Capacitary Estimates for Dirichlet Eigenvalues

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Let $H$ be the non-negative definite selfadjoint operator associated to a regular irreducible Dirichlet form on $L^2(X, m)$. Assume that $H$ has discrete spectrum. We study perturbations of this operator which arise through the imposition of Dirichlet boundary conditions on a compact subset of $X$. The eigenvalues of the perturbed operator are of course shifted to the right. Under an ultracontractivity condition, we show that the magnitude of this shift can be estimated by the capacity. We also obtain a capacitary lower bound for the ground-state shift under suitable conditions. An application to the “crushed ice” problem is described.

1. Introduction

Let $H$ be the non-negative definite selfadjoint operator associated to a regular irreducible Dirichlet form on $L^2(X, m)$. Assume that $H$ has discrete spectrum. We study perturbations of this operator which arise through the imposition of Dirichlet boundary conditions on a compact subset $K$ of $X$. By the min-max principle we know a priori that the perturbed Dirichlet operator has discrete spectrum and that the perturbed eigenvalues are shifted to the right. Under an ultracontractivity condition, we show that the magnitude of the shift in the $k$th eigenvalue can be estimated from above by the capacity of order 1. According to [7] the groundstate is shifted to the right if and only if the set $K$ has positive capacity. We show that the ground-state shift also admits a capacitary lower bound under suitable conditions. Estimates in terms of the capacity of order zero hold if the bottom eigenvalue of $H$ is positive.

The case when $H$ is the non-negative definite Laplace-Beltrami operator on a compact Riemannian manifold $X$ and $K$ is a tubular neighbourhood of a submanifold of codimension at least two has been studied in [1, 2, 8–11]. For arbitrary compact $K$, estimates in terms of the capacity are given in [3].

In the following section we present essential tools from the theory of probabilistic potential theory. Sections 3 and 4 are devoted to the proof of the
upper and lower bounds for the ground-state shift respectively, while the upper bound for the shift in the higher eigenvalues is treated in Section 5. Lastly we describe an application to the so-called “crushed-ice” problem.

2. Some Probabilistic Potential Theory

In this section we fix notation, describe the details of our set-up and present some preliminary material on probabilistic potential theory. For background on Dirichlet forms we refer the reader to the monographs [5, 6, 14, 17].

Let $X$ be a locally compact separable metric space and $m$ an everywhere positive Radon measure defined on the Borel $\sigma$-algebra of $X$. We write $\mathcal{M} = L^2(X, m)$ for the Hilbert space of Borel measurable $m$-square integrable functions equipped with the usual inner product $(\cdot, \cdot)$. We work over the real numbers until otherwise specified. We are given an irreducible Dirichlet form $(\mathcal{F}, \mathcal{E})$ on $\mathcal{M}$. The space $\mathcal{F}$ together with the scalar product $\mathcal{E}(u, v) := \mathcal{E}(u, v) + (u, v)$ constitutes a Hilbert space. The associated non-negative definite selfadjoint operator is denoted $H$; thus $\mathcal{F} = D(H^{1/2})$ and $\mathcal{E}(u, u) = (H^{1/2}u, H^{1/2}u)$ for $u \in \mathcal{F}$. Let $\rho(H)$ and $\sigma(H)$ stand for the resolvent set and the spectrum of $H$ respectively. The resolvent is given by $R(z) := (H - z)^{-1}$, $z \in \rho(H)$ while a strongly continuous submarkovian semigroup is defined via $T_t := \exp(-tH)$, $t > 0$. The spectral measure associated to $H$ is written $E(\cdot)$. We assume throughout that

(A.1) $H$ has discrete spectrum.

By [4], Proposition 1.4.3 the lowest eigenvalue $\lambda$ of $H$ is simple.

Recall that the capacity (of order 1) associated to the Dirichlet form $(\mathcal{F}, \mathcal{E})$ is defined by

$$\text{Cap}(A) := \begin{cases} \inf \{ \mathcal{E}(u, u); u \in \mathcal{F}, u \geq 1 \text{ a.e. on } A \} & A \in X \text{ open}, \\ \inf \{ \text{Cap}(B); A \subset B, B \in X \text{ open} \}, & A \in X \text{ arbitrary}, \end{cases}$$

where the convention $\inf \emptyset = \infty$ applies. A function $u$ is said to be quasi-continuous (q.c.) if for any $\varepsilon > 0$ there exists an open set $G$ with $\text{Cap}(G) < \varepsilon$ such that the restriction of $u$ to $F = X \setminus G$ is continuous. We shall assume that $(\mathcal{F}, \mathcal{E})$ is regular i.e. $\mathcal{F} \cap C_0(X)$ is dense both in the Hilbert space $(\mathcal{F}, \mathcal{E}^{1/2})$ and the Banach space $C_0(X)$ of continuous functions on $X$ with compact support. Under this condition each $u \in \mathcal{F}$ has a quasi-continuous $m$-version $\tilde{u}$. It follows from regularity and Urysohn’s lemma that each compact set has finite capacity. A statement is said to be valid quasi-everywhere (q.e.) if it holds on the complement of a set of zero capacity.
Regularity further ensures the existence of a Hunt process $M = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x)$ properly associated to $(\mathcal{F}, \mathcal{E})$ with life-time $\zeta$. Define

