A counter-example to a conjecture of Félix

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ABSTRACT

If $X$ is a simply connected space of finite type, then the rational homotopy groups of the based loop space of $X$ possess the structure of a graded Lie algebra, denoted $L_X$. The radical of $L_X$, which is an important rational homotopy invariant of $X$, is of finite total dimension if the Lusternik–Schnirelmann category of $X$ is finite.

Let $X$ be a simply connected space with finite Lusternik–Schnirelmann category. If $\dim L_X < \infty$, i.e., if $X$ is elliptic, then $L_X$ is its own radical, and therefore the total dimension of the radical of $L_X$ in odd degrees is less than or equal to its total dimension in even degrees (Friedlander and Halperin (1979) [8]). Félix conjectured that this inequality should hold for all simply connected spaces with finite Lusternik–Schnirelmann category.

We prove Félix’s conjecture in some interesting special cases, then provide a counter-example to the general case. © 2010 Elsevier B.V. All rights reserved.

1. Introduction

If $X$ is a simply connected space with the homotopy type of a CW-complex of finite type, then the graded $\mathbb{Q}$-vector space $\pi_*(\Omega X) \otimes \mathbb{Q}$ naturally admits the structure of a graded Lie algebra, where the bracket is given by the Samelson product of homotopy classes. This graded Lie algebra, called the homotopy Lie algebra of $X$, and denoted $L_X$, is an important rational homotopy invariant, the structure of which has been studied in great detail, in particular in a long series of papers by Félix, Halperin, and Thomas.

We contribute in this paper to the knowledge of $L_X$, by studying properties of its radical $\text{Rad} L_X$, i.e., the ideal of $L_X$ that is the union of all solvable subideals. We consider only spaces $X$ of finite Lusternik–Schnirelmann category (see the Appendix), as it is known that then $\dim \text{Rad} L_X < \infty$ (see Theorems 3 and 23) [4]. Moreover, the spaces $X$ of finite Lusternik–Schnirelmann category such that $\dim L_X < \infty$, known as elliptic spaces, are extremely well understood. In particular, if $X$ is elliptic, then $\dim (L_X)_{\text{odd}} \leq \dim (L_X)_{\text{even}}$ [8].

If $X$ is an elliptic space, then easy degree arguments show that $\text{Rad} L_X = L_X$, and so $\dim (\text{Rad} L_X)_{\text{odd}} \leq \dim (\text{Rad} L_X)_{\text{even}}$. Félix conjectured that the same inequality should hold for hyperbolic spaces as well.

Conjecture 1. Let $X$ be a simply connected space of finite type. If the Lusternik–Schnirelmann category of $X$ is finite, then $\dim (\text{Rad} L_X)_{\text{odd}} \leq \dim (\text{Rad} L_X)_{\text{even}}$.

We begin in Section 2 by recalling the definition of the radical and a number of key results concerning bounds on its total dimension. We also provide a new bound on the total dimension in odd degrees of the radical of the homotopy Lie algebra of the total space in a nice enough fibration (Proposition 6).

In Section 3, we show first that Conjecture 1 holds in certain special cases (Corollary 8, Propositions 9 and 10), and then provide a counter-example to the general case (Example 11). We also construct a space with nonzero radical for which Conjecture 1 holds (Example 14).

The reader will find the definitions and results from rational homotopy theory that we need in our proofs in the Appendix.
2. The radical of a graded Lie algebra

In this section, we recall the definition and a few key properties of the radical of a graded Lie algebra. We also establish a new bound on the total dimension in odd degrees of the radical of the homotopy Lie algebra of the total space in a nice enough fibration (Proposition 6).

Definition 2 ([2]). Let $L$ be a graded Lie algebra over $\mathbb{Q}$.

(1) An ideal $I$ of $L$ is solvable if $I_{[n]} = 0$ for some $n \geq 0$, where $I_{[0]} = I$ and $I_{[k]} = [I_{k-1}, I_{k-1}]$ for all $k > 0$.

(2) The radical of $L$, denoted $\text{Rad} L$, is the union of all the solvable ideals of $L$.

It is obvious that, if $L$ is of finite total dimension, and $L_0 = 0$, then $\text{Rad} L = L$, and that the center $Z(L)$ of $L$ is always contained in $\text{Rad} L$.

There are strong restrictions on the radical of a Lie algebra of finite depth (Definition 21).

Theorem 3 ([4]). Let $L$ be a graded Lie algebra of finite type. If $\text{dim} L = m$, then

(1) $\text{Rad} L$ is solvable and of finite dimension;

(2) $\text{dim} (\text{Rad} L)_{\text{even}} \leq m$; and

(3) if $\text{dim} (\text{Rad} L) = m$, then $L = \text{Rad} L$.

