) = \int_{\Delta_i} \int_{\Delta_j} \log \frac{1}{|x-y|} \, \mathrm{d}\mu_j(x) \, \mathrm{d}\mu_i(y)$$

is the mutual energy of μ_i and μ_j. If all μ_k are probability measures, then $W = \inf J(\mu)$ is finite or ∞, where the infimum is taken over all probability vector measures μ. It can be shown that in the case $W < \infty$ there exists a unique vector measure μ_{e} such that $W = J(\mu_{\mathrm{e}})$ and for obvious reasons this vector measure is known as the equilibrium measure. This equilibrium measure is quite interesting in studying simultaneous rational approximation, in particular Hermite–Padé approximation. For instance, the logarithmic asymptotic behavior (nth root asymptotics) for the denominator polynomial $Q_n(x)$ of Hermite–Padé simultaneous approximants

$$\left(\frac{P_{n_1}(x)}{Q_n(x)}, \frac{P_{n_2}(x)}{Q_n(x)}, \ldots, \frac{P_{n_k}(x)}{Q_n(x)}\right)$$

to the k Stieltjes functions

$$\left(\int_{\Delta_1} \frac{\mathrm{d}\mu_1(t)}{x-t}, \int_{\Delta_2} \frac{\mathrm{d}\mu_2(t)}{x-t}, \ldots, \int_{\Delta_k} \frac{\mathrm{d}\mu_k(t)}{x-t}\right),$$

where the Δ_j are disjoint intervals on the real line (the *Angelesco* case [35, p. 196] for simultaneous Hermite–Padé approximation) is given in terms of the equilibrium problem for vector potentials with a matrix A given by

$$a_{i,j} = \begin{cases} 2p_i^2 & \text{if } i = j, \\ p_i p_j & \text{if } i = j, \end{cases}$$

where

$$p_i = \lim_{n\to\infty} \frac{n_i}{n}$$

[35, p. 196]. Another important system of Stieltjes functions for simultaneous Hermite–Padé approximation is a *Nikishin system* (MT-system) [35, p. 202] and for such systems the logarithmic asymptotic behavior is in terms of the equilibrium problem for vector potentials with a tridiagonal matrix A of the form

$$A = \begin{pmatrix} 2q_1^2 & -q_1q_2 & & & 0 \\ -q_1q_2 & 2q_2^2 & -q_2q_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -q_{k-2}q_{k-1} & 2q_{k-1}^2 & -q_{k-1}q_k \\ 0 & & & -q_{k-1}q_k & 2q_k^2 \end{pmatrix},$$

where $1 = q_1 > q_2 > \cdots > q_k > 0$ [35, p. 207].

3. Other special polynomials

In the work of Stieltjes one can find many examples of orthogonal polynomials. Some of these polynomials were later rediscovered and most of the time no reference to Stieltjes is made. However, in all honesty, Stieltjes does not always give the polynomials explicitly. Of interest to those studying orthogonal polynomials are the following:

- an explicit expression for the orthogonal polynomials;
- an explicit expression for the orthogonality measure or the weight function;
- an explicit expression of the coefficients in the three-term recurrence relation.

In a few cases Stieltjes provides all three of the ingredients but usually only part is given. Often the Stieltjes transform of the measure is given instead of the measure itself (which can be deduced from the Stieltjes transform by an inversion formula which was for that purpose also derived by Stieltjes). The coefficients of the three-term recurrence relation can be obtained from the J-fraction for the Stieltjes transform.

Let us consider monic polynomials $M_n(x)$ with recurrence relation

$$\begin{cases} M_{n+1}(x) = (x-\beta_n)M_n(x) - \gamma_n M_{n-1}(x), & n \geqslant 1 \\ M_0(x) = 1, \qquad M_1(x) = x - \beta_0. \end{cases} \tag{2}$$

Provided that $\{\beta_n, n \geqslant 0\}$ are real and $\{\gamma_n, n \geqslant 1\}$ are strictly positive, there exists a positive orthogonality measure Ψ such that

$$\int_{-\infty}^{\infty} M_m(x) M_n(x)\,\mathrm{d}\Psi(x) = \rho_n\,\delta_{mn}, \qquad \rho_0 = 1, \quad \rho_n = \gamma_1 \cdots \gamma_n, \quad n \geqslant 1. \tag{3}$$

The Stieltjes transform of the orthogonality measure is

$$\mathscr{S}(z) = \int \frac{\mathrm{d}\Psi(x)}{z+x}, \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{4}$$

and it is written as a continued fraction in the so-called Grommer form (J-fraction)

$$\mathscr{S}(z) = \underline{1}/z + \beta_0 - \underline{\gamma_1}/z + \beta_1 - \underline{\gamma_2}/z + \cdots \tag{5}$$

from which we read off the recurrence coefficients in (2).

In Stieltjes' work there is a complete identification between the continued fraction (5) and the holomorphic function $\mathscr{S}(z)$ (4). We will follow Stieltjes' terminology even if nowadays an object like (5) is called a continued fraction which may converge to $\mathscr{S}(z)$ the Stieltjes transform of the orthogonality measure. No confusion can arise from such an abuse of terminology.

Most of the examples worked out by Stieltjes correspond to birth and death processes, i.e., particular β_n and γ_n which can be written as

$$\beta_n = \lambda_n + \mu_n, \quad n \geqslant 0, \qquad \gamma_n = \lambda_{n-1}\mu_n, \quad n \geqslant 1, \tag{6}$$

with positive λ_n and μ_n.

Defining new polynomials F_n by

$$F_0(x) = 1, \qquad F_n(x) = \frac{(-1)^n}{\mu_1 \cdots \mu_n} M_n(x), \quad n \geqslant 1,$$

one gets the recurrence

$$\begin{cases} (\lambda_n + \mu_n - x)F_n(x) = \mu_{n+1}F_{n+1}(x) + \lambda_{n-1}F_{n-1}(x), & n \geqslant 1 \\ F_0(x) = 1, \qquad F_1(x) = \dfrac{\lambda_0 + \mu_0 - x}{\mu_1}. \end{cases} \tag{7}$$

The orthogonality relation becomes

$$\int F_m(x) F_n(x)\,\mathrm{d}\Psi(x) = \pi_n\,\delta_{mn}, \tag{8}$$

with

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \quad n \geqslant 1.$$

Following Stieltjes' articles is difficult because he uses several different ways to write the continued fractions. The first is the Grommer form (5) but he uses also what we call today Stieltjes continued fractions (S-fraction)

$$T(z) = \underline{1}/z + \underline{\lambda_0}/z + \underline{\mu_1}/z + \underline{\lambda_1}/z + \underline{\mu_2}/z + \cdots \tag{9}$$

from which we read the recurrence coefficients λ_n and μ_n. Elementary reductions give

$$\frac{T(\sqrt{z})}{\sqrt{z}} = \underline{1}/z + \underline{\lambda_0}/1 + \underline{\mu_1}/z + \underline{\lambda_1}/1 + \underline{\mu_2}/z + \cdots, \tag{10}$$

and use of the identity

$$z + \underline{\lambda_0}/1 + \underline{\mu_1}/X = z + \lambda_0 - \underline{\lambda_0\mu_1}/\mu_1 + X$$

brings (10) to the Grommer form

$$\frac{T(\sqrt{z})}{\sqrt{z}} = \underline{1}/z + \lambda_0 - \underline{\lambda_0\mu_1}/z + \lambda_1 + \mu_1 - \underline{\lambda_1\mu_2}/z + \cdots, \tag{11}$$

where we observe that $\mu_0 = 0$ by comparing (11) with (5) and (6).