$$V_t f(x) := E_x \left( \int_0^\zeta e^{-f(X_s)} \, ds \right).$$

Then $V_t f$ is a q.c. version of $R(-1)f, f \in \mathcal{E}$. Let $K$ be a compact subset of $X$. Given a function $f$ defined q.e. the hitting operator is defined by

$$P^1_K f(x) := E_x (e^{-\sigma(K)} f(X_{\tau_K}))$$

where $\sigma(K) := \inf \{ t > 0 : X_t \in K \}$ is the first hitting time of $K$.

A positive Radon measure $\mu$ on $X$ is said to be of finite energy integral if there exists a finite constant $c$ with

$$\int_X |u| \, d\mu \leq \epsilon \sqrt{\mathcal{E}_1(u,u)} \quad \text{for all} \quad u \in \mathcal{F} \cap C_0(X),$$

and the collection of such measures is denoted $S_0$. The Riesz representation theorem entails that there exists a function $U_1 \mu \in \mathcal{F}$, called the (1-) potential of $\mu$, such that

$$\mathcal{E}_1(U_1 \mu, u) = \langle u, \mu \rangle \quad \text{for all} \quad u \in \mathcal{F} \cap C_0(X) \quad (2.1)$$

where $\langle u, \mu \rangle := \int_X u \, d\mu$. In fact, the last inequality holds for all $u \in \mathcal{F}$ if $u$ is replaced by a quasi continuous version $\tilde{u}$.

We shall find the following descriptions of the capacity particularly useful. According to [6], Theorem 4.2.5 and (2.2.13) for instance,

$$\text{Cap}(K) = \mathcal{E}_1(p^1_K, p^1_K) = \mu_K(X) \quad (2.2)$$

where $p^1_K := P^1_K 1 = U_1 \mu_K$ is the (1-) equilibrium potential of $K$ and $\mu_K \in S_0$ is the (1-) equilibrium measure of $K$. Using (2.1) and (2.2) the capacity of a compact set $K$ may also be expressed as

$$\text{Cap}(K) = \sup \{ \mu(K) : \mu \in S_{00}, \text{Supp}[\mu] \subset K, \widetilde{U}_1 \mu \leq 1 \text{ q.e.} \} \quad (2.3)$$

(see [6], Problem 2.2.2) where

$$S_{00} := \{ \mu \in S_0 : \mu(X) < \infty, \|U_1 \mu\|_\infty < \infty \}.$$ 

Given a (1-)potential $f \in \mathcal{F}$ define

$$\mathcal{L}_{f,K} := \{ u \in \mathcal{F} : \tilde{u} \geq f \text{ q.e. on } K \}.$$
Then there exists a unique element \( f_{K} \in \mathcal{L}_{L_{K}} \) minimising \( \mathcal{E}_{1}(u, u) \) over \( \mathcal{L}_{L_{K}} \). The function \( P^{1}_{K}f \) is a quasi-continuous version of \( f_{K} \) and \( f_{K} = U_{1} \mu \) for a unique measure \( \mu \in S_{0} \) supported in \( K \) (see [6], Section 2.3 and Theorem 4.3.1).

Given a compact subset \( K \) of \( X \) put \( Y := X \setminus K \). Define

\[
\mathcal{F}_{Y} := \{ u \in \mathcal{F} : \hat{u} = 0 \text{ q.e. on } K \}
\]

The restriction of the form \( \mathcal{E} \) to the domain \( \mathcal{F}_{Y} \) is again a regular Dirichlet form; the associated non-negative definite selfadjoint operator is written \( H_{Y} \).

Notice that the spectrum of \( H_{Y} \) is again discrete by the min-max principle (see [13], Theorem XIII.1). Its resolvent and semigroup are denoted \( R_{Y}(z) \), \( z \in \rho(H_{Y}) \) resp. \( T_{Y}^{t} \), \( t > 0 \). The corresponding spectral measure is written \( E_{Y}(\cdot) \). Functions of the operator \( H_{Y} \) defined via the spectral theorem are of course mappings of \( L^{2}(Y) \) into itself, but can be extended to \( \mathcal{H} \) by introducing restriction and extension operators in the obvious way. This extension will be written simply \( R_{Y}(z) \) etc. to avoid cumbersome notation. The adjoint of the extended operator \( R_{Y}(z) \) is \( R_{Y}(\bar{z}) \).

