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ON TOPOI AS CLOSED CATEGORIES

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0. Introduction

There have been two main avenues of work on categorical set theory. The first, the theory of closed categories, deals with the unique position of the category of sets as the category in which all hom functors land and considers other categories that can play the same role [1]. The second, Lawvere and Tierney's theory of topoi, concentrates on the internal structure of the category of sets and derives much of this structure from a set of elementary axioms (see e.g. [2, 4]). But a topos is a closed category, and in this paper we use this fact to prove two theorems about topoi. The first is a characterisation of those topoi which have a geometric morphism into a given topos. The other is a generalization from the category of sets to arbitrary topoi of Lawvere's theorem that any complete model of the Elementary Theory of the Category of Sets (ETCS) is equivalent to the category of sets [5, p. 1510]. Section 1 contains a review of known results, which concludes with the statement of the theorems. Their proofs use the following lemmas, which are proved in Section 2. We suppose that E and E' are topoi and $L\colon E \to E'$ is a functor left adjoint to R which preserves finite products (including the terminal object 1). We let t be the unit of the adjunction and s is the characteristic function in E of the map $RT'\colon 1 \to R\Omega'$, where $T'\colon 1 \to \Omega'$ is the subobject generator of E'.

Lemma 0.1. *If L preserves the pullbacks*

(1)
$$\begin{array}{ccc} 1 & \xrightarrow{(T,T)} & \Omega \times \Omega \\ \downarrow & & \downarrow \wedge \\ 1 & \xrightarrow{\;\;T\;\;} & \Omega \end{array}$$

and

$$(2)\qquad \begin{array}{ccc} 1 & \xrightarrow{RT'} & R\Omega' \\ \downarrow & & \downarrow{\scriptstyle s} \\ 1 & \xrightarrow{T} & \Omega \end{array}$$

then L is left exact.

Lemma 0.2. *If R preserves epis,* $t(\Omega)\colon \Omega \to RL\Omega$ *is an isomorphism, and there are isomorphisms d and s such that*

$$\begin{array}{ccc} & \overset{RT'}{\nearrow} & R\Omega' \\ 1 & & \downarrow{\scriptstyle s} \\ & \underset{T}{\searrow} & \Omega \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \overset{LT}{\nearrow} & L\Omega \\ 1 & & \downarrow{\scriptstyle d} \\ & \underset{T'}{\searrow} & \Omega' \end{array}$$

commute, then L is an equivalence.

1.

The following simple proposition provides an introduction to our discussion:

Proposition 1.1. *If E' is a topos, then there is (up to natural isomorphism) at most one geometric morphism from E' to the category of sets.*

Proof. If (L, R) is a geometric morphism from E' to **Sets**, then for all x in E', $Rx \cong (Rx)^1 \cong E'(L1, x) \cong E'(1, x)$, so R is already determined up to isomorphism by the hom functor.

On the other hand, the two projections from **Sets** × **Sets** to **Sets** are the left adjoints of distinct geometric morphisms, so the proposition is not true in general. The proof used the fact that the hom functor $E'(\text{-},\text{-})$ lands in sets; this suggests the use of the theory of closed categories. In fact, the proof works without change of wording to show that if E' is an E-category, then there can be at most one geometric morphism that forms an E-adjunction. Conversely, if (L, R) is a geometric morphism from E' to E, then the functor $E'(X, Y) = R(Y^X)$ makes E' an E category. The following paragraph considers this in more detail. The ideas are paraphrased from [1] and [3] and make no essential use of the structure on E and E' beyond the fact that they are closed.

Let E' be an E-category with hom functor $E'(\text{-}, \text{-})\colon E'^{\mathrm{op}} \times E' \to E$. Then we say that E' is E-cocomplete (tensored in the language of [3]) if for each object X of E and W of E', there is an object $X \otimes W$ of E', which should be thought of as the sum of X copies of W, and a map $\bar{t}(X, W)$: $1 \to E'(W, X \otimes W)^X$ such that for each object B of E, the E-natural transformation

$$E'(X \otimes W, B) \xrightarrow{(\bar{t}(X, W), 1)} E'(W, X \otimes W)^X \times E'(X \otimes W, B) \xrightarrow{\circ} E'(W, B)^X$$

is an isomorphism, where $\circ$ is the composition map. If $X \otimes 1$ exists for all $X \in E$, then $- \otimes 1 : E \to E'$ is a functor from E to E' which is a left E-adjoint to $E'(1, -)$. Suppose further that E' is an E-closed E-category; that is, that $E'(X \otimes Y, Z) \cong E'(X, Z^Y)$ is E-naturally in X, Y and Z. Then for all X and Y in E, and B in E',

$$\begin{aligned} E'((X \otimes 1) \times (Y \otimes 1), B) &\cong E'(X \otimes 1, B^{Y \otimes 1}) \cong E'(1, B^{Y \otimes 1})^X \\ &\cong E'(Y \otimes 1, B)^X \cong E'(1, B)^{X \times Y}, \end{aligned}$$

all E-naturally in B, so $(X \times Y) \otimes 1 \cong (X \otimes 1) \times (Y \otimes 1)$. Similarly, $1 \otimes 1 \cong 1$.