In his articles Stieltjes presents his results in all possible writings (9), (10) and (11) and he often gives f such that

$$\int \frac{\sqrt{z}}{z+u}\, \mathrm{d}\Psi(u) = \int_0^\infty f(u) \mathrm{e}^{-u\sqrt{z}}\, \mathrm{d}u,$$

where f is the Laplace transform of the measure Ψ

$$f(u) = \int_0^\infty \mathrm{e}^{-tu}\, \mathrm{d}\Psi(t),$$

which expresses the fact that a Stieltjes transform is an iterated Laplace transform. If f has a Taylor expansion at zero of the form $f(u) = \sum_{n\geqslant 0}(-1)^n \sigma_n u^n/n!$, then clearly one gets the asymptotic series

$$\int \frac{\mathrm{d}\Psi(u)}{z+u} \sim \sum_{n\geqslant 0} \frac{(-1)^n \sigma_n}{z^{n+1}},$$

from which we see that the σ_n are the orthogonality measure moments $\sigma_n = \int t^n\, \mathrm{d}\Psi(t)$. In a similar way, if

$$\int_0^\infty \frac{z}{z^2+u^2}\, \mathrm{d}\Psi(u) = \int_0^\infty g(u) \mathrm{e}^{-zu}\, \mathrm{d}u, \tag{12}$$

where g has a Taylor expansion at zero with only even powers

$$g(u) = \sum_{n\geqslant 0} (-1)^n \sigma_n \frac{u^{2n}}{(2n)!},$$

then the moments of the measure $\Psi(\sqrt{u})$ on $[0,\infty)$ are $\int_0^\infty u^n \, \mathrm{d}\Psi(\sqrt{u}) = \sigma_n$. In this case

$$g(u) = \int_0^\infty \cos(ux) \, \mathrm{d}\Psi(x).$$

In all what follows, the page number for each example is taken from Vol. II of the new edition of Stieltjes' collected works [40] and the articles we have considered are

(1) "Sur la réduction en fraction continue d'une série procédant suivant les puissances descendantes d'une variable", pp. 188–204;

(2) "Sur quelques intégrales définies et leur développement en fractions continues", pp. 382–395;

(3) "Recherches sur les fractions continues", English translation, pp. 609–745.

Let us point out that nearly all of the continued fractions given by Stieltjes were recorded in Wall's book [45], but their connection with orthogonal polynomial theory was lost. As the reader will observe many of the examples worked out by Stieltjes are special cases of orthogonal polynomials in the Askey–Wilson scheme.

3.1. Indeterminate moment problems

Stieltjes was certainly the first to exhibit examples of indeterminate measures. He gives on p. 696 the orthogonality measure

$$\Psi'(u) = \chi_{[0,\infty[}(u) \frac{1}{\sqrt{\pi}} (1 + \lambda \sin(2\pi \ln u)) \mathrm{e}^{-(\ln u)^2}$$

for $-1 \leqslant \lambda \leqslant +1$, whose moments $\sigma_n = \sqrt{\pi} \mathrm{e}^{(n+1)^2/4}$ are independent of the parameter λ. He also gives

$$\lambda_n = q^{-2n-1}, \quad \mu_n = q^{-2n} - q^{-n}, \qquad q = \mathrm{e}^{-1/2} < 1.$$

The explicit form of the corresponding orthogonal polynomials was given in [25, p. 89] and can be written

$$p_n(x;q) = {}_1\Phi_1 \left(\begin{matrix} q^{-n} \\ 0 \end{matrix} ; q \; ; -xq^{n+3/2} \right).$$

In [43] we mentioned that Askey gave another weight function with respect to which these polynomials are orthogonal, but Askey mistakenly writes $x^{-5/2}$ instead of $\sqrt{x}$ in the numerator, and the weight function should read

$$\frac{\sqrt{x}}{(-x;q)_\infty (-q/x;q)_\infty}, \quad 0 < x < \infty.$$

Yet another weight function can be obtained by observing that $p_n(x) = (q;q)_n S_n(x\sqrt{q})$, where

$$S_n(x) = \frac{1}{(q;q)_n} {}_1\Phi_1 \left(\begin{matrix} q^{-n} \\ 0 \end{matrix} ; q \; ; -xq^{n+1} \right),$$

and the latter polynomials are the ones given in [25]. Using the weight function in [25] and taking into account the scaling $x \mapsto x\sqrt{q}$ gives the weight function

$$\frac{1}{(-x\sqrt{q};q)_\infty(-\sqrt{q}/x;q)_\infty}, \quad 0 < x < \infty.$$

Chihara has weight functions like this [8]. Stieltjes was aware that the special choice $q = e^{-1/2}$ was not the only possible choice and explicitly mentions that it sufficies to have $q < 1$. Stieltjes–Wigert polynomials are also limit cases of q-Laguerre polynomials, as was shown in [3]. These q-Laguerre polynomials are also corresponding to an indeterminate moment problem and they were studied [15, 31].

Two other examples are mentioned, the first on p. 695 with

$$\Psi'(u) = \chi_{[0,\infty[}(u)\left[1 + \lambda\sin(u^{1/4})\right] e^{-u^{1/4}}, \quad -1 \leqslant \lambda \leqslant +1,$$

for which $\sigma_n = 4(4n+3)!$, and the second on p. 707

$$\Psi'(u) = \chi_{[0,\infty[}(u) u^{a-1} e^{-bu^\lambda}, \quad a > 0,\ b > 0,\ \lambda > 0,$$

for which $\sigma_n = (b^{-(n+a)/\lambda}/\lambda)\Gamma((n+a)/\lambda)$, and he claims (without proof) this to be indeterminate for $\lambda < \frac{1}{2}$. In both cases the recurrence relation is not given. Perron [37, Ch. IV, Section 39] showed that the weight functions

$$e^{-bt^\lambda}\left[1 + \alpha\sin(bt^\lambda\tan(\lambda\pi))\right]$$

have for $-1 \leqslant \alpha \leqslant 1$ the same moments for all λ with $\lambda \in]0, \frac{1}{2}[$. Perron refers to Hamburger [16] who treated this case. These weight functions are nowadays known as Freud weights (on $[0,\infty)$) and have received quite some attention in the 1980s, mostly due to Freud's conjecture regarding that growth of the corresponding recurrence coefficients; see [32, Section 4.18, 28].

On p. 707 Stieltjes gives an isolated example of a determined orthogonality measure

$$\Psi'(u) = \chi_{[0,\infty[}(u)\,\frac{1}{2\pi\sqrt{u}}\,\ln\left(\frac{1}{1 - e^{-2\pi\sqrt{u}}}\right), \qquad \int \frac{d\Psi(u)}{z+u} = \frac{J(\sqrt{z})}{\sqrt{z}}, \quad \operatorname{Re} z > 0, \tag{13}$$

where J is related to Stirling's series for the logarithm of the Γ function

$$J(z) = \ln\Gamma(z) - (z - \tfrac{1}{2})\ln z + z - \tfrac{1}{2}\ln(2\pi).$$

This function J is related to the Binet–Stirling asymptotic expansion for the logarithm of the Γ function

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + \sum_{r=1}^{\infty}\frac{(-1)^r B_{2r}}{2r(2r-1)} z^{2r-1},$$

where the B_n are the Bernoulli numbers, see [10, p. 36], hence the moments are

$$\sigma_0 = \tfrac{1}{12}, \qquad \sigma_n = (-1)^{n+1}\frac{B_{2n}}{2n(2n+1)}, \quad n \geqslant 1.$$

Stieltjes gives only the first few coefficients in the recurrence, which are rational numbers, but nothing seems to be known about the relevant orthogonal polynomials.