Define

\[
V_{Y}^{1}f(x) := E_{Y} \left( \int_{0}^{e^{\mathcal{E}_{1}(K)}} e^{-f(X_{t})} dt \right).
\]

Then \( V_{Y}^{1}f \) is a quasi-continuous version of \( R_{Y}(-1)f \), \( f \in \mathcal{H} \). The Hilbert space \( (\mathcal{F}, \mathcal{E}) \) admits the orthogonal decomposition

\[
\mathcal{F} = \mathcal{F}_{Y} \oplus \mathcal{H}_{K}, \quad \mathcal{H}_{K} := \mathcal{F}_{Y}^{\perp}.
\]

Moreover, the orthogonal decomposition of \( V_{1}f \in \mathcal{F} \) is given by Dynkin’s formula

\[
V_{1}f = V_{Y}^{1}f + P^{1}_{K}V_{1}f
\]

and each of the above functions is quasi-continuous.

A zero-order version \( \text{Cap}_{0} \) of the capacity can be defined as above by replacing \( \mathcal{E}_{1} \) with \( \mathcal{E} \). In this case, we shall refer to the assumption

\[
(A.1)' \quad H \text{ has discrete spectrum and } \lambda > 0.
\]

By the min-max principle \( \|u\|_{2}^{2} \leq \lambda^{-1} \mathcal{E}(u, u) \), \( u \in \mathcal{F} \); hence \( (\mathcal{F}, \mathcal{E}) \) is a Hilbert space and the resolvent at zero \( R(0) \) is a bounded operator on \( \mathcal{H} \). Analogues of (2.2), (2.3), (2.4) and (2.5) also hold in the zero-order case [6].

The letters \( c, c_{1} \) etc. stand for generic constants and may change their value from one appearance to the next.
3. Upper Bound for the Ground-State Shift

We first estimate the ground-state shift in terms of the capacity of order 1. For this we require (A.1) as well as the following condition:

(A.2) \((T_t)_t > 0\) is ultracontractive i.e. \(\|T_t\|_{2,\infty} < \infty\) for all \(t > 0\).

Let \(\phi\) be the unique real-valued normalised eigenfunction of \(H\) corresponding to the ground-state energy \(\lambda\), which is strictly positive \(m\)-a.e. on \(X\) by [4]. By the spectral theorem \(\phi = \exp(\lambda t) T_t \phi\) and hence \(\|\phi\|_{\infty} \leq \exp(\lambda t) \|T_t\|_{2,\infty}\) by (A.2).

**Theorem 3.1.** Assume (A.1) and (A.2). Then there exist positive finite constants \(c_1, c_2\) such that

\[0 \leq \lambda^* - \lambda \leq c_1 \text{Cap}(K)\]

for all compact subsets \(K\) of \(X\) with capacity smaller than \(c_2\).

**Proof.** Since \(P^1_K V_1 \phi\) is a quasi-continuous version of the reduced function \((V_1 \phi)_K\) there exists a unique measure \(\mu \in S_0\) concentrated on \(K\) such that \(P^1_K V_1 \phi = U_1 \mu\) q.e. In fact, with the aid of (2.1) we see that \(\mu\) belongs to \(S_{\infty}(X, m)\) and \(K\) is compact. Using the characterisation of capacity (2.3) we have

\[(V_1 \phi, P^1_K V_1 \phi) = \delta(V_1 V_1, U_1 \mu) = \langle V_1 \phi, \mu \rangle = (1 + \lambda)^{-2} \langle \phi, \mu \rangle \leq (1 + \lambda)^{-2} \exp(2\lambda \|T_1\|_{2,\infty} \text{Cap}(K)) \quad (3.6)\]

We now estimate \(\lambda^*\) using the min-max principle:

\[1 + \lambda^* \leq \frac{\delta(V_1 \phi, V_1 \phi)}{(V_1 \phi, V_1 \phi)} = \frac{\delta(V_1 \phi, V_1 \phi) - \delta(V_1 \phi, P^1_K V_1 \phi)}{(V_1 \phi, V_1 \phi) - 2(V_1 \phi, P^1_K V_1 \phi) + (P^1_K V_1 \phi, P^1_K V_1 \phi)}\]

where we have used orthogonality (2.4) in the numerator,

\[\leq \frac{\delta(V_1 \phi, V_1 \phi)}{(V_1 \phi, V_1 \phi) - 2(V_1 \phi, P^1_K V_1 \phi)}\]

since \(\delta(V_1 \phi, P^1_K V_1 \phi) = \langle \phi, P^1_K V_1 \phi \rangle \geq 0\). Inserting the estimate (3.6) the calculation continues...
\[
\frac{1 + \lambda}{1 - 2 \exp{(2\lambda)} \| T_1 \|_{L^\infty} \text{Cap}(K)} \\
\leq 1 + \lambda + \frac{(1 + \lambda) \exp{(2\lambda)} \| T_1 \|_{L^\infty} \text{Cap}(K)}{1 - 2 \exp{(2\lambda)} \| T_1 \|_{L^\infty} \text{Cap}(K)} \\
\leq 1 + \lambda + \frac{4(1 + \lambda) \exp{(2\lambda)} \| T_1 \|_{L^\infty} \text{Cap}(K)}{1 - 2 \exp{(2\lambda)} \| T_1 \|_{L^\infty} \text{Cap}(K)}
\]

provided \( \text{Cap}(K) \leq \frac{1}{4} \exp(-2\lambda) \| T_1 \|_{L^\infty}^{-2} \).

