The use of Cornu spirals in drawing planar curves of controlled curvature

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Abstract: Cornu spirals or clothoids have been used in highway design for many years. In the past the spirals have been found manually by draftsmen. The purpose of this paper is to show that it is practical to find these spirals with a micro-computer. The design curve will be made up of arcs of circles and segments of Cornu spirals joined in such a way that the curvature is continuous throughout, and takes its largest values on the arcs of circles. Thus, the radii of the circles used will limit, and control the curvature of the whole design curve.

Keywords: Cornu spiral, controlled curvature.

1. Introduction

Cornu spirals or clothoids have been used in highway design for many years [2]. In the past the spirals have been found manually by draftsmen. This could be a tedious process, and according to Baass [1], it is much easier to use templates to fit the spirals to arcs of circles. Even that method involves some trial and error. A further step in automating the process is to use a computer to calculate the position of the spirals. Recently published approximations to the Fresnel integrals by Heald [3] make it easy to work with Cornu spirals. The purpose of this paper is to show that it is practical to find these spirals with a micro-computer. The design curve will be made up of arcs of circles, straight line segments, and segments of Cornu spirals joined in such a way that the curvature is continuous throughout, and takes its largest values on the arcs of circles. Thus the radii of the circles used will limit, and control the curvature of the whole design curve.

The curves described here provide an alternative to cubic B-splines. Like B-splines, the new curves have the desirable property that local changes to the controlling curve affect the resulting curve locally. The new curves have controlled curvature, but are harder to find and evaluate, while B-splines are very easy to evaluate, but can have unbounded curvature. When the curvature of a B-spline is bounded, the maximum curvature is reasonably hard to find (see [4], [8], and [9]). Some previous work done with Cornu spirals can be found in [5], [6], and [7]. Mehlum finds approximations to Cornu spirals to use in interpolation, and Stoer does interpolation and least squares fits with curves made up of Cornu spirals. The method described here finds a curve made up of Cornu spirals which is tangent to given circles and lines because this problem is considerably simpler than interpolation.
2. Properties of Cornu spirals

The Cornu spiral is defined parametrically in terms of Fresnel integrals as

\[ x(t) = aC(t), \quad y(t) = aS(t), \quad (1) \]

where the scaling factor \( a \) is positive, the parameter \( t \) is non-negative, and the Fresnel integrals are

\[ C(u) = \int_0^u \cos \frac{1}{2} \pi t^2 \, dt \quad \text{and} \quad S(u) = \int_0^u \sin \frac{1}{2} \pi t^2 \, dt. \]

The Cornu spiral (1) is in the first quadrant, starts at the origin at \( t = 0 \), and approaches the limiting point \((\frac{1}{2}a, \frac{1}{2}a)\) as \( t \) approaches infinity. Let the integrals of the Fresnel integrals be

\[ C_1(u) = \int_0^u C(t) \, dt = uC(u) - \frac{1}{\pi} \sin \frac{1}{2} \pi u^2, \]

and

\[ S_1(u) = \int_0^u S(t) \, dt = uS(u) + \frac{1}{\pi} \cos \frac{1}{2} \pi u^2 - \frac{1}{\pi}. \]

Both \( C_1(u) \) and \( S_1(u) \) are monotone increasing with increasing \( u \), and both functions approach infinity as \( u \) goes to infinity. Some formulae for the Cornu spiral follow.

Angle of tangent: \( \frac{1}{2} \pi t^2 \).

Curvature: \( \pi t/a \).

Arclength: \( ds = a \, dt \).

Centre of circle of curvature:

\[ \left( a/tC_1(t), \, a/t(S_1(t) + 1/\pi) \right). \quad (2) \]

Integrals:

\[ \frac{\pi}{u} C_1(u) = \int \left( \frac{1}{u^2} \sin \frac{1}{2} \pi u^2 \right) \, du, \]

\[ - \frac{\pi}{u} S_1(u) - \frac{1}{u} = \int \left( \frac{1}{u^2} \cos \frac{1}{2} \pi u^2 \right) \, du. \quad (3) \]

Inequalities for \( u \) in \([0,1]\) (from simple arguments with sin and cos curves):

\[ u^2 \leq \sin \frac{1}{2} \pi u^2 \leq \frac{1}{2} \pi u^2, \quad 1 - u^2 \leq \cos \frac{1}{2} \pi u^2 \leq 1, \quad (4) \]

\[ \frac{1}{2} u^3 \leq S(u) \leq \frac{1}{6} \pi u^3, \quad u - \frac{1}{2} u^3 \leq C(u) \leq u, \quad (5) \]

\[ \frac{1}{2} u^4 \leq S_1(u) \leq \frac{1}{24} \pi u^4, \quad \frac{1}{2} u^2 - \frac{1}{12} u^4 \leq C_1(u) \leq \frac{1}{2} u^2. \quad (6) \]