The converse is also valid: If $L \dashv R$, $L: E \to E'$, and L preserves finite products (including 1), then the functor E' defined by $E'(X, Y) = R(Y^X)$ makes E' an E-cocomplete, E-closed E-category, and $- \otimes 1 = L$. These results, combined with Lemmas 0.1 and 0.2, yield the following theorems:

Theorem 1.2. *If E and E' are topoi, then there is a geometric morphism from E' to E if and only if there is a hom functor making E' into an E-cocomplete, E-closed E-category such that $- \otimes 1$ preserves the following pullbacks:*

$$\begin{array}{ccc} 1 & \xrightarrow{(T, T)} & \Omega \times \Omega \\ \downarrow & & \downarrow \wedge \\ 1 & \xrightarrow{T} & \Omega \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{E'(1, T')} & E'(1, \Omega') \\ \downarrow & & \downarrow s \\ 1 & \xrightarrow{T} & \Omega \end{array}$$

Theorem 1.3. *If E and E' are topoi, then the following implies that E' is equivalent to E:*

(i) *E' is an E-cocomplete, E-closed E-category with hom functor $E'(-,-)$: $E'^{\mathrm{op}} \times E' \to E$.*

(ii) *$E'(1, -)$ preserves epis.*

(iii) *$t(\Omega)$: $1 \to E'(1, \Omega \otimes 1)$ is an isomorphism.*

(iv) *There are isomorphisms s and d such that the following diagrams commute:*

$$\begin{array}{ccc} 1 & \xrightarrow{E'(1, T')} & E'(1, \Omega') \\ & {\scriptstyle T} \searrow & \downarrow s \\ & & \Omega \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{T \otimes 1} & \Omega \otimes 1 \\ & {\scriptstyle T'} \searrow & \downarrow d \\ & & \Omega' \end{array}$$

If E is the category of sets and E' is a complete model of ETCS, then (ii) holds because epis split in E', s is an isomorphism because E' is two valued, d is an isomorphism because E' is boolean, and condition (iii) becomes redundant when E is two valued. Thus Theorem 1.3 implies that any cocomplete model of ETCS is equivalent to the category of sets.

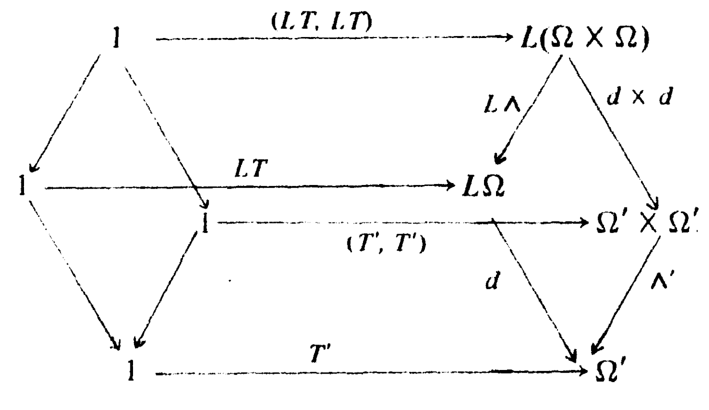
2.

We now proceed to the proof of Lemma 0.1. L and R are as stated in the hypothesis, and we will show that L preserves equalizers. Objects of E' will be distinguished from the corresponding objects of E by adding a prime, thus Ω' is the subobject generator of E'. Since all functors preserve finite products, we make no distinction between, for example, $L(X \times Y)$ and $LX \times LY$. We define d: $L\Omega \to \Omega'$ to be the characteristic function in E' of LT: $1 \to L\Omega$; and s: $R\Omega' \to \Omega$ is the characteristic function in E of RT'. The unit of adjunction is t. If f, g: $X \to R\Omega'$, then $f \leqslant g$ iff $R \wedge' (f, g) = f$. The relation $\leqslant$ is a partial order.