There is also a zero-order counterpart to the above result, whose proof is along the same lines, namely

**Theorem 3.2.** Assume (A.1)' and (A.2). Then there exist positive finite constants \( c_1, c_2 \) such that

\[
0 \leq \lambda^* - \lambda \leq c_1 \text{Cap}(K)
\]

for all compact subsets \( K \) of \( X \) with capacity smaller than \( c_2 \).

## 4. Lower Bound for the Ground-State Shift

In this section we obtain lower bounds accompanying Theorems 3.1 and 3.2. For the companion to the former result, we shall need two further assumptions:

(A.3) \( m(X) < \infty \)

(A.4) \( R_{-1} \) is an integral operator whose kernel \( r_{-1}(x, y) \) satisfies \( \text{ess-inf}_{x, y \in U} r_{-1}(x, y) > 0 \) for any relatively compact open set \( U \) in \( X \).

Let \( \phi^* \) stand for the unique normalised eigenfunction corresponding to the ground-state energy \( \lambda^* \) of \( H_Y \).

**Lemma 4.1.** Assume (A.1)–(A.4). Let \( V \) be a relatively compact open subset of \( X \) with the property that \( W := X \setminus \bar{V} \) is non-empty. Then there exists a positive constant \( \delta \) such that

\[ V, \phi^* \geq \delta \text{ q.e. on } V \]

for all compact subsets \( K \) of \( V \).

**Proof.** Let \( U \) be a relatively compact open subset of \( X \) with \( V \subset U \subset X \). Since \( \bar{V} \) is compact and \( \mathcal{F}_U \neq \emptyset \) it follows that \( \lambda^* \) is finite. Then \( \| \phi^* \|_{L^\infty} \leq \exp(\lambda^*) \| T_1 \|_{L^\infty} \); thus,

\[ \| 1_{U^c} \phi^* \|_{L^1} \leq \exp(\lambda^*) \| T_1 \|_{L^\infty} m(U^c) \quad (4.7) \]
We also have that

\[ 1 = \| \phi^Y \|_2 = \exp(\lambda^W) \| T_1 \|_{L_\infty} \| \phi^Y \|_1 \]

(4.8)

Combining (4.7) and (4.8) we obtain

\[ \| 1_c \phi^Y \|_1 \geq \exp(-\lambda^W) \| T_1 \|_{L_\infty}^{-1} - \exp(\lambda^W) \| T_1 \|_{L_\infty} m(U'), \]

where all terms are finite by (A.3). By choosing \( U \) to be sufficiently large the right-hand side can be made positive. In virtue of the assumption (A.4) and positivity of \( \phi^Y \) there exists a positive constant \( \delta \) such that

\[ \text{ess-inf}_{x \in E} R(-1) \phi^Y(x) \geq \text{ess-inf}_{x \in E} R(-1)(1_c \phi^Y)(x) \]

\[ \geq \text{ess-inf}_{x, y \in U} r_{-1}(x, y) \| 1_c \phi^Y \|_1 \geq \delta > 0. \]

Since \( V_1 \phi^Y \) is a quasi-continuous version of \( R(-1) \phi^Y \), the result follows.

**Theorem 4.2.** Assume (A.1)–(A.4). Let \( V \) be a relatively compact open subset of \( X \) with the property that \( W := X \setminus V \) is non-empty. Then there exists a positive finite constant \( \epsilon \) such that

\[ \lambda^Y - \lambda \geq \epsilon \text{Cap}(K) \]

for all compact subsets \( K \) of \( V \).

**Proof.** By Dynkin’s formula (2.5), orthogonality (2.4) and the min-max principle, we have

\[ (1 + \lambda^Y)^{-1} = \delta_1(V_1^Y \phi^Y, V_1^Y \phi^Y) \]

\[ = \delta_1(V_1 \phi^Y, V_1 \phi^Y) - \delta_1(V_1 \phi^Y, P_k^1 V_1 \phi^Y) \]

\[ \leq (1 + \lambda)^{-1} - \delta_1(V_1 \phi^Y, P_k^1 V_1 \phi^Y) \]

(4.9)

Using Lemma 4.1 the last term in the above expression obeys an estimate of the form

\[ \delta_1(V_1 \phi^Y, P_k^1 V_1 \phi^Y) = \delta_1(V_1 \phi^Y, P_k^1 V_1 \phi^Y) \]

\[ = \delta \| V_1 \phi^Y, P_k^1 \phi^Y \| \geq \delta \| \phi^Y, \mu_k \| \geq \delta \text{Cap}(K). \]