3.2. Co-recursive Chebyshev

On p. 698 Stieltjes gives the continued fraction

$$F(z) = \frac{\sqrt{\alpha\beta}}{z} + \frac{1}{2\pi}\int_\alpha^\beta \frac{\sqrt{(u-\alpha)(\beta-u)}}{u}\frac{du}{z+u},$$

where

$$\alpha = (\sqrt{p}-\sqrt{q})^2, \qquad \beta = (\sqrt{p}+\sqrt{q})^2, \qquad \sqrt{\alpha\beta} = p-q, \qquad p>0,\ q>0,$$

and the mass at $z=0$ is present for $p>q$ and absent for $p \leqslant q$. The corresponding orthogonal polynomials have a recurrence of type (7) with

$$\lambda_n = q, \qquad \mu_n = p(1-\delta_{n0}).$$

The changes

$$F_n(x) = \left(-\sqrt{\frac{q}{p}}\right)^n U_n(y), \quad x = p+q+2\sqrt{pq}\,y,$$

bring the recurrence to the form

$$2yU_n(y) = U_{n+1}(y) + U_{n-1}(y), \qquad n \geqslant 1$$

$$U_0(y) = 1, \quad U_1(y) = 2y + \sqrt{\frac{p}{q}},$$

from which we recognize a particular case of a co-recursive perturbation of the second kind Chebyshev polynomials (see [9, p. 205]) with co-recursivity parameter $b = -\sqrt{p/q} < 0$. Switching to the variables v, ζ such that

$$u = p+q+2\sqrt{pq}\ v, \qquad z = -p-q+2\sqrt{pq}\ \zeta,$$

one obtains

$$F(z) = \tilde{F}(\zeta) = -\frac{b}{2}\left\{\frac{1-1/b^2}{\zeta+\frac{1}{2}(b+1/b)} + \frac{2}{\pi}\int_{-1}^{+1}\frac{\sqrt{1-v^2}}{1+b^2-2bv}\,\frac{dv}{\zeta+v}\right\},$$

where now the mass term is present for $b<-1$ and absent for $-1 \leqslant b < 0$. This result is in perfect agreement with relation (13.7) in [9, p. 205] up to an overall factor $-\frac{1}{2}b$.

Stieltjes also gives the explicit form

$$\tilde{F}(\zeta) = \frac{1}{2}\frac{\sqrt{\zeta^2-1}-\zeta-b}{\zeta+\frac{1}{2}(b+1/b)},$$

from which one can check that the first moment is $-\frac{1}{2}b$, showing that the probability orthogonality measure is

$$d\Psi(u) = (1-1/b^2)\, d\delta_{(b+1)/2b}(u) + \frac{2}{\pi}\chi_{[-1,+1]}(u)\frac{\sqrt{1-u^2}}{1+b^2-2bu}\,du,$$

in agreement with Chihara's result. (Here δ_α means the Dirac measure located at $u = \alpha$.) This example worked out by Stieltjes corresponds in queueing theory to the simplest queue M/M/1.

3.3. Meixner polynomials

On p. 730 Stieltjes gives

$$\frac{1}{m^a}F(z,\lambda,a,b) = \int_0^\infty \left(\frac{1-\lambda^b}{\mathrm{e}^{(1-\lambda)u}-\lambda^b}\right)^a \mathrm{e}^{-zu}\,\mathrm{d}u, \quad \operatorname{Re} z > 0,$$

where

$$m = \frac{1-\lambda}{1-\lambda^b}, \quad \lambda > 0, \quad \lambda \neq 1, \quad a > 0, \quad b > 0.$$

Setting $u = v/m$ the integral becomes

$$\int_0^\infty \left(\frac{1-\lambda^b}{\mathrm{e}^{(1-\lambda^b)v}-\lambda^b}\right)^a \mathrm{e}^{-zv/m}\,\mathrm{d}v \tag{14}$$

from which it is obvious that b can be scaled to 1 without any loss of generality. In this case the recurrence coefficients are

$$\lambda_n = n + a, \quad \mu_n = \lambda n, \qquad n \geqslant 0,$$

which correspond to Meixner polynomials. Their explicit hypergeometric form

$$F_n(x) = \lambda^n \frac{(a)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, x/(\lambda-1)\\ a\end{matrix}; 1 - 1/\lambda\right) \tag{15}$$

is not given but the orthogonality measure is derived as follows. For $\lambda > 1$ expanding

$$\left(\frac{\lambda-1}{\lambda-\mathrm{e}^{(1-\lambda)u}}\right)^a = \left(\frac{\lambda-1}{\lambda}\right)^a \sum_{n\geqslant 0} \frac{(a)_n}{n!\lambda^n}\mathrm{e}^{-n(\lambda-1)u}$$

and integrating term by term gives

$$F(z,\lambda,a,1) = \left(\frac{\lambda-1}{\lambda}\right)^a \sum_{n\geqslant 0} \frac{(a)_n}{n!}\frac{(1/\lambda)^n}{z+(\lambda-1)n}, \quad \operatorname{Re} z > 0.$$

Analytic continuation shows that this function is meromorphic with simple poles and we conclude that the orthogonality measure is

$$\Psi = \left(\frac{\lambda-1}{\lambda}\right)^a \sum_{n\geqslant 0} \frac{(a)_n}{n!}(1/\lambda)^n \delta_{(\lambda-1)n}, \quad \lambda > 1.$$

For $\lambda < 1$ the expansion

$$\left(\frac{1-\lambda}{\mathrm{e}^{(1-\lambda)u}-\lambda}\right)^a = (1-\lambda)^a \sum_{n\geqslant 0} \frac{(a)_n}{n!}\lambda^n \mathrm{e}^{-(n+a)(1-\lambda)u}$$

leads to

$$F(z,\lambda,a,1)=(1-\lambda)^a\sum_{n\geqslant 0}\frac{(a)_n}{n!}\frac{\lambda^n}{z+(1-\lambda)(n+a)}$$

and the orthogonality measure becomes

$$\Psi=(1-\lambda)^a\sum_{n\geqslant 0}\frac{(a)_n}{n!}\lambda^n\delta_{(1-\lambda)(n+a)},\quad \lambda<1.$$

Forty years later, Meixner [30] gives the generating function

$$\sum_{n\geqslant 0}w^nF_n(x)=(1-w)^{x/(1-\lambda)}(1-\lambda w)^{-a-x/(1-\lambda)}$$

from which he derives the orthogonality measure for $\lambda\neq 1$.

For $\lambda\to 1$ the Meixner polynomials reduce to Laguerre polynomials. In this limit Stieltjes writes

$$F(z,1,a,1)=\int_0^\infty\frac{e^{-uz}}{(1+u)^a}\,du=\frac{1}{\Gamma(a)}\int_0^\infty\frac{u^{a-1}e^{-u}}{z+u}\,du,\quad \operatorname{Re}z>0,$$

from which the orthogonality measure is explicit. In modern notation the first integral is nothing else but $\Psi(1,2-a;z)$ while the second is $z^{a-1}\Psi(a,a;z)$ and their equality is a special case of

$$\Psi(A,C;z)=z^{1-C}\Psi(A-C+1,2-C;z).$$

Another example is given on p. 386 with the continued fraction

$$\int_0^\infty\frac{e^{-zu}}{(\cosh u+a\sinh u)^m}\,du,\qquad m>0,\quad 0\leqslant a<1.$$

In terms of $\mu=(1-a)/(1+a)$, $u=v/(1+a)$ and $p=(z-m)/(1+a)$ this integral becomes

$$\frac{1+\mu}{2}\int_0^\infty\left(\frac{1+\mu}{e^{(1+\mu)v}+\mu}\right)^m e^{-pv}\,dv$$

which corresponds to Meixner polynomials with $\lambda=-\mu<0$ (compare with (14)), a case for which positive orthogonality is excluded. Here Stieltjes gives the recurrence coefficients but no orthogonality measure.