Moreover, the radicals of the homotopy Lie algebras of the total space and fibre in a fibration are closely related.

Proposition 4 ([2]). Let $F \xrightarrow{i} E \xrightarrow{\rho} B$ be a fibration of rational spaces with $B$, $E$ and $F$ simply connected spaces of finite type. If $\text{depth} L_F < \infty$, then

(1) $\text{Rad} L_F$ is a solvable ideal; and

(2) $[\pi_\ast (\Omega i) \otimes \mathbb{Q}]^{-1} (\text{Rad} L_F) = \text{Rad} L_F$.

We can now bound the dimension of the odd radical of the total space of a fibration, in terms of the following classical notion.

Definition 5 ([5]). Let $L$ be a graded Lie algebra. An element $\alpha \in L$ is an Engel element if, for all $\beta \in L$, there exists $p \geq 1$ such that $[\text{ad}(\alpha)]^p(\beta) = 0$. The set of all Engel elements in $L$ is denoted $\text{Eng} L$.

Proposition 6. Let $F \xrightarrow{i} E \xrightarrow{\rho} B$ be a fibration with $B$, $E$ and $F$ simply connected spaces of finite type. If $\text{cat}_0 (F) < \infty$ and $\text{depth} L_F < \infty$, then

\[
\text{dim} (\text{Rad} L_F)_{\text{odd}} \leq \text{dim} (\text{Rad} L_E)_{\text{odd}}
\]

and

\[
\rho_\ast (\text{Rad} L_E)_{\text{odd}} \subset \text{Eng} L_B.
\]

Proof. We write $i_\ast$, $\rho_\ast$ and $\delta_\ast$ for $\pi_\ast (\Omega i) \otimes \mathbb{Q}$, $\pi_\ast (\Omega \rho) \otimes \mathbb{Q}$ and $\delta \otimes \mathbb{Q}$, respectively. Here, $\delta$ denotes the connecting map in the long exact sequence of homotopy groups of the looped fibration.

By Proposition 4,

\[
(\text{Rad} L_F)_{\text{odd}} = [\text{odd}]^{-1} (\text{Rad} L_E)_{\text{odd}}.
\]

Moreover, $(\ker i_\ast)_{\text{odd}} = (\text{im} \delta_\ast)_{\text{odd}} \subset s^{-1} G_{\text{even}} (F) \otimes \mathbb{Q}$ by Proposition 27, where $G_{\text{even}} (F)$ denotes the graded Gottlieb group of $F$ (Definition 24). As $F$ is of finite category, Proposition 25 implies that $G_{\text{even}} (F) \otimes \mathbb{Q} = 0$. It follows that $i_\ast$ is injective, and therefore

\[
\text{dim} (\text{Rad} L_F)_{\text{odd}} \leq \text{dim} (\text{Rad} L_E)_{\text{odd}}.
\]

Let $\alpha \in (\text{Rad} L_E)_{\text{odd}}$, and let $\beta = \rho_{\ast \text{odd}} (\alpha)$. If $\beta = 0$, then it is clearly an Engel element. If not, suppose that there exists $\psi \in \pi_\ast (\Omega B) \otimes \mathbb{Q}$ such that $[\text{ad}(\beta)]^n (\psi) \neq 0$, for all $n$.

Case 1: If $\psi$ is of odd degree, then $\delta_\ast (\psi) = 0$, because $G_{\text{even}} (F) \otimes \mathbb{Q} = 0$ by Proposition 25. There exists therefore $\gamma \in L_E$ such that $\rho_\ast (\gamma) = \psi$. We have

\[
\rho_\ast ([\text{ad}(\alpha)]^n (\gamma)) = [\text{ad}(\beta)]^n (\psi).
\]

On the other hand, since $\alpha$ is in the radical of $L_E$, there exists $N > 0$ such that $[\text{ad}(\alpha)]^N (\gamma) = 0$, which is a contradiction.

Case 2: If $\psi$ is of even degree, then the degree of $[\beta, \psi]$ is odd. For the same reasons as above, we have that $\delta_\ast ([\beta, \psi]) = 0$, so there exists $\phi \in L_E$ such that $\rho_\ast (\phi) = [\beta, \psi]$. It follows that

\[
\rho_\ast ([\text{ad}(\alpha)]^n (\phi)) = [\text{ad}(\beta)]^{n+1} (\psi),
\]

and, as above, we have a contradiction.

We conclude that $\beta \in (\text{Eng} L_B)_{\text{odd}}$. □
3. Félix’s conjecture

We show in this section that Conjecture 1 is false. Before presenting our counter-example, we establish certain special cases in which it does hold.