The inequality \( \text{Cap}(K) \leq \delta^{-1}(1 + \lambda)^{-1} \) is a consequence of this and (4.9). Putting all these facts together we get

\[ \lambda^Y - \lambda \geq (1 + \lambda)^{-1} \frac{\delta^2}{1 - (1 + \lambda) \delta \text{Cap}(K)} \text{Cap}(K) \geq (1 + \lambda)^{-1} \frac{\delta^2 \text{Cap}(K)}{1 - (1 + \lambda) \delta \text{Cap}(K)}. \]
The zero-order counterpart of this result requires a modification to the assumption (A.4):

\[(A.4)' \quad R(0) \text{ is an integral operator whose kernel } r_0(x,y) \text{ satisfies } \quad \text{ess-inf}_{x,y \in U} r_0(x,y) > 0 \text{ for any relatively compact open set } U \text{ in } X.\]

The result may then be formulated as follows.

**Theorem 4.3.** Assume (A.1)', (A.2), (A.3), (A.4)'. Let \( V \) be a relatively compact open subset of \( X \) with the property that \( W := X \setminus V \) is non-empty. Then there exists a positive constant \( c \) such that

\[\lambda^V - \lambda \geq c \text{Cap}_0(K)\]

for all compact subsets \( K \) of \( V \).

5. **Deviation of Higher Eigenvalues**

It is a consequence of the min-max principle that each eigenvalue—not only the lowest—gets shifted to the right under the imposition of additional Dirichlet boundary conditions. We show that the magnitude of this shift can be estimated by the capacity. Our assumptions shall be slightly stronger than in Section 3. Throughout this section we work in the complexification of \( \mathcal{H} \), but retain our previous notation for the sake of simplicity.

Together with (A.1), we shall assume that a stronger version of (A.2) holds, namely

\[(A.5) \quad (T_t)_{t > 0} \text{ is ultracontractive and for all } m \geq m_0\]

\[\int_0^\infty t^m e^{-t} \| T_t \|_{L^2 \to L^2} \, dt < \infty\]

**Remark.** The above inequality is valid if a Sobolev inequality holds, see [4], Theorem 2.4.2.

Suppose that \( z, w \in \rho(H) \cap \rho(H_Y) \) and \( k \in \mathbb{N} \). By the first resolvent formula, \( R(w)^k = (I + (w - z) R(w))^k R(z)^k \) and likewise for the Dirichlet resolvent. On taking the difference we are led to the identity

\[R(z)^k - R_Y(z)^k = (I - (w - z) R_Y(z))^k (R(w)^k - R_Y(w)^k)\]

\[- \sum_{j=1}^{k} \binom{k}{j} (w - z)^j (I - (w - z) R_Y(z))^k\]

\[\times (R(w)^j - R_Y(w)^j) R(z)^k \quad (5.10)\]
Lemma 5.1. The inequality
\[ \| (R(-1) - R_T(-1)) f \|_2 \leq (1 + \lambda)^{-1/2} \text{Cap}(K)^{1/2} \| f \|_\infty, \ f \in \mathcal{H} \cap L^\infty(X, m) \]
holds for all compact subsets K of X.

Proof. Let \( f \in \mathcal{H} \cap L^\infty(X, m) \). By the min-max principle,
\[
\| (R(-1) - R_T(-1)) f \|_2 \leq \| p_k^1 V_1 |f| \|_2 \\
\leq \| p_k \|_2 \| f \|_\infty \leq (1 + \lambda)^{-1/2} \text{Cap}(K)^{1/2} \| f \|_\infty.
\]

For convenience of notation we set
\[ C(k) := \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-t} \| T_{p_k} \|_\infty \ dt, \quad k \in \mathbb{N}; \]
as well as
\[ F(z) := 1 + (1 + |z|) d(z, \sigma(H))^{-1}, \quad z \in \rho(H); \]
and define \( F_T(z) \) similarly. Here \( d \) denotes the usual distance in the complex plane.

Lemma 5.2. For \( z \in \rho(H) \) and \( k \geq m_0 \), \( \| R(z)^k \|_\infty \leq C(k) F(z)^k \) and an analogous inequality holds for the perturbed resolvent.

Proof. Writing the resolvent as the Laplace transform of the associated semigroup and differentiating, we arrive at the estimate \( \| R(-1)^k \|_\infty \leq (1 + |1 + z|)^{k-1} \text{Cap}(K)^{1/2} \| f \|_\infty \).

Theorem 5.3. Fix \( m \in \mathbb{N}, m > 2m_0 \). Then there exists a finite constant \( c \) depending only on \( H \) and \( m \) such that
\[ \| R(z)^m - R_T(z)^m \| \leq c(1 + |1 + z|)^m F(z)^m F_T(z)^m \text{Cap}(K)^{1/2} \]
for all \( z \in \rho(H) \cap \rho(H_T) \) and all compact subsets K of X.