3.4. Continuous dual Hahn polynomials

These polynomials satisfy the recurrence relation (see [25, p. 27])

$$-(x^2+a^2)Q_n(x^2)=\lambda_nQ_{n+1}(x^2)-(\lambda_n+\mu_n)Q_n(x^2)+\mu_nQ_{n-1}(x^2),\tag{16}$$

with

$$\lambda_n=(n+a+b)(n+a+c),\qquad \mu_n=n(n+b+c-1)\tag{17}$$

and the explicit hypergeometric form

$$Q_n(x^2;a,b,c)=\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n}={}_3F_2\left(\begin{matrix}-n,a+\mathrm{i}x,a-\mathrm{i}x\\ a+b,a+c\end{matrix};1\right).$$

For $a\geqslant 0$ and $b>0$, $c>0$ they are orthogonal with respect to the probability measure

$$\Psi'(u)=\chi_{[0,\infty[}(u)\frac{1}{2\pi\Gamma(a+b,a+c,b+c)}\left|\Gamma\left(\begin{matrix}a+\mathrm{i}u,b+\mathrm{i}u,c+\mathrm{i}u\\ 2\mathrm{i}u\end{matrix}\right)\right|^2, \tag{18}$$

where we use the notation

$$\Gamma\left(\begin{matrix}a_1,\ldots,a_n\\ b_1,\ldots,b_l\end{matrix}\right)=\frac{\prod_{i=1}^n\Gamma(a_i)}{\prod_{i=1}^l\Gamma(b_i)}.$$

The first example in this class appears on p. 738 in the form

$$\mathscr{S}(z)=\tfrac{1}{2}\left\{\psi\left(\tfrac{1}{2}(z+2-\lambda)\right)+\psi\left(\tfrac{1}{2}(z+1+\lambda)\right)-\psi\left(\tfrac{1}{2}(z+\lambda)\right)-\psi\left(\tfrac{1}{2}(z+1-\lambda)\right)\right\},$$

with $\psi(z)=\Gamma'(z)/\Gamma(z)$ the logarithmic derivative of the Γ function. Stieltjes also gives

$$\mathscr{S}(z)=\int_0^\infty\frac{\cosh((\lambda-\frac{1}{2})u)}{\cosh(\frac{1}{2}u)}\mathrm{e}^{-uz}\,\mathrm{d}u=2\sin(\lambda\pi)\int_0^\infty\frac{\cosh(\pi u)}{\cosh^2(\pi u)-\cos^2(\pi\lambda)}\,\frac{z}{z^2+u^2}\,\mathrm{d}u, \tag{19}$$

where $\operatorname{Re}z>0$ and

$$\lambda_n=(n+\lambda)(n+1-\lambda),\qquad \mu_n=n^2,\qquad 0<\lambda<\tfrac{1}{2},$$

which corresponds to the polynomials $S_n(x^2;0,\lambda,1-\lambda)$. It is easy to check that the appropriate limits in (18) do give the orthogonality measure (19).

Using the generating function of the Euler polynomials (see [10, p. 40]) Stieltjes gets

$$\frac{\cosh((\lambda-\frac{1}{2})u)}{\cosh(\frac{1}{2}u)}=\sum_{n\geqslant 0}E_{2n}(\lambda)\frac{u^{2n}}{2n!}$$

which implies the moments

$$\sigma_n=2\sin(\lambda\pi)\int_0^\infty\frac{\cosh(\pi u)}{\cosh^2(\pi u)-\cos^2(\pi\lambda)}u^{2n}\mathrm{d}u=(-1)^nE_{2n}(\lambda),\quad n\geqslant 0,$$

in agreement with relation (19) in [10, p. 43].

The second example, on p. 739, is

$$\mathscr{S}(z)=\frac{1}{(2\lambda-1)}\left\{\psi\left(\frac{z+2-\lambda}{2}\right)-\psi\left(\frac{z+1+\lambda}{2}\right)+\psi\left(\frac{z+\lambda}{2}\right)-\psi\left(\frac{z+1-\lambda}{2}\right)\right\}$$

and

$$\mathscr{S}(z)=\frac{1}{\lambda-\frac{1}{2}}\int_0^\infty \frac{\sinh((\lambda-\frac{1}{2})u)}{\cosh(\frac{1}{2}u)}\mathrm{e}^{-uz}\,du=\frac{2\cos(\lambda\pi)}{\frac{1}{2}-\lambda}\int_0^\infty \frac{u\sinh(\pi u)}{\cosh^2(\pi u)-\cos^2(\pi\lambda)}\,\frac{z}{z^2+u^2}\,du. \tag{20}$$

The recurrence coefficients are

$$\lambda_n=(n+\tfrac{1}{2}+\lambda)(n+\tfrac{3}{2}-\lambda), \qquad \mu_n=n^2,$$

whose polynomials are

$$S_n(x^2;\tfrac{1}{2},\lambda,1-\lambda), \quad 0\leqslant\lambda\leqslant\tfrac{1}{2}.$$

The moments follow from

$$\frac{\sinh((\lambda-\frac{1}{2})u)}{\cosh(\frac{1}{2}u)}=\sum_{n\geqslant 0}E_{2n+1}(\lambda)\frac{u^{2n+1}}{(2n+1)!}$$

and are

$$\sigma_n=\frac{2\cos(\lambda\pi)}{\frac{1}{2}-\lambda}\int_0^\infty \frac{u\sinh(\pi u)}{\cosh^2(\pi u)-\cos^2(\pi\lambda)}u^{2n}du=(-1)^n\frac{E_{2n+1}(\lambda)}{E_1(\lambda)}, \quad n\geqslant 0.$$

This result agrees with relation (20) in [10, p. 43].

The third example is given on p. 196 [1]

$$\int_0^\infty \frac{z\,\mathrm{d}\Psi(u)}{z^2+u^2}=\int_0^\infty (\cosh(\tfrac{1}{2}u))^{-k}e^{-uz}\,\mathrm{d}u, \quad \mathrm{Re}\,z>0, \tag{21}$$

with

$$\lambda_n=(n+\tfrac{1}{2}k)(n+\tfrac{1}{2}k), \qquad \mu_n=n\left(n+\tfrac{1}{2}(k-1)\right),$$

and corresponds to the polynomials

$$S_n(x^2;0,\tfrac{1}{2},\tfrac{1}{2}k), \quad k>0.$$

Let us point out that in (21) the orthogonality measure is not explicit. To get it one needs relation (26) in [10, p. 11]

$$\int_0^\infty \frac{\cos(ut)}{(\cosh(\frac{1}{2}t))^k}\mathrm{d}t=\frac{2^{k-1}}{\Gamma(k)}\left|\Gamma(\tfrac{1}{2}k+\mathrm{i}u)\right|^2,$$

from which the Fourier inversion theorem gives

$$(\cosh(\tfrac{1}{2}u))^{-k}=\frac{2^k}{\pi\Gamma(k)}\int_0^\infty \cos(ut)\left|\Gamma(\tfrac{1}{2}k+\mathrm{i}t)\right|^2\,\mathrm{d}t.$$

[1] Note that in his continued fraction Stieltjes has $4\lambda_n$ and $4\mu_n$. One has to scale z and the continued fraction and this explains the $\cosh(\frac{1}{2}u)$ instead of $\cosh u$.

Inserting this relation into (21) and interchanging the order of summation leads to

$$\int \frac{z\,\mathrm{d}\Psi(u)}{z^2+u^2} = \frac{2^k}{\pi\Gamma(k)}\int_0^\infty \left|\Gamma(\tfrac{1}{2}k+\mathrm{i}u)\right|^2 \frac{z}{z^2+u^2}\,\mathrm{d}u, \quad \mathrm{Re}\, z > 0,$$

and the orthogonality measure agrees with (18) for this particular choice of the parameters.

Defining, following [9, p. 181], generalized Euler numbers by $(\cosh u)^{-k} = \sum_{n\geqslant 0} E_n^{(k)} u^n/n!$, gives for the moments $\sigma_n = (-4)^{-n} E_{2n}^{(k)}$, $n \geqslant 0$. This result, for $k \in \mathbb{N}$, is attributed to Carlitz in [9, p. 181].

This example of Stieltjes is also of interest because it corresponds to the case where the continuous dual Hahn degenerate into symmetric Meixner–Pollaczek polynomials $P_n^{(\lambda)}(x; \frac{1}{2}\pi)$ by duplication according to

$$S_n(x^2; 0, \tfrac{1}{2}, \tfrac{1}{2}k) = \frac{(-1)^n n!}{(\frac{1}{2}k)_n} P_{2n}^{(k/2)}(x; \tfrac{1}{2}\pi), \quad k > 0.$$

Such a relation is implicit in [9, p. 181].

3.5. *Associated continuous dual Hahn*

These polynomials are defined by the recurrence (16) with

$$\lambda_n = (n+a)(n+b), \quad \mu_n = (n+\tilde{a})(n+\tilde{b})(1-\eta\delta_{n0}), \qquad n \geqslant 0,$$

where either $\eta = 0$ or $\eta = 1$, the second case corresponding to the zero-related polynomials. If either $\tilde{a}$ or $\tilde{b}$ vanishes one recovers the continuous dual Hahn polynomials. For both values of η the Stieltjes transform and the absolutely continuous part of the orthogonality measure have been derived in [20] and involve hypergeometric functions at the level ${}_3F_2$.