**Lemma 7.** Let $L$ be a connected, graded Lie algebra of finite type. If depth $L < \infty$, then

$$Z(L) = 0 \iff \text{Rad} L = 0.$$ 

As an immediate consequence of this lemma, the proof of which is obvious, we obtain a first nontrivial class of graded Lie algebras for which Conjecture 1 is verified.

**Corollary 8.** Let $L$ be a graded Lie algebra of finite type and of finite depth. If $Z(L) = 0$, then $\dim(\text{Rad} L) \subseteq \dim(\text{Rad} L)$.

The next nontrivial class of graded Lie algebras satisfying Conjecture 1 is characterized in terms of associated coformal spaces (Definition 22).

**Proposition 9.** Let $X$ be a simply connected space of finite type. If the Lusternik–Schnirelmann category of the coformal space associated to $X$ is finite, then

$$\dim(\text{Rad} L_X) \subseteq \dim(\text{Rad} L_X).$$

Note that Theorem 23 implies that the hypothesis of this proposition is stronger than that of Conjecture 1.

**Proof.** Applying the cochain functor (Definition 15) to the inclusion $i$ of $\text{Rad} L_X$ into $L_X$, we obtain a morphism of commutative differential graded algebras

$$(AV, d_2) = C^*(L_X) \rightarrow C^*(\text{Rad} L_X) = AW, d_2),$$

where $(AV, d_2)$ is the Sullivan model of the coformal space $\tilde{X}$ associated to $X$. This algebra map corresponds to a continuous map $f : Y \rightarrow \tilde{X}$ of rational spaces, where $L_Y = \text{Rad} L_X$.

Observe that $\pi_0 (f) = i$ and is injective. Theorem 19 therefore implies that

$$\text{cat}_0 (Y) \subseteq \text{cat}_0 (\tilde{X}) < \infty.$$ 

As depth $L_X \subseteq \text{cat}_0 (\tilde{X}) < \infty$, Theorem 3 implies that the dimension of $\text{Rad} L_X$ is finite, and therefore that $Y$ is elliptic, whence

$$\dim(\text{Rad} L_X) \subseteq \dim(\text{Rad} L_X).$$

Another case in which Conjecture 1 holds can be expressed in terms of the evaluation map (Definition 28).

**Proposition 10.** Let $X$ be a simply connected space of finite type such that $L_X$ is of finite depth. If the evaluation map of the associated coformal space is nonzero, then

$$\dim(\text{Rad} L_X) \subseteq \dim(\text{Rad} L_X).$$

**Proof.** The exact sequence of Lie algebras

$$\text{Rad} L_X \rightarrow L_X \rightarrow L_X / \text{Rad} L_X$$

corresponds to a fibration

$$F \rightarrow \tilde{X} \rightarrow B$$

of rational spaces, with $\pi_0 (\Omega F) \subseteq \text{Rad} L_X$, $\pi_0 (\Omega \tilde{X}) \subseteq L_X$, and $\pi_0 (\Omega B) \subseteq L_X / \text{Rad} L_X$.

Observe that $\pi_0 (\Omega F)$ is of finite dimension, because $L_X$ is of finite depth, and that $\rho$ is surjective. By Theorem 30, $\text{ev}_F$ is nonzero, and, by Theorem 29, $\dim H^*(F; \mathbb{Q})$ is finite.

It follows that $F$ is an elliptic space, and so

$$\dim(\text{Rad} L_X) \subseteq \dim(\pi_0 (\Omega F)) \subseteq \dim(\pi_0 (\Omega F)) = \dim(\text{Rad} L_X).$$

As explained in [10, Proposition 4.4.7], actually follows from Proposition 10.

We now present our counter-example to Conjecture 1.
Example 11. Consider the following fibre product.

\[
\begin{array}{ccc}
K(\mathbb{Z}, 3) & \longrightarrow & K(\mathbb{Z}, 3) \\
\downarrow & & \downarrow i \\
y & \longrightarrow & * \\
\tilde{p} & \downarrow & \downarrow p \\
X = (S^3 \vee S^4) \times K(\mathbb{Z}, 2) & \longrightarrow & K(\mathbb{Z}, 4).
\end{array}
\]

Here the map \( f \) corresponds to the class \([f] = y^2 - x \in H^4(X; \mathbb{Z})\), where \( x \) is a generator of \( H^4(S^4; \mathbb{Z}) \) and \( y \) is a generator of \( H^2(K(\mathbb{Z}, 2); \mathbb{Z}) \).

Then \( Y \) is an hyperbolic space that does not satisfy Félix’s conjecture.

If \((\Lambda Z, d_2)\) is the Sullivan model of \( S^3 \vee S^4\), the Sullivan model of \( X \) is

\[
(\Lambda Z \oplus x, d),
\]

where \( d|_2 = d_2 \), \( d(y) = 0 \) and \( |y| = 2 \). The model of \( f \) is then

\[
(Av, 0) \rightarrow (\Lambda(Z \oplus y), d) : v \mapsto y^2 - x,
\]

where \( |v| = 4 \).