3. Joining the Cornu spiral to circles and lines

The curve described in this paper is formed by joining given circles and line segments with Cornu spirals. Five situations in which Cornu spirals are used are given by Baass [1]. Each will be examined in turn to show how to find the Cornu spirals. In all but the last case, it is necessary to
solve a nonlinear equation to find the scaling factor \( a \). In two cases, a pair of Cornu spirals which match at their origins are used. An undesirable curve can arise if the tangent of the spiral is allowed to rotate through too large an angle. The rotation of the tangent in each spiral will be arbitrarily limited to an angle of \( \frac{1}{2} \pi \).

3.1. Line to circle with a single spiral

Given a circle of radius \( R_1 \), and a line which is distance \( E_1 \) from the centre of the circle, the problem is to find a Cornu spiral joining the line to the circle in such a way that the tangents and curvatures of the spiral match the corresponding tangents and curvatures of the line and circle at the joining points. It will be shown that this can always be done when the circle does not intersect the line, or when \( R_1 < E_1 \).

Choose the coordinate system so that the line is the \( X \)-axis, the centre of the circle is in the first quadrant, and the spiral is in the standard form \( (1) \) (see Fig. 1). Let the centre of the circle be \((D_1, E_1)\), and let the scaling factor in the spiral be \( a = \pi R_1 A \), where \( A \) will be determined later. The parameter value for which the spiral touches the line is \( t = 0 \), since the curvature of a line is zero, and the spiral must match that value. Let the parameter value for which the spiral touches the circle be \( t_1 \). The curvature of the spiral must match the curvature of the circle at \( t = t_1 \), thus,

\[
\pi t_1 / (\pi R_1 A) = 1/R_1 \quad \text{or} \quad t_1 = A.
\]

Also, the circle must be tangent to the spiral at the point of contact on the spiral. In other words, the circle must be the circle of curvature at its point of contact. Thus, matching the centres,

\[
(D_1, E_1) = (\pi R_1 C_1(A), \pi R_1 S_1(A) + R_1).
\]

Since \( E_1 \) is given, matching the \( Y \)-coordinate gives a non-linear equation for \( A \). \( S_1(t) \) is a positive, monotone increasing function so it is clear that there is a unique solution for any \( E_1 > R_1 \). Once \( A \) is found, \( D_1 \) can be calculated, and the coordinate system is then completely determined.

![Fig. 1. Line to circle with a single spiral.](image)
The angle difference of the tangents at the two end-points of the spiral is $\frac{1}{2} \pi A^2$. To keep this angle difference less than or equal to $\frac{1}{2} \pi$, take $A$ in $(0,1]$. There will be a solution for $A$ in $(0,1]$ if $S_1(1) \geq \left( \frac{E_1}{R_1} - 1 \right) / \pi > 0$.

If $A$ is in $(0,1]$, closer bounds on $A$ are, from (6),

$$\left( \frac{24}{\pi^2} \left( \frac{E_1}{R_1} - 1 \right) \right)^{1/4} \leq A \leq \left( \frac{12}{\pi} \left( \frac{E_1}{R_1} - 1 \right) \right)^{1/4}.$$

3.2. Circle to circle with a single spiral

Given two circles of radii $R_0$ and $R_1$, and the distance between their centres $T$, the problem is to find a single Cornu spiral joining the two circles so that the tangents and curvatures of the spiral match the corresponding tangents and curvatures of the circles at the joining points. This problem cannot always be solved, so it is worthwhile to state two simple conditions which must be met. Firstly, it is clear that $R_0$ and $R_1$ cannot be equal. The radius of the circle of curvature at parameter value $t$ is $a/(\pi t)$. Thus, there cannot be two circles of curvature with equal radii contacting a spiral at two different points. Henceforth, assume that $R_0 > R_1$.

Secondly, the smaller circle must be entirely contained inside the larger, or $T < R_0 - R_1$. The proof of this result can be found in [7], where it also shown that there always exists a spiral between two circles when one is contained inside the other. Unfortunately, the spiral is not always uniquely determined. Below it is shown that, with a restricted scaling factor, there is a unique spiral.