Surprisingly enough, Stieltjes has studied on p. 394 a particular subclass of these polynomials with

$$\lambda_n = (n+\tfrac{1}{2}\alpha)(n+\tfrac{1}{2}\beta), \quad \mu_n = (n+\tfrac{1}{2}(\alpha-1))(n+\tfrac{1}{2}(\beta-1))(1-\delta_{n0}), \qquad n \geqslant 0,$$

i.e., in the zero-related case. He gives the continued fraction [2]

$$\mathscr{S}(z) = \int \frac{z\,\mathrm{d}\Psi(u)}{z^2+u^2} = \frac{2f(\alpha,\beta;z)}{(\beta-1)f(\alpha-1,\beta-1;z)}, \tag{22}$$

with

$$f(\alpha,\beta;z) = \int_0^\infty (\sinh u)^{\beta-1}(\cosh u)^{-\alpha}\mathrm{e}^{-2uz}\mathrm{d}u, \quad \alpha > 1, \;\; \beta > 1, \;\; \mathrm{Re}\, z > 0. \tag{23}$$

[2] We take into account that Stieltjes considers $4\lambda_n$ and $4\mu_n$ instead of λ_n and μ_n.

The change of variable $v = 1 - e^{-2u}$ transforms (23) into Euler's integral for the hypergeometric function with variable $z = \frac{1}{2}$ and (22) becomes

$$\mathscr{S}(z) = \frac{1}{\left(\frac{1}{2}(\alpha+\beta-1)+z\right)} \frac{{}_2F_1\left(\begin{matrix}\alpha,\beta \\ \frac{1}{2}(\alpha+\beta+1)+z\end{matrix};\frac{1}{2}\right)}{{}_2F_1\left(\begin{matrix}\alpha-1,\beta-1 \\ \frac{1}{2}(\alpha+\beta-1)+z\end{matrix};\frac{1}{2}\right)}. \tag{24}$$

It is interesting to compare this with the expression for $\mathscr{S}(z)$ which follows from [20]. One gets

$$\mathscr{S}(z) = \frac{2}{\alpha-1} \frac{{}_3F_2\left(\begin{matrix}z,\frac{1}{2}+z,\frac{1}{2}(\beta-\alpha+1)+z \\ 1+2z,\frac{1}{2}(\beta+1)+z\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}1+z,\frac{1}{2}+z,\frac{1}{2}(\beta-\alpha+1)+z \\ 1+2z,\frac{1}{2}(\beta+1)+z\end{matrix};1\right)} \tag{25}$$

and Stieltjes results show that each ${}_3F_2$ should reduce to a ${}_2F_1$. Let us prove

Theorem 1. *If* $-1-2a \notin \mathbb{N}$ *one has the relation*

$${}_3F_2\left(\begin{matrix}a,\frac{1}{2}+a,c \\ 1+2a,d\end{matrix};1\right) = 2^{2a}\Gamma\left(\begin{matrix}d,d-c+\frac{1}{2} \\ \frac{1}{2},2d-c\end{matrix}\right){}_2F_1\left(\begin{matrix}2d-2a-1,2d-2c \\ 2d-c\end{matrix};\frac{1}{2}\right)$$

provided that $\operatorname{Re} d > \operatorname{Re} c > 0$ *and*

$${}_3F_2\left(\begin{matrix}1+a,\frac{1}{2}+a,c \\ 1+2a,d\end{matrix};1\right) = 2^{2a}\Gamma\left(\begin{matrix}d,d-c-\frac{1}{2} \\ \frac{1}{2},2d-c-1\end{matrix}\right){}_2F_1\left(\begin{matrix}2d-2a-2,2d-2c-1 \\ 2d-c-1\end{matrix};\frac{1}{2}\right)$$

provided that $\operatorname{Re} d > \operatorname{Re} c + \frac{1}{2}$, $\operatorname{Re} c > 0$.

Proof. Let us begin with the first relation. Since $-1-2a \notin \mathbb{N}$ and $\operatorname{Re}(d-c) > -\frac{1}{2}$ the ${}_3F_2$ is well defined. Let us use Theorem 28 in [38, p. 85]

$${}_3F_2\left(\begin{matrix}a,\frac{1}{2}+a,c \\ 1+2a,d\end{matrix};1\right) = \Gamma\left(\begin{matrix}d \\ d-c,c\end{matrix}\right)\int_0^1 (1-u)^{d-c-1}u^{c-1}{}_2F_1\left(\begin{matrix}a,\frac{1}{2}+a \\ 1+2a\end{matrix};u\right)\,du$$

for $\operatorname{Re} d > \operatorname{Re} c > 0$. We change the variable by $\sqrt{u} = \sin\theta$ to get

$$2\Gamma\left(\begin{matrix}d \\ d-c,c\end{matrix}\right)\int_0^{\pi/2} (\sin\theta)^{2c-1}(\cos\theta)^{2d-2c-1}{}_2F_1\left(\begin{matrix}a,\frac{1}{2}+a \\ 1+2a\end{matrix};\sin^2\theta\right)\,d\theta$$

and then use the quadratic transformation (2) in [10, p. 111]

$$ {}_2F_1\left(\begin{matrix} a, \frac{1}{2}+a \\ 1+2a \end{matrix}; \sin^2\theta\right) = {}_2F_1\left(\begin{matrix} 2a, 1+2a \\ 1+2a \end{matrix}; \sin^2(\tfrac{1}{2}\theta)\right) = (\cos(\tfrac{1}{2}\theta))^{-4a}. $$

The remaining integral is reduced, via the change of variable $u = 2\sin^2(\frac{1}{2}\theta)$, to Euler's integral for the hypergeometric function

$$ {}_2F_1\left(\begin{matrix} 2a-c+1, c \\ 2d-c \end{matrix}; \tfrac{1}{2}\right) $$

which gives the first relation upon use of the transformation relation (2) in [10, p. 105].

The second relation is proved similarly. The only difference is that now we use the quadratic transformation (13) in [10, p. 111] which gives

$$ {}_2F_1\left(\begin{matrix} 1+a, \frac{1}{2}+a \\ 1+2a \end{matrix}; \sin^2\theta\right) = \frac{(\cos(\frac{1}{2}\theta))^{-4a}}{\cos\theta}. $$

In Eq. (25) we use the first relation (resp. the second one) of Theorem 1 for the numerator (resp. the denominator) to obtain (24). The validity conditions are satisfied because we can choose $\operatorname{Re} z$ large enough (and positive) and then use analytic continution to extend the equality to $z \in \mathbb{C}\backslash\mathbb{R}$. $\square$

Stieltjes does not give the absolutely continuous part of the orthogonality measure but from his results we expect that the ${}_3F_2$ which appears should simplify similarly. This is indeed the case. From the results of [20] and upon use of Theorem 1 we get the absolutely continuous part of the orthogonality measure with weight function

$$ \frac{2\chi_{[0,\infty)}(u)}{\Gamma\left(\frac{1}{2}\alpha, \frac{1}{2}\beta, \frac{1}{2}(\alpha+1), \frac{1}{2}(\beta+1)\right)} \left| \frac{\Gamma\left(\frac{1}{2}(\alpha+\beta-1)+iu\right)}{{}_2F_1\left(\begin{matrix} \alpha-1, \beta-1 \\ \frac{1}{2}(\alpha+\beta-1)+iu \end{matrix}; \tfrac{1}{2}\right)} \right|^2. $$

For $\beta \to 1$ we are back to the third example of continuous dual Hahn polynomials given in the previous subsection.