It is well known that the model of \( p \) is given by

\[
(Av, 0) \rightarrow (A(v \oplus sv), d),
\]

where \( d(sv) = v \), and therefore that the model of \( \tilde{p} \) is

\[
(\Lambda(Z \oplus y) \oplus sv, d),
\]

where \( D|_{\Omega y} = d, D(sv) = y^2 - x \), and \( sv \) is of degree 3 [7].

We have then \( H^*(Y; \mathbb{Q}) \cong (\Lambda(a) \oplus \Lambda(x)) \otimes \Lambda(y) / (y^2 = x, x^2 = 0) \). The nilpotency is 3, the formal dimension is 6, and \( Y \) is simply connected. Hence, Propositions 16 and 17 together imply that \( \text{cat}_0(Y) = 3 \).

Write \( Z = \mathbb{Q}x \oplus \mathbb{Z}^\ell \). There exists a quasi-isomorphism

\[
(\Lambda(Z \oplus y \oplus sv), D) \xrightarrow{\tilde{F}} (\Lambda(Z' \oplus y), D')
\]

that sends \( sv \) to \( 0, x \to y^2, y \to y, \) and \( z \to z \), where \( D' = \rho \circ D \). Simple computations show then that \( D'(y) = 0 \), and that

\[
D'(z') \subset A^2Z' \oplus A^2y \otimes A^3Z' \oplus A^3y.
\]

so that \( D_1Z' \subset A^2Z' \). In other words, the element \( \hat{\gamma} \) of \( L_Y \) corresponding to \( y \) under the cochain functor is in the centre, and therefore in the radical. Moreover, \( \hat{\gamma} \) is of odd degree, i.e., \( \dim(\text{Rad} L_Y)_{\text{odd}} \geq 1 \). In fact, equality holds, since \( \dim(\text{Rad} L_{K(\mathbb{Z}, 3)})_{\text{odd}} = 0 \) and \( \dim(\text{Eng}(L_{K(\mathbb{Z}, 2)})_{\text{odd}} = 1 \) (see Proposition 6).

Furthermore, depth \( L_Y \leq 3 \). If depth \( L_Y \) were exactly 3, then \( \text{cat}_0(\tilde{Y}) \) would also be 3, where \( \tilde{Y} \) is the coformal space associated to \( Y \), which is not the case, since \( D_1y = 0 \) and \( y^2 \) is never a \( D_1 \)-boundary. We conclude that depth \( L_Y \leq 2 \).

It follows from Theorem 3 that depth \( \text{Rad} L_Y \leq 1 \), because \( \dim L_Y = \infty \), and therefore

\[
\dim(\text{Rad} L_Y)_{\text{even}} = \text{depth} \text{Rad} L_Y \leq 1.
\]

If we had equality, as \( \hat{\gamma} \) is of minimal degree in \( L_Y \), there would have to be an even element \( \hat{\alpha} \) of \( \text{Rad} L_Y \) that actually belonged to the centre of \( L_Y \). It would correspond to an element \( \alpha \in (Z')^{\text{odd}} \) not in \( \text{Im} D_2 \), i.e., there would exist no \( z' \in Z' \) with \( D_2z' \in A^4 + \alpha \mathbb{Z}^\ell \oplus A^3Z' \).

The dimension of the ideal generated by \( \hat{\alpha} \) in \( L_X \) would be infinite, since \( L_X = L_{S^3 \vee S^4} \times \mathbb{Q} \hat{\alpha} \), and \( L_{S^3 \vee S^4} \) is a free Lie algebra. There would therefore be infinitely many elements in \( L_X \) whose bracket with \( \hat{\alpha} \) was nonzero. In particular, there would necessarily exist \( \hat{y} \neq \hat{\alpha} \), where \( \hat{\alpha} \) is the element corresponding to \( x \) in \( L_{S^3 \vee S^4} \), such that \( [\hat{y}, \hat{\alpha}] \neq 0 \). In other words, there would exist \( \beta \in Z' \) such that

\[
d_2\beta = \alpha y + \phi, \quad \phi \in \mathbb{Z}^\ell,
\]

and therefore

\[
D'\beta = \alpha y + \rho(\phi).
\]

We arrive at a contradiction, because then \( \alpha \in \text{Im} D_2 \).

In short, we have

\[
\text{depth} L_Y \leq 2, \quad \dim(\text{Rad} L_Y)_{\text{even}} = 0, \quad \dim(\text{Rad} L_Y)_{\text{odd}} = 1.
\]
Remark 12. Even though the conjecture is false, the following inequality is verified by the space constructed above.

\[ \dim(\text{Rad } L