Lemma 2.1. *If f, g: $X \to \Omega$ and $f \leqslant g$, then $Rd\ t(\Omega) f \leqslant Rd\ t(\Omega) g$.*

Proof. In the diagram

(3)



the upper left square is a pullback by (1) and all others are pullbacks by definition, so $d\, L\wedge = \wedge' (d \times d)$ since they are characteristic functions for the same map. Thus

$$\begin{aligned} R\wedge' (Rd\ t(\Omega) f, Rd\ t(\Omega) g) &= R(\wedge' (d \times d))\ t(\Omega \times \Omega)(f, g) \\ &= Rd\ RL\wedge\ t(\Omega \times \Omega)(f, g) \\ &= Rd\ t(\Omega) \wedge (f, g) = Rd\ t(\Omega) f. \end{aligned}$$

Lemma 2.2. *If $h = Rd\ t(\Omega)s$: $R\Omega' \to R\Omega'$, then $h \leqslant$* id.

Proof. Both squares of the diagram

$$\begin{array}{ccc}
1 & \xrightarrow{LRT'} & LR\Omega' \\
\downarrow & & \downarrow{\scriptstyle Ls} \\
1 & \xrightarrow{LT} & L\Omega \\
\downarrow & & \downarrow{\scriptstyle d} \\
1 & \xrightarrow{T'} & \Omega'
\end{array}$$

are pullbacks. If υ is the counit of the adjunction, then $\upsilon(\Omega')\, LRT' = T'$, so $\upsilon(\Omega') \geqslant d\, Ls$. But then

$$\begin{aligned}
\mathrm{id} &= R\upsilon(\Omega')\, t(R\Omega') \geqslant R(d\, Ls)\, t(R\Omega') \\
&= Rd\, RLs\, t(R\Omega') = Rd\, t(\Omega)\, s = h.
\end{aligned}$$

Lemma 2.3. *If $i\colon X \rightarrowtail Y$ in E and $L(i)$ is monic, then $\varphi' = d\, L\varphi$, where φ and φ' are the characteristic functions of i and $L(i)$, respectively.*

Proof. From the pullback

$$\begin{array}{ccc}
RLX & \overset{RLi}{\rightarrowtail} & RLY \\
\downarrow & & \downarrow{\scriptstyle R\varphi'} \\
1 & \xrightarrow{RT'} & R\Omega' \\
\downarrow & & \downarrow{\scriptstyle s} \\
1 & \xrightarrow{T} & \Omega
\end{array}$$

$sR\varphi'$ is the characteristic function of RLi, so $\varphi \leqslant s\, R\varphi'\, t(Y)$. Then

$$\begin{aligned}
Rd\, RL\varphi\, t(Y) &= Rd\, t(\Omega)\, \varphi \leqslant Rd\, t(\Omega)\, s\, R\varphi'\, t(Y) \\
&= h\, R\varphi'\, t(Y) \leqslant R\varphi\, t(Y) \leqslant Rd\, RL\varphi\, t(Y),
\end{aligned}$$

where the last step follows from $d\, L\varphi\, Li = d\, LT = T'$, so $\varphi' \leqslant d\, L\varphi$. But now we have $R(d\, L\varphi)\, t(Y) = R\varphi'\, t(Y)$, so $\varphi' = d\, L\varphi$.

Proof of Lemma 0.1. If Y is in E, let $\approx_Y$ be the characteristic function of $\Delta\colon Y \rightarrowtail Y \times Y$. $L(\Delta)\colon L(Y) \to L(Y \times Y)$ is $\Delta'\colon L(Y) \rightarrowtail L(Y) \times L(Y)$, so $L(\Delta)$ is monic and by Lemma 2.3, $\approx'_{L(Y)} = d\,L\approx_Y$. Now let $f, g\colon X \to Y$ in E and let $e\colon Q \rightarrowtail X$ be their equalizer. Then Le is monic, for

$$\begin{aligned} R\approx'_{L(Y)} RLe\; t(X \times X) &= Rd\; RL\approx_Y RLe\; t(X \times X) = Rd\; t(\Omega) \approx_Y e \\ &= Rd\; t(\Omega) \approx_X = Rd\; RL\approx_X t(X \times X) \\ &= R\approx'_{L(X)} t(X \times X), \end{aligned}$$

so $\approx'_{L(Y)} Le = \approx'_{L(X)}$. Then if $e'\colon Q' \to L(X)$ is the equalizer of (Lf, Lg), φ' is the characteristic function of e', and φ is the characteristic function of e, then

$$\varphi' = \approx'_{L(Y)} (Lf, Lg) = d\,L\approx_Y (Lf, Lg) = d\,L(\approx_Y (f, g)) = d\,L\varphi.$$

By Lemma 2.3, this is the characteristic function of Le, so $L(Q) = Q'$ and so L preserves equalizers. But L preserves products by assumption, so L is left exact.