3.6. *Wilson polynomials*

These are defined by

$$ Q_n(x^2; a,b,c,d) = \frac{W_n(x^2; a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3\left(\begin{matrix} -n, n+s-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1\right), $$

where $s = a+b+c+d$ and provided that $\mathrm{Re}\, a \geqslant 0$ and that $\mathrm{Re}\, b, \mathrm{Re}\, c, \mathrm{Re}\, d > 0$. The recurrence relation of the $Q_n(x^2)$ is of the form (16) with

$$\begin{cases} \lambda_n = \dfrac{(n+a+b)(n+a+c)(n+a+d)(n+s-1)}{(2n+s-1)(2n+s)}, \\ \mu_n = \dfrac{n(n+b+c-1)(n+c+d-1)(n+d+b-1)}{(2n+s-2)(2n+s-1)}. \end{cases} \tag{26}$$

For $a \geqslant 0$, and $b, c, d > 0$ they are orthogonal with respect to the probability measure

$$\Psi'(u) = A\chi_{[0,\infty[}(u) \left| \Gamma \begin{pmatrix} a+\mathrm{i}u, b+\mathrm{i}u, c+\mathrm{i}u, d+\mathrm{i}u \\ 2\mathrm{i}u \end{pmatrix} \right|^2, \tag{27}$$

with

$$A = \frac{1}{2\pi} \Gamma \begin{pmatrix} s \\ a+b, a+c, a+d, b+c, c+d, d+b \end{pmatrix}.$$

The first example is given by Stieltjes on p. 736 (it also appears on p. 387 with a slightly different notation)

$$\mathscr{S}(z) = \frac{1}{\lambda(1-\lambda)} \{\psi(z+\lambda) + \psi(z+1-\lambda) - \psi(z) - \psi(z+1)\},$$

and he also gives

$$\begin{aligned} \mathscr{S}(z) &= \frac{2z}{\lambda(1-\lambda)} \int_0^\infty \frac{\sinh(\frac{1}{2}\lambda u)\sinh(\frac{1}{2}(1-\lambda)u)}{\sinh(\frac{1}{2}u)} \mathrm{e}^{-uz}\, \mathrm{d}u \\ &= \frac{2\sin^2(\lambda\pi)}{\lambda(1-\lambda)} \int_0^\infty \frac{u}{\tanh(\pi u)[\cosh^2(\pi u) - \cos^2(\pi\lambda)]} \frac{z}{z^2+u^2}\, \mathrm{d}u. \end{aligned} \tag{28}$$

The recurrence coefficients are

$$\lambda_n = \frac{(n+1)(n+\lambda)(n+1-\lambda)}{2(2n+1)}, \qquad \mu_n = \frac{n(n+\lambda)(n+1-\lambda)}{2(2n+1)}, \tag{29}$$

corresponding to the particular case

$$W_n(x^2; 0, 1, \lambda, 1-\lambda), \quad 0 < \lambda \leqslant \tfrac{1}{2}.$$

Taking the appropriate limits in (27) one indeed obtains the orthogonality measure in (28).

Using the generating function of the Bernoulli polynomials

$$\frac{u\mathrm{e}^{\lambda u}}{\mathrm{e}^u - 1} = \sum_{n \geqslant 0} B_n(\lambda) \frac{u^n}{n!}, \quad B_n = B_n(0).$$

Stieltjes finds the moments

$$\sigma_n = \frac{2\sin^2(\lambda\pi)}{\lambda(1-\lambda)} \int_0^\infty \frac{u^{2n+1}}{\tanh(\pi u)[\cosh^2(\pi u) - \cos^2(\pi\lambda)]} \mathrm{d}u = \frac{(-1)^n}{n+1} \frac{B_{2n+2} - B_{2n+2}(\lambda)}{B_2 - B_2(\lambda)}, \quad n \geqslant 0.$$

Surprisingly this relation does not seem to appear in [10] (beware that Stieltjes uses a different definition of Bernoulli polynomials).

The second example of Wilson polynomials appears on p. 737 (also on p. 390 with a slightly different notation)

$$\mathscr{S}(z) = \frac{1}{(2\lambda - 1)}\{\psi(z+\lambda) - \psi(z+1-\lambda)\}$$

and the orthogonality measure is given by

$$\begin{aligned}\mathscr{S}(z) &= \frac{1}{(2\lambda-1)}\int_0^\infty \frac{\sinh((\lambda-\frac{1}{2})u)}{\sinh(\frac{1}{2}u)}\mathrm{e}^{-uz}\,\mathrm{d}u \\ &= -\frac{\sin(2\pi\lambda)}{(2\lambda-1)}\int_0^\infty \frac{1}{\cosh^2(\pi u) - \cos^2(\pi\lambda)}\frac{z}{z^2+u^2}\,\mathrm{d}u. \end{aligned} \tag{30}$$

The recurrence coefficients are

$$\lambda_n = \frac{(n+\frac{1}{2})^2(n+\lambda)(n+1-\lambda)}{(2n+\frac{1}{2})(2n+\frac{3}{2})}, \qquad \mu_n = \frac{n^2(n+\frac{1}{2}-\lambda)(n-\frac{1}{2}+\lambda)}{(2n-\frac{1}{2})(2n+\frac{1}{2})},$$

and correspond to the polynomials

$$W_n(x^2; 0, \tfrac{1}{2}, \lambda, 1-\lambda), \qquad 0 < \lambda < 1.$$

Taking the appropriate limits in (27) gives (30) and here too the moments are given in terms of Bernoulli polynomials

$$\sigma_n = \frac{\sin(2\lambda\pi)}{1-2\lambda}\int_0^\infty \frac{u^{2n+1}}{\cosh^2(\pi u) - \cos^2(\pi\lambda)}\mathrm{d}u = \frac{(-1)^n}{2n+1}\frac{B_{2n+1}(\lambda)}{B_1(\lambda)}, \quad n \geqslant 0,$$

which is relation (21) in [10, p. 38].

On p. 387 Stieltjes takes the limit $\lambda \to \frac{1}{2}$ for which he gets

$$\mathscr{S}(z) = \psi'(z+\tfrac{1}{2}) = \pi\int_0^\infty \frac{1}{\cosh^2(\pi u)}\frac{z}{z^2+u^2}\,\mathrm{d}u$$

corresponding to $W_n(x^2; 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. One has also for the moments

$$\mathscr{S}(z) = \int_0^\infty \frac{u}{\sinh(\frac{1}{2}u)}\mathrm{e}^{-uz}\,\mathrm{d}u \quad \Longrightarrow \quad \sigma_n = (-1)^n\left(1 - 2^{-2n+1}\right)B_{2n}, \qquad n \geqslant 0.$$

The more general case of

$$W_n(x^2; 0, \tfrac{1}{2}, \tfrac{1}{2}\alpha, \tfrac{1}{2}\beta), \qquad \alpha > 0, \ \beta > 0,$$

is considered in relation (30) on p. 393. Here Stieltjes gives the continued fraction (but not the orthogonality measure) in the form[3]

$$\int \frac{z\,\mathrm{d}\Psi(u)}{z^2+u^2} = \int_0^\infty {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \frac{1}{2}(\alpha+\beta+1)\end{matrix}; -\sinh^2(\tfrac{1}{2}u)\right)\mathrm{e}^{-uz}\,\mathrm{d}u, \quad \operatorname{Re} z > 0. \tag{31}$$

[3] Here he takes $16\lambda_n$ and $16\mu_n$ instead of λ_n and μ_n.

Since we know the orthogonality measure, which is

$$\Psi'(u) = \frac{2^{\alpha+\beta-1}}{\pi}\Gamma\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1)\\ \frac{1}{2},\alpha,\beta,\frac{1}{2}(\alpha+\beta)\end{matrix}\right)\chi_{[0,\infty[}(u)\left|\Gamma(\tfrac{1}{2}\alpha+\mathrm{i}u,\tfrac{1}{2}\beta+\mathrm{i}u)\right|^2, \tag{32}$$

Stieltjes' result can be stated as

Theorem 2. *For* $\alpha > 0, \beta > 0$ *and* $\operatorname{Re} z > 0$ *one has relation* (31) *where the measure* Ψ *is given by* (32).