Proof. Define
\[
D_k(z) := R(z)^k - R_T(z)^k; \\
T_k(z) := (I + (1 + z) R_T(z)) D_k(-1); \\
T_k(z) := (I + (1 + z) R_T(z)) D_k(-1) R(z)^m, \quad k = 1, \ldots, m - 1.
\]
By \((5.10)\) we have
\[
\|D_m(z)\| \leq \|T_0(z)\| + \sum_{k=1}^{m} \binom{m}{k} |1 + z|^k \|T_k(z)\|.
\] (5.11)

We make use of the identity
\[
D_k(-1) = \sum_{j=0}^{k-1} R_j(-1)^j D_j(-1) R(-1)^{k-1-j}.
\] (5.12)

We may thus expand \(T_0(z)\) as follows:
\[
T_0(z) = \sum_{k=0}^{m-1} (I + (1 + z) R_y(z))^m R_y(-1)^j D_j(-1) R(-1)^{m-1-k}.
\]

Suppose \(m\) is even, so may be written \(m = 2(m_1 + 1)\) with \(m_1 \geq m_0\). Put
\[
S_{0,k} := R_y(-1)^k D_1(-1) R(-1)^{m_1-k}.
\]

Using the fact that the norms of \(S_{0,k}\) and its adjoint are identical and Lemma 5.2 we see that
\[
\|S_{0,k}\| \leq (1 + \lambda)^{-1/2} Cap(K)^{1/2} C(m - 1 - k)
\]
provided \(0 \leq k \leq m_1\). We therefore have that
\[
\|T_0(z)\| \leq 2(1 + \lambda)^{-1/2} \sum_{k=0}^{m_1} C(m - 1 - k) F_y(z)^m Cap(K)^{1/2}.
\] (5.13)

The case of odd \(m\) goes through similarly. We now turn to the estimation of \(T_k(z)\). On expanding using the identity \((5.12)\) we get
\[
T_k(z) = \sum_{j=0}^{k-1} (I + (1 + z) R_y(z))^m R_y(-1)^j D_j(-1) R(-1)^{k-1-j} R(z)^m.
\]

Applying Lemmas 5.1 and 5.2 once more we see that the norm is dominated by
\[
\|T_k(z)\| \leq k(1 + \lambda)^{-1/2} C(m) F(z)^m F_y(z)^m Cap(K)^{1/2}.
\] (5.14)

Combining \((5.11)\), \((5.13)\) and \((5.14)\) leads finally to
\[
\|D_m(z)\| \leq c(1 + |1 + z|)^m F(z)^m F_y(z)^m Cap(K)^{1/2}
\]
where \(c\) is a finite constant depending only on \(H\) and \(m\).
Let \( \lambda_1 \leq \lambda_2 \leq \cdots \) be an ordering of the eigenvalues of \( H \) counting multiplicity and \( \mu_1 < \mu_2 < \cdots \) be an ordering of the distinct eigenvalues of \( H \) with multiplicities \( m_j \). Define \( m_0 := 0, \mu_0 := -\eta, p_j := \frac{1}{2}(\mu_j + \mu_{j+1}), \delta_j := \frac{\theta}{2}(\mu_{j+1} - \mu_j), \theta := \sqrt{5/36} \) where \( \eta \) is some fixed positive number. We set \( s_0 := \sum_0^k m_j \). Let \( \lambda'_1 \leq \lambda'_2 \leq \cdots \) be an ordering of the eigenvalues of \( H_Y \) counting multiplicity.

**Lemma 5.4.** Fix \( n \in \mathbb{N} \). Then there exists a positive constant \( c \) such that

\[
\sigma(A_Y) \cap \bigcup_{j=1}^n (p_j - \theta \delta_j, p_j + \theta \delta_j) = \emptyset
\]

for all compact subsets \( K \) of \( X \) with capacity smaller than \( c \).

**Proof.** The proof is similar to that of [12], Theorem VIII.23. Define \( z_j := p_j + \frac{1}{2} \delta_j i \) where \( i := \sqrt{-1} \). Then the norm of the unperturbed resolvent obeys the estimate \( |R(z_j)^m| \leq \delta_j^{-m} \). We choose \( m > 2m_0 \). By Theorem 5.3 there exists a finite constant \( c \) such that \( \|R_Y(z_j)^m\| \leq 2\delta_j^{-m} \) for \( j = 1, \ldots, n \) whenever the capacity of \( K \) is smaller than \( c \). The perturbed resolvent powers have a norm convergent power series expansion in a neighbourhood of \( z_j \). Its norm can be estimated by

\[
\|R_Y(w)^m\| \leq \sum_{k=0}^m \binom{m+k-1}{k} |w - z_j|^k 2 \tilde{1}^{k/m} \delta_j^{-(m+k)}
\]

with the aid of the spectral mapping theorem. Using elementary analysis the above series is seen to converge absolutely in the disc \( |w - z_j| < 2^{-1}\delta_j \) and in particular if \( |w - z_j| < 2^{-1}\delta_j \), it follows that the intervals \( (p_j - \theta \delta_j, p_j + \theta \delta_j), j = 1, \ldots, n \) are contained in the resolvent set of the perturbed operator.