Choose a coordinate system so that the centres of the circles are in the first quadrant, and the spiral is in the standard form (1) (see Fig. 2). Let the centres of the circles be $(D_0, E_0)$ and $(D_1, E_1)$, and let the scaling factor of the spiral be $a = \pi R_0 R_1 A$, where $A$ will be determined later. Let the parameter values at which the spiral meets the two circles be $t_0$ and $t_1$. The curvature of the spiral must match the curvature of the circles at the points of contact, so

$$\frac{\pi t_0}{(\pi R_0 R_1 A)} = \frac{1}{R_0} \quad \text{or} \quad t_0 = R_1 A,$$

and

$$\frac{\pi t_1}{(\pi R_0 R_1 A)} = \frac{1}{R_1} \quad \text{or} \quad t_1 = R_0 A.$$

Also the centres of curvature must match the centres of the circles,

$$(D_0, E_0) = (\pi R_0 C_1(R_1 A), \pi R_0 S_1(R_1 A) + R_0),$$

and

$$(D_1, E_1) = (\pi R_1 C_1(R_0 A), \pi R_1 S_1(R_0 A) + R_1).$$

Since the distance between the centres is given, a single non-linear equation for $A$ is

$$T = f(A) = \sqrt{\left\{ (D_1 - D_0)^2 + (E_1 - E_0)^2 \right\}}$$

$$= \left\{ \left[ \pi (R_1 C_1(R_0 A) - R_0 C_1(R_1 A)) \right]^2$$

$$+ \left[ \pi (R_1 S_1(R_0 A) - R_0 S_1(R_1 A)) + R_1 - R_0 \right]^2 \right\}^{1/2}. \quad (7)$$
Using (3), the derivative of $f(A)$ with respect to $A$ is

$$f'(A) = \frac{\pi R_0^2 R_1^2 A}{f(A)} \left( \int_{R_0 A}^{R_0 A} \frac{\sin \frac{1}{2} \pi t^2}{t^2} \, dt \cdot \int_{R_0 A}^{R_0 A} \cos \frac{1}{2} \pi t^2 \, dt ight) \right.$$

$$- \left. \int_{R_0 A}^{R_0 A} \cos \frac{1}{2} \pi t^2 \, dt \cdot \int_{R_0 A}^{R_0 A} \sin \frac{1}{2} \pi t^2 \, dt \right).$$

It will now be shown that $f'(A)$ is negative for $A$ in the interval $(0, 1/(R_0^2 - R_1^2)^{1/2}]$. Although this interval could be expanded to $(0, (2/(R_0^2 - R_1^2))^{1/2}]$ in the following proof, the above value gives an acceptable angle of rotation for the tangent of the spiral. Let $g(R)$ be defined

$$g(R) = \int_{R_0 A}^{R_0 A} \frac{\sin \frac{1}{2} \pi t^2}{t^2} \, dt \cdot \int_{R_0 A}^{R_0 A} \cos \frac{1}{2} \pi t^2 \, dt$$

$$- \int_{R_0 A}^{R_0 A} \cos \frac{1}{2} \pi t^2 \, dt \cdot \int_{R_0 A}^{R_0 A} \sin \frac{1}{2} \pi t^2 \, dt.$$

The fact that $g(R)$ is negative for $R_1 < R \leq R_0$ follows by noting that $g(R_1) = 0$, and by showing that $g'(R)$ is negative for $R_1 < R \leq R_0$. Differentiating the above expression, and simplifying,

$$g'(R) = -A \int_{R_0 A}^{R_0 A} \left( \frac{1}{t^2} - \frac{1}{(RA)^2} \right) \sin \frac{1}{2} \pi \left[(RA)^2 - t^2\right] \, dt.$$
As \( t \) varies from \( R, A \) to \( R, A \), the factor in front of the sin function varies from \( \frac{1}{2} \pi (R^2 - R_1^2) A^2 \) to 0. The biggest argument occurs when \( R = R_0 \), and is less than or equal to \( \frac{1}{2} \pi \) since \( A \leq 1/(R_0^2 - R_1^2)^{1/2} \). Thus, the integrand is positive for \( R_1 < R < R_0 \), and \( g'(R) \) is negative for \( R_1 < R \leq R_0 \). Consequently, \( g(R_0) \) is negative, which implies that \( f'(A) \) is negative for \( 0 < A < 1/(R_0^2 - R_1^2)^{1/2} \).