Alternative Proof. We start from

$$H(t) = \int_0^\infty \cos(tu)\,\mathrm{d}\Psi(u), \tag{33}$$

where Ψ is given by (32). Inserting (33) in the integral $\int_0^\infty H(t)\mathrm{e}^{-tz}\,\mathrm{d}t$ and reversing the order of the integration gives

$$\int_0^\infty H(t)\mathrm{e}^{-tz}\,\mathrm{d}t = \int \frac{z\,\mathrm{d}\Psi(u)}{z^2+u^2}.$$

To prove the theorem we just need to compute $H(t)$. We use relation (36) in [11, p. 93][4]

$$\left|\Gamma(\tfrac{1}{2}\alpha+\mathrm{i}u,\tfrac{1}{2}\beta+\mathrm{i}u)\right|^2 = 2^{3-(\alpha+\beta)/2}\Gamma\left(\tfrac{1}{2}(\alpha+\beta)\right)\int_0^\infty K_{2\mathrm{i}u}(x)K_{(\alpha-\beta)/2}(x)x^{(\alpha+\beta)/2-1}\,\mathrm{d}x.$$

In (33) one can reverse the order of integration using

$$\int_0^\infty \cos(tu)K_{2\mathrm{i}u}(x)\,\mathrm{d}u = \tfrac{1}{4}\pi\mathrm{e}^{-x\cosh(t/2)}$$

one gets

$$H(t) = 2^{(\alpha+\beta)/2}\Gamma\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1)\\ \frac{1}{2},\alpha,\beta\end{matrix}\right)\int_0^\infty x^{(\alpha+\beta)/2-1}\mathrm{e}^{-x\cosh(t/2)}K_{(\alpha-\beta)/2}(x)\,\mathrm{d}x.$$

This integral is given by relation (26) in [11, p. 50] and leads to a ${}_2F_1$ which, after a Kummer transformation, is nothing but

$$H(t) = {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix};-\sinh^2(\tfrac{1}{4}t)\right),$$

and this proves Theorem 2. □

[4] Beware that in the validity conditions there is a misprint: the parameter ρ should be changed to $-\rho$.

This remarkable result of Stieltjes shows how complicated the moments for this particular case of Wilson polynomials already are. Indeed they are implicitly given by

$$ {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix};-\sinh^2(\tfrac{1}{2}t)\right)=\sum_{n\geqslant 0}(-1)^n\sigma_n(\alpha,\beta)\,\frac{t^{2n}}{(2n)!}. $$

To conclude this section let us examine more closely the example (18) considered on p. 390. Stieltjes gives the continued fraction in Grommer form for the integral

$$ \frac{c}{ab}\int_0^\infty \frac{\sinh(au)\sinh(bu)}{\sinh(cu)}\mathrm{e}^{-uz}\,\mathrm{d}u, $$

with the coefficients

$$ \beta_n = 2c^2(n^2+n+\tfrac{1}{2})-a^2-b^2, \qquad \gamma_n = 4\frac{n^2(c^2n^2-a^2)(c^2n^2-b^2)}{(2n-1)(2n+1)}. $$

Trial and error shows that

$$ \beta_n=\lambda_n+\mu_n-4c^2A^2, \qquad \gamma_n=\lambda_{n-1}\mu_n, \qquad A=\tfrac{1}{2}(1-(a+b)/c), $$

where

$$ \lambda_n = 2c^2\frac{(n+1)(n+1-a/c)(n+1-b/c)}{2n+1}, \qquad \mu_n=2c^2\frac{n(n+a/c)(n+b/c)}{2n+1}. $$

For $c=0$ one gets Legendre polynomials in the variable $(x+a^2+b^2)/2ab$, while for $c\neq 0$ the corresponding orthogonal polynomials are Wilson polynomials with parameters

$$ A=\frac{1}{2}\left(1-\frac{a+b}{c}\right),\quad B=\frac{1}{2}\left(1+\frac{a+b}{c}\right),\quad C=\frac{1}{2}\left(1+\frac{a-b}{c}\right),\quad D=\frac{1}{2}\left(1-\frac{a-b}{c}\right), $$

i.e., with $A+B=C+D=1$.

In view of these important simplifications pointed out by Stieltjes, let us examine more closely the case of Wilson polynomials $W_n(x^2;a,b,c,d)$ when $a+b=c+d$. The coefficients of the recurrence (26) in which we get rid of the parameter a are

$$ \begin{cases} \lambda_n = \dfrac{(n+2c+d-b)(n+c+2d-b)(n+2c+2d-1)}{2(2n+2c+2d-1)},\\[2ex] \mu_n = \dfrac{n(n+b+c-1)(n+b+d-1)}{2(2n+2c+2d-1)}, \end{cases} $$

and unexpectedly

$$ \lambda_n+\mu_n=\tfrac{1}{2}\left\{n^2+n(2c+2d-1)+(2c+d-b)(c+2d-b)\right\}. $$

We can go a step further by imposing $b=c+\frac{1}{2}$ in which case we have

$$ \lambda_n=\tfrac{1}{4}(n+2d-\tfrac{1}{2})(n+2c+2d-1), \qquad \mu_n=\tfrac{1}{4}n(n+2c-\tfrac{1}{2}), $$

and these are the recurrence coefficients of the continuous dual Hahn polynomials (see (17)), but in the variable $4x^2$. We therefore obtain an interesting reduction of a ${}_4F_3$ to a ${}_3F_2$ which can be written as

Theorem 3. *Provided that $d > \frac{1}{4}$, $c + d > \frac{1}{2}$ one has the identity*

$$W_n(x^2; d - \tfrac{1}{2}, c + \tfrac{1}{2}, c, d) = \frac{(c+d)_n (c+d-\frac{1}{2})_n}{(2c+2d-1)_n} S_n(4x^2; 2d-1, \tfrac{1}{2}, 2c). \tag{34}$$

A similar phenomenon happens for $a + b = c + d - 1$. Specializing further to $b = c - \frac{1}{2}$ leads again to relation (34) with c shifted to $c - \frac{1}{2}$.

3.7. Associated Wilson polynomials

They are defined by replacing $n \to n + \gamma$, with $\gamma \geqslant 0$ in the recurrence relation of the Wilson polynomials. This results in two families with

$$\begin{cases} \lambda_n = \dfrac{(n+\gamma+a+b)(n+\gamma+a+c)(n+\gamma+a+d)(n+\gamma+s-1)}{(2n+2\gamma+s-1)(2n+2\gamma+s)}, \\ \mu_n = \dfrac{(n+\gamma)(n+\gamma+b+c-1)(n+\gamma+c+d-1)(n+\gamma+d+b-1)}{(2n+2\gamma+s-2)(2n+2\gamma+s-1)}(1-\eta\delta_{n0}), \end{cases} \tag{35}$$

where $s = a+b+c+d$ and $\eta = 0$ or 1 for the zero related case ($\mu_0 = 0$). In both cases the Stieltjes transform and the absolutely continuous part of the orthogonality measure were given in [22]. Let us mention the Stieltjes transform of the zero-related case:

$$\begin{aligned} \mathcal{S}_0(z) = \int_0^\infty \frac{\mathrm{d}\Psi_0(u)}{z^2+a^2+u^2} &= \frac{(2\gamma+s-1)(\gamma+1+2t)}{(a+t)(\gamma+b+t)(\gamma+c+t)(\gamma+d+t)} \\ &\quad \times \frac{W(\gamma+1+2t; \gamma+1, 1-a+t, 1-b+t, 1-c+t, 1-d+t)}{W(\gamma+2t; \gamma, -a+t, 1-b+t, 1-c+t, 1-d+t)} \end{aligned} \tag{36}$$

with $t = \sqrt{a^2+z^2}$ and the very well-poised ${}_7F_6$ defined by

$$W(a; b, c, d, e, f) = {}_7F_6\left(\begin{matrix} a, 1+\frac{1}{2}a, b, c, d, e, f \\ \frac{1}{2}a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f \end{matrix}; 1\right).$$

The right-hand side of (36) is well defined for $\gamma+s > 1$, and analytic continuation in the parameter space extends its validity at least to $\gamma \geqslant 0$ and $a, b, c, d > 0$.