It will be convenient to refer to the intervals \( I_j := [p_{j-1}, p_j] \) resp. \( F_j := [p_{j-1} - \theta \delta_j, p_j + \theta \delta_j] \).

**Lemma 5.5.** Fix \( n \in \mathbb{N} \). Then there exists a positive constant \( c \) such that

\[
\lambda'_k \in [\mu_j, p_j - \theta \delta_j] \quad \text{for} \quad s_{j-1} < k \leq s_j \quad \text{and} \quad j = 1, \ldots, n
\]

for each compact \( K \) with capacity smaller than \( c \).

**Proof.** We fix \( m = 2m_0 + 1 \). According to the Stone–Weierstrass theorem, the algebra

\[
\mathcal{A} := \{P((x+i)^{-m}, (x-i)^{-m}) : P \text{ complex polynomial}\}
\]
is dense in the Banach space \( C_\infty(\mathbb{R}; C) \) of continuous complex-valued functions on \( \mathbb{R} \) vanishing at infinity. For each \( 1 \leq j \leq n \), choose a continuous real-valued function \( f_j \) which is equal to 1 on the closed interval \( I_j \) and vanishes outside \( I_j \). We may choose \( Q_j \in \mathcal{A} \) satisfying \( \|Q_j - f_j\|_\infty < 1/3 \). Assume that the capacity of \( K \) is smaller than the positive constant \( c \) appearing in the last Lemma. A “three-epsilon” argument delivers the estimate

\[
\|E(I_j) - E_y(I_j)\| = \|f_j(H) - f_j(H_Y)\| < 2/3 + \|Q_j(R(i)^m, R(-i)^m) - Q_j(R_y(i)^m, R_Y(-i)^m)\|
\]

Suppose that \( Q_j \) has the form \( Q_j(z, w) = \sum_{k+l \in N} a_k^j z^k w^l \). To proceed further it is necessary to estimate a difference of mixed resolvent powers. But

\[
\|R(i)^m R(-i)^m - R_y(i)^m R_Y(-i)^m\| \leq \|D_m(-i)\| + \|D_m(i)\|
\]

Appealing to Theorem 5.3 once more and by choosing \( c \) to be smaller if necessary, we can arrange that \( \|E(I_j) - E_y(I_j)\| < 1 \) for \( j = 1, ..., n \). In particular, \( \dim E(I_j) = \dim E_y(I_j) \) which gives the result.

The normalised real-valued eigenfunction corresponding to the \( k \)-th perturbed eigenvalue \( \lambda_k^Y \) is denoted \( \psi_k \).

**Lemma 5.6.** The inequality

\[
\|(H + I)^{1/2} (R(-1) - R_y(-1)) \psi_k\|_2 \leq \exp(\lambda_k^Y) \|T_1\|_{2, \infty} \text{Cap}(K)^{1/2}
\]

holds for all compact subsets \( K \) of \( X \) and all natural numbers \( k \).

**Proof.** Recall the identity \( \psi_k = \exp(\lambda_k^Y) T_1^Y \psi_k \) and the fact that \( P_k^1 V_1 |\psi_k| \) is a potential. The square of the expression on the left-hand side above is exactly

\[
\epsilon_k(P_k^1 V_1 \psi_k, P_k^1 V_1 \psi_k) \leq \epsilon_k(P_k^1 V_1 |\psi_k|, P_k^1 V_1 |\psi_k|) \leq \exp(2\lambda_k^Y) \|T_1\|_{2, \infty} \text{Cap}(K).
\]

**Lemma 5.7.** Let \( n \in \mathbb{N} \) and \( c \) be the positive constant mentioned in Lemma 5.5. Then there exists a finite constant \( c_1 \) such that

\[
\|(H + 1)^{1/2} (I - E(F_j)) \psi_k\|_2 \leq c_1 \text{Cap}(K)^{1/2}
\]
for \( s_{j-1} < k \leq s_j \) and \( j = 1, \ldots, n \) whenever the capacity of the compact \( K \) is smaller than \( c \).