I am indebted to B. Day for the observation that if $p\colon A \to B$ is any finite product-preserving functor of finitely complete small categories, then the functor

$$[p, I]\colon [B, \textbf{Sets}] \to [A, \textbf{Sets}]$$

has a left adjoint p* that preserves products and which preserves equalizers only if p does. Hence the assumption that L preserves the pullbacks (1) and (2) is necessary. A calculation shows that in this example the left adjoint always preserves (1); we do not know if this is true in general.

Proof of Lemma 0.2. The pullback (2) is trivially preserved. The isomorphism $s\,Rd\; t(\Omega)\colon \Omega \to \Omega$ preserves T and hence is its characteristic function, so $s\,Rd\; t(\Omega)$ is the identity. Thus, in the diagram

$$\begin{array}{ccccccc} \Omega \times \Omega & \xrightarrow[\cong]{t(\Omega\times\Omega)} & RL(\Omega \times \Omega) & \xrightarrow[\cong]{R(d\times d)} & R(\Omega' \times \Omega') & \xrightarrow[\cong]{s\times s} & \Omega \times \Omega \\ \downarrow{\scriptstyle \wedge} & & \downarrow{\scriptstyle RL\wedge} & & \downarrow{\scriptstyle R\wedge'} & & \downarrow{\scriptstyle \wedge} \\ \Omega & \xrightarrow[\cong]{t(\Omega)} & RL(\Omega) & \xrightarrow[\cong]{Rd} & R(\Omega') & \xrightarrow[\cong]{s} & \Omega \end{array}$$

the two end squares and the outside commute, so the middle commutes, and so $d\,L\wedge = \wedge'\,(d \times d)$. Thus (1) is preserved and L is left exact by Lemma 0.1. If υ is the counit of adjunction, then $\upsilon(L\Omega)$ is an isomorphism since $Lt(\Omega)$ is. So from

$$\begin{array}{ccc} LR(\Omega') & \xrightarrow{\upsilon(\Omega')} & \Omega' \\ \| \wr\; {\scriptstyle LRd} & & \| \wr\; {\scriptstyle d} \\ LRL(\Omega) & \xrightarrow[\cong]{\upsilon(L(\Omega))} & L(\Omega) \end{array},$$

$\upsilon(\Omega')$ is an isomorphism. Since $\upsilon(\Omega')$ and $d\,Ls$ both take LRT' to T', they are both characteristic functions for LRT' and hence are equal. Also, $\approx'_{L(X)} = d\,L(\approx_X)$ from the pullback

$$(4)\qquad \begin{array}{ccc} L(X) & \xrightarrow{\;L\Delta\;} & L(X \times X) \\ \downarrow & & \downarrow{\scriptstyle L\approx_X} \\ 1 & \xrightarrow{\;LT\;} & L(\Omega) \\ \downarrow & & \downarrow{\scriptstyle d} \\ 1 & \xrightarrow{\;T'\;} & \Omega' \end{array}$$

and similarly $\approx_{R(X)} = s\,R\approx'_X$. Bringing these together, we see that for all X in E', $\upsilon(X)$: $LR(X) \to X$ is monic:

$$\begin{aligned} \approx'_X (\upsilon(X) \times \upsilon(X)) &= \upsilon(\Omega')\,LR\approx'_X = \upsilon(\Omega')\,Ls^{-1}\,L\approx_{R(X)} \\ &= \upsilon(\Omega')\,L(s)^{-1}\,d^{-1} \approx'_{LR(X)} = \upsilon(\Omega')\,(d\,Ls)^{-1} \approx'_{LR(X)} \\ &= \approx'_{LR(X)}. \end{aligned}$$

We claim that $\upsilon(X)$ is also epi: Let φ be the characteristic function of $\upsilon(X)$ and let p be the unique map from X to 1. Since $R\upsilon(X)$ splits, it is epi and so $R(\varphi) = R(T'\,p)$. Let i be the monic part of φ and j the monic part of $T'\,p$. Then since R preserves epis, Ri and Rj are the monic part of $R\varphi$ and $R(T'\,p)$, respectively, so $Ri = Rj$. Let φ_1 and φ_2 be the characteristic functions of i and j, respectively. Then $R\varphi_1 = R\varphi_2$, but since $\Omega' \cong L\Omega$, both φ_1 and φ_2 take $L\Omega$ to Ω', so this implies that $\varphi_1 = \varphi_2$. Thus $i = j$ and $\upsilon(X)$ is epi.

We now know that $\upsilon(X)$ is an isomorphism for all X, so $Lt(Y) = \upsilon(L(X))^{-1}$ is an isomorphism for all Y. But the topology associated with (L, R) is $s\,Rd\,t(\Omega)$ which is, as was pointed out at the beginning of the proof, the identity; so by the Lawvere–Tierney factorization theorem for geometric morphisms [4, Theorem 4.3] L reflects isomorphisms. Thus t is an isomorphism and L is an equivalence.

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