It has now been shown that \( f(A) \) is monotone decreasing as \( A \) increases for \( A \) in \( (0, 1/(R_0^2 - R_1^2)^{1/2}) \), thus, there is a unique solution to (7) as long as

\[
f\left(\frac{1}{\sqrt{(R_0^2 - R_1^2)}}\right) \leq T < R_0 - R_1.
\]

Once \( A \) is found, \( D_0, E_0, D_1 \) and \( E_1 \) can be calculated, and the coordinate system is then completely determined.

The angle difference between the tangents at the two end-points of the spiral is

\[
\frac{1}{2} \pi (R_0^2 - R_1^2) A^2 \leq \frac{1}{2} \pi,
\]

which satisfies the previously stated requirements.

### 3.3 Circle to circle forming an S-curve with a pair of spirals

Given two circles of radii \( R_0 \) and \( R_1 \), and the distance between their centres \( T \), the problem is to find a pair of Cornu spirals from one circle to the other forming an S-curve, and joining the circles in such a way as to match tangents and curvatures at the joining points. It will be shown that this can always be done when circles have no common area, or when \( T > R_0 + R_1 \).

Choose a coordinate system so that the centre \((D_0, E_0)\) of the circle of radius \( R_0 \) is in the third quadrant, the centre \((D_1, E_1)\) of the circle of radius \( R_1 \) is in the first quadrant, and the two spirals join at the origin, have tangents of slope zero there, and have curvatures of zero there (see Fig. 3). Let the scaling factors of the two spirals be \( \pi R_0 A \) and \( \pi R_1 A \), where \( A \) has to be determined. Different scaling factors for the two spirals are chosen because it leads to simpler expressions later. This choice gives two spirals whose tangents rotate through the same total angle. Let the parameter values at which the spirals meet the circles be \( t_0 \) and \( t_1 \). The curvature of each spiral must match the curvature of the circle it meets, so

\[
\pi t_0 / (\pi R_0 A) = 1/R_0 \quad \text{or} \quad t_0 = A.
\]

and

\[
\pi t_1 / (\pi R_1 A) = 1/R_1 \quad \text{or} \quad t_1 = A.
\]

Also, the given circles must be circles of curvature of the spirals at \( t = t_0 \) at \( t = t_1 \), thus

\[
(D_0, E_0) = (-\pi R_0 c_1(A), -\pi R_0 s_1(A) - R_0),
\]

and

\[
(D_1, E_1) = (\pi R_1 c_1(A), \pi R_1 s_1(A) + R_1).
\]

Since the distance between the centres of the two circles, \( T \), is given, a single non-linear equation for \( A \) is

\[
T = f(A) = \sqrt{\{(D_1 - D_0)^2 + (E_1 - E_0)^2\}} = (R_0 + R_1) \sqrt{\left\{\left[\pi c_1(A)\right]^2 + \left[\pi s_1(A) + 1\right]^2\right\}}.
\]
The right side of this equation is a monotone increasing function of $A$ with minimum value $R_0 + R_1$ at $A = 0$. Thus, this equation has a unique solution when $T > (R_0 + R_1)$. Once $A$ is found, $D_0$, $E_0$, $D_1$ and $E_1$ can be calculated, and the coordinate system is then completely determined.

Total angle which the tangent rotates through along both spirals from one circle to the other is 0, but the tangent in each spiral rotates through an angle $\frac{1}{2} \pi A^2$. To keep the angles within the previously stated requirement, take $A$ in $(0, 1]$. There is a solution for $A$ if

$$f(1) > T > R_0 + R_1.$$ 

### 3.4. Circle to circle forming a broken-back or C-curve with a pair of spirals

Given two circles of radii $R_0$ and $R_1$, and the distance between their centres $T$, the problem is to find a pair of Cornu spirals from one circle to the other forming a broken-back or C-curve, and joining the circles in such a way as to match tangents and curvatures at the joining points. It will be shown that this can always be done when neither circle is entirely inside the other, or when $T > |R_0 - R_1|$.

Choose a coordinate system so that the centre $(D_0, E_0)$ of the circle of radius $R_0$ is in the second quadrant, the centre $(D_1, E_1)$ of the circle of radius $R_1$ is in the first quadrant, and the two spirals join at the origin, have tangents of slope zero there, and have curvatures of zero there (see Fig. 4). Let the scaling factors of the two spirals be $\pi R_0 A$ and $\pi R_1 A$. As before, choose different scaling factors to simplify subsequent formulae. With this choice, the tangents of both spirals rotate through the same total angle. Let the parameter values at which the spirals meet the circles be $t_0$ and $t_1$. The curvature of each spiral must match the curvature of the circle it meets, so

$$\pi t_0 / (\pi R_0 A) = 1 / R_0 \quad \text{or} \quad t_0 = A.$$
Fig. 4. Circle to circle forming a C-curve.