**Proof.** Let \( \Gamma_j \) be a simple contour in the complex plane circumscribing the interval \( F_j \) in a counterclockwise direction and cutting the real line in the points \( p_{j-1}, p_j \). The spectral measure of this interval is given by

\[
E(F_j) = -\frac{1}{2\pi i} \oint_{\Gamma_j} R(z) \, dz
\]

and likewise for the perturbed spectral measure. Due to the choice of \( c \) the \( k \)th perturbed eigenvalue lies in the interval \( F_j \) and hence

\[
(I - E(F_j)) \psi_k = \frac{1}{2\pi i} \oint_{\Gamma_j} (R(z) - R_Y(z)) \, dz \, \psi_k
\]

By the Hilbert identity the integrand is equal to

\[
(1 + (1 + z)(\lambda^Y_k - z)^{-1})(I + (1 + z) R(z)) \, P_k^Y V_1 \psi_k.
\]

Using Lemma 5.6 and the min-max principle the norm of \((H + I)^{1/2}\) applied to this expression can be bounded by some finite multiple \( c' \) of \( \text{Cap}(K)^{1/2} \) independently of \( z \) in the contour \( \Gamma_j \) and for \( j = 1, \ldots, n \). This means that

\[
\| (H + I)^{1/2} (I - E(F_j)) \psi_k \|_2 \leq \frac{1}{2\pi} L(\Gamma_j) \, c' \, \text{Cap}(K)^{1/2}
\]

where \( L(\Gamma_j) \) stands for the length of the contour \( \Gamma_j \). \( \square \)

**Theorem 5.8.** Fix \( n \in \mathbb{N} \). Let \( c \) and \( c_1 \) be the constants appearing in Lemma 5.7. Then

\[
\lambda^Y_k \leq \lambda_k + c_1^2 \, \text{Cap}(K)
\]

for \( k = 1, \ldots, s_n \) for all compact subsets \( K \) whose capacity is smaller than \( c \).

**Proof.** By the min-max principle and Lemma 5.7.

\[
1 + \lambda^Y_k = \psi_k^*(H + I) \psi_k
\]

\[
= \| (H + I)^{1/2} E(F_j) \psi_k \|_2^2 + \| (H + I)^{1/2} (I - E(F_j)) \psi_k \|_2^2
\]

\[
\leq 1 + \lambda_j + c_1^2 \, \text{Cap}(K). \quad \square
\]
The zero-order version of this result is

**Theorem 5.9.** Assume \((A.1')\) and \((A.5)\). Fix \(n \in \mathbb{N}\). Then there exist positive finite constants \(c_1, c_2\) for which

\[
\lambda_k^d \leq \lambda_k + c_1 \text{Cap}_0(K)
\]

for \(k = 1, \ldots, s_n\) for all compact subsets \(K\) whose capacity is smaller than \(c\).

**Remark.** One cannot expect in general a lower bound for higher eigenvalues as in Theorem 4.2. See [7] for an example in which the second eigenvalue is unaffected by an extra Dirichlet boundary condition.

6. The “Crushed Ice” Problem

We now describe an application to the so-called “crushed ice” problem. Our results extend those of [15, 16, 18].

Let \(D\) be a bounded domain in \(\mathbb{R}^d (d \geq 1)\). Denote by \(\Delta^d\) the non-negative Laplace operator in \(L^2(D, dx)\) with Dirichlet boundary conditions defined via quadratic forms. According to [4], Theorem 1.6.8, this operator has compact resolvent, the bottom eigenvalue \(\lambda_d\) is strictly positive and the corresponding heat semigroup satisfies \((A.5)\). Write the associated capacity of order zero \(\text{Cap}^d_0(K)\) and the lowest eigenvalue \(\lambda_d\). Given a compact subset \(K\) of \(D\) the operator obtained by imposing additional Dirichlet boundary conditions on \(K\) is denoted \(\Delta^d_{D-K}\). The lowest eigenvalue of this operator is \(\lambda^d_{D-K}\). In virtue of [4], Theorem 3.3.5, our assumption \((A.4)\) is fulfilled. We therefore have by Theorems 3.2 and 4.3

**Theorem 6.1.** Let \(V\) be a relatively compact open subset of \(D\). Then there exist positive finite constants \(c_1, c_2, c_3\) such that

\[
\lambda_d + c_1 \text{Cap}^d_0(K) \leq \lambda_{D-K}^d \leq \lambda_d + c_2 \text{Cap}^d_0(K)
\]

whenever \(K\) is a compact subset of \(V\) whose capacity is smaller than \(c_3\).

A parallel result holds for the (non-negative) Neumann Laplacian \(\Delta^n\) in \(L^2(D, dx)\) defined via quadratic forms. However to ensure compactness of the resolvent and ultracontractivity \((A.5)\) we require that \(D\) has the extension property [4], Theorem 1.7.12 and Lemma 1.7.11. We have

**Theorem 6.2.** The same result holds for the Neumann Laplacian \(\Delta^n\) and the capacity \(\text{Cap}^n\) of order 1 provided the domain \(D\) has the extension property.
The above has the following physical interpretation. A fluid is contained in a region $D$ whose boundary is insulating (Neumann boundary conditions) or is maintained at constant temperature zero (Dirichlet boundary conditions). A cooler (ice) occupying the region $K$ is placed in the container and the boundary of $K$ is maintained at constant temperature zero. The problem is to determine the improvement in cooling efficiency resulting from the addition of the extra cooler $K$. We choose the lowest eigenvalue as a measure of cooling efficiency because it governs the decay of the heat semigroup norm. According to our results the improvement in cooling efficiency is determined by the capacity.

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