Amazingly, Stieltjes has obtained, on p. 393, a special case of this result. He considers the zero-related associated Wilson polynomials whose λ_n and μ_n are given by (35) with the particular choice of parameters

$$a = 0, \quad b = \tfrac{1}{2}, \quad c = \tfrac{1}{2}(\alpha - k), \quad d = \tfrac{1}{2}(\beta - k), \quad \gamma = \tfrac{1}{2}k, \quad \eta = 1,$$

and gives the continued fraction

$$\int \frac{z}{z^2+a^2}\,\mathrm{d}\Psi_0(u) = \frac{4}{k}\frac{f(\alpha,\beta,k;z)}{f(\alpha-1,\beta-1,k-1;z)}, \tag{37}$$

where

$$f(\alpha,\beta,k;z) = \int_0^\infty (\sinh u\cosh u)^k {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix};-\sinh^2 u\right)\mathrm{e}^{-4uz}\,\mathrm{d}u \tag{38}$$

provided that $k>0$, $\alpha-k>0$, $\beta-k>0$, $\mathrm{Re}\,z > k-\max(\alpha,\beta)$. Specializing (36) to Stieltjes' choice of parameters we get

$$\mathscr{S}_0(z) = \frac{(\alpha+\beta-1)(1+\frac{1}{2}k+2z)}{2z\left(\frac{1}{2}(k+1)+z\right)\left(\frac{1}{2}\alpha+z\right)\left(\frac{1}{2}\beta+z\right)}$$
$$\times\frac{W(1+\frac{1}{2}k+2z;\frac{1}{2}k+1,1+z,\frac{1}{2}+z,1+\frac{1}{2}(k-\alpha)+z,1+\frac{1}{2}(k-\beta)+z)}{W(\frac{1}{2}k+2z;\frac{1}{2}k,z,\frac{1}{2}+z,1+\frac{1}{2}(k-\alpha)+z,1+\frac{1}{2}(k-\beta)+z)}. \tag{39}$$

Let us now prove that (37) and (39) coincide. In (38) we use the quadratic transformation (18) in [10, p. 112] followed by the change of variable $u\to\frac{1}{2}u$ to get

$$f(\alpha,\beta,k;z) = 2^{-k-1}\int_0^\infty (\sinh u)^k {}_2F_1\left(\begin{matrix}\frac{1}{2}\alpha,\frac{1}{2}\beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix};-\sinh^2 u\right)\mathrm{e}^{-2uz}\,\mathrm{d}u. \tag{40}$$

We then use the Mellin–Barnes integral representation of the hypergeometric function

$${}_2F_1\left(\begin{matrix}\frac{1}{2}\alpha,\frac{1}{2}\beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix};-\sinh^2 u\right) = \Gamma\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1)\\ \frac{1}{2}\alpha,\frac{1}{2}\beta\end{matrix}\right)$$
$$\times\frac{1}{2\mathrm{i}\pi}\int_{-i\infty}^{+i\infty}\Gamma\left(\begin{matrix}\frac{1}{2}\alpha+\zeta,\frac{1}{2}\beta+\zeta,-\zeta\\ \frac{1}{2}(\alpha+\beta+1)+\zeta\end{matrix}\right)(\sinh u)^{2\zeta}\mathrm{d}\zeta,$$

which, when inserted in (40), gives a double integral. In view of its absolute convergence for $\mathrm{Re}\,z>\frac{1}{2}k$ one can reverse the order of integrations. The first integral

$$\int_0^\infty (\sinh u)^{k+2\zeta}\mathrm{e}^{-2uz}\,\mathrm{d}u$$

reduces, setting $x = \mathrm{e}^{-2u}$, to

$$2^{-1-k-2\zeta}\Gamma\begin{pmatrix} -\frac{1}{2}k+z-\zeta, k+1+2\zeta \\ \frac{1}{2}k+1+z+\zeta \end{pmatrix}.$$

We use the duplication formula in this result and we are left with

$$f(\alpha,\beta,k;z) = 2^{-k-2}\Gamma\begin{pmatrix} \frac{1}{2}(\alpha+\beta+1) \\ \frac{1}{2}, \frac{1}{2}\alpha, \frac{1}{2}\beta \end{pmatrix}$$
$$\times\frac{1}{2\mathrm{i}\pi}\int\Gamma\begin{pmatrix} -\zeta, -\frac{1}{2}k+z-\zeta, \frac{1}{2}\alpha+\zeta, \frac{1}{2}\beta+\zeta, \frac{1}{2}k+1+\zeta, \frac{1}{2}(k+1)+\zeta \\ \frac{1}{2}(\alpha+\beta+1)+\zeta, \frac{1}{2}k+1+z+\zeta \end{pmatrix}\mathrm{d}\zeta.$$

This gives a very well-poised ${}_7F_6$ by relation (2) in [4, p. 44]. Apart from Γ factors we get

$$W\left(\tfrac{1}{2}(\alpha+\beta)+z; \tfrac{1}{2}k+1, \tfrac{1}{2}+z, \tfrac{1}{2}\alpha, \tfrac{1}{2}\beta, \tfrac{1}{2}(\alpha+\beta-k)\right). \tag{41}$$

Applying to this object the connection relation (1) in [4, p. 62] (with $c = \frac{1}{2}k+1$, $d = \frac{1}{2}+z$) gives

$$f(\alpha,\beta,k;z) = A\ W\left(\tfrac{1}{2}k+1+2z; \tfrac{1}{2}k+1, \tfrac{1}{2}+z, 1+z, 1+\tfrac{1}{2}(k-\alpha)+z, 1+\tfrac{1}{2}(k-\beta)+z\right), \tag{42}$$

where

$$A = 2^{-2z-k-2}\Gamma\begin{pmatrix} \frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(k+1), \frac{1}{2}k+1, \frac{1}{2}(\alpha-k)+z, \frac{1}{2}(\beta-k)+z, 2+\frac{1}{2}k+2z \\ 1+\frac{1}{2}k+z, \frac{1}{2}(3+k)+z, 1+\frac{1}{2}\alpha+z, 1+\frac{1}{2}\beta+z, \frac{1}{2}(\alpha+\beta-k-1) \end{pmatrix}.$$

To deal with the denominator one should substitute $\alpha \to \alpha-1, \beta \to \beta-1, k \to k-1$ in the intermediate form (41). This gives, up to factors

$$W\left(\tfrac{1}{2}(\alpha+\beta)-1+z; \tfrac{1}{2}(k+1), \tfrac{1}{2}+z, \tfrac{1}{2}(\alpha-1), \tfrac{1}{2}(\beta-1), \tfrac{1}{2}(\alpha+\beta-k-1)\right).$$

One has to use again the connection relation (1) in [4, p. 62] (with $c = \frac{1}{2}(\alpha-1)$ and $d = \frac{1}{2}(\beta-1)$ to first get

$$W\left(\tfrac{1}{2}(\alpha+\beta-3)+z; \tfrac{1}{2}k, z, \tfrac{1}{2}(\alpha-1), \tfrac{1}{2}(\beta-1), \tfrac{1}{2}(\alpha+\beta-k)-1\right),$$

and the same connection relation (with $c = \frac{1}{2}k$ and $d = z$) to obtain

$$f(\alpha-1,\beta-1,k-1;z) = B\ W\left(\tfrac{1}{2}k+2z; \tfrac{1}{2}k, z, \tfrac{1}{2}+z, 1+\tfrac{1}{2}(k-\alpha)+z, 1+\tfrac{1}{2}(k-\beta)+z\right), \tag{43}$$

where

$$B = 2^{-2z-k-1}\Gamma\begin{pmatrix} \frac{1}{2}(\alpha+\beta-1), \frac{1}{2}(k+1), \frac{1}{2}k, \frac{1}{2}(\alpha-k)+z, \frac{1}{2}(\beta-k)+z, 1+\frac{1}{2}k+2z \\ 1+\frac{1}{2}k+z, \frac{1}{2}(k+1)+z, \frac{1}{2}\alpha+z, \frac{1}{2}\beta+z, \frac{1}{2}(\alpha+\beta-k-1) \end{pmatrix}.$$

Combining (42) and (43) shows that (37) and (39) coincide, up to analytic continuation in the parameters and in the variable z.

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