and

$$\pi t_i/(\pi R_i A) = 1/R_i \quad \text{or} \quad t_i = A.$$  

Also, the given circles must be circles of curvature of the spirals at \( t = t_i \) and \( t = t_i' \), thus

$$ (D_0, E_0) = (-\pi R_0 C_i(A), \pi R_0 S(A) + R_0), $$

and

$$ (D_1, E_1) = (\pi R_1 C_i(A), \pi R_1 S(A) + R_1). $$

Since the distance between the centres of the two circles, \( T \), is given, a single non-linear equation for \( A \) is

$$ T = f(A) = \sqrt{\left( (D_1 - D_0)^2 + (E_1 - E_0)^2 \right)} $$

$$ = \sqrt{\left( \pi(R_1 + R_0) C_i(A) \right)^2 + \left( (R_1 - R_0) (\pi S(A) + 1) \right)^2}. $$

The right side of this equation is a monotone increasing function of \( A \) with minimum value \( |R_0 - R_1| \) at \( A = 0 \). Thus, this equation has a unique solution when \( T > |R_0 - R_1| \). Once \( A \) is found, \( D_0, E_0, D_1 \) and \( E_1 \) can be calculated, and the coordinate system is then completely determined.

The angle difference of the tangents from starting point to ending point of each spiral is \( \frac{1}{2} \pi A^2 \). To keep the angle difference within the previously stated requirement, take \( A \) in (0,1]. There is a solution for \( A \) if

$$ f(1) \geq T > |R_0 - R_1|. $$

3.5. Line to line with a pair of spirals

Given two line segments forming an angle \( \Omega \) in (0, \( \pi \)), the problem is to find a pair of Cornu spirals between them so that the spirals join each other, and the lines with continuous tangent and continuous curvature. This can always be done.

Choose a coordinate system so that one of the line segments is along the \( x \)-axis, and the other line segment is in the upper half plane (see Fig. 5). Choose the scaling factors of each spiral to be the same because of the obvious symmetry, and, without loss of generality, choose the scaling factor for both spirals to be 1. The spirals join the lines at parameter value \( t = 0 \). Let \( t_0 \) be the
value of the parameter at which the two spirals meet. By symmetry, the angle of the tangent at \( t = t_0 \) must be \( \frac{1}{2} (\pi - \Omega) \), or

\[
\frac{1}{2} \pi t_0^2 = (\pi - \Omega)/2 + 2k\pi, \quad k = \ldots, -2, -2, 0, 1, 2, \ldots.
\]

There is an infinite set of values for \( t_0 \), but it is desirable to take the smallest positive value of \( t_0 \). The value of \( t_0 \) is thus,

\[
t_0 = \sqrt{\left\{1 - \Omega/\pi\right\}}.
\]

The distance along the lines from their intersection to place where the Cornu spirals meet the lines is

\[
C(t_0) + S(t_0) \cot \frac{1}{2} \Omega.
\]

The curve can now be scaled as desired, and the coordinate system is then completely determined.

The angle difference of the tangents from starting point to ending point of each spiral is of course \( \frac{1}{2} \Omega \).

4. Drawing a curve with spirals joining arcs of circles and lines

The previous section shows how to find the transition Cornu spirals. In the curve drawing problem, one is presented with a set of circles and lines. The problem is to draw a curve from a starting line or circle to an ending line or circle touching the other lines or circles in a specified order. It is convenient to let the circles have signed radii, a positive radius indicates that the centre of the circle is on the left as the curve touches that circle, and a negative radius indicates that the centre is on the right of the curve. Thus two positive or two negative radii in a succession indicate that a C-curve is desired, while two opposite signed radii in succession indicate that an S-curve is desired. It is only possible to connect a circle to line if the line does not intersect the circle.

There are situations in which it is not possible to join the circles with spirals as requested. For instance, the restriction that the total angle of rotation of the tangent in each spiral be less than \( \frac{1}{2} \pi \) may be impossible to meet. Depending on the situation, the restriction could be relaxed, or intermediate circles could be introduced. Another problem is that in joining three circles in a
row, for example, the spiral from left one to the centre one may touch the centre circle past the starting point of the spiral joining the centre circle to the right circle. This overlapping is a mathematical problem, and means that a sensible curve does not exist; that is, a curve without loops does not exist. A possible solution is to change the radii of the circles in question or move the circles slightly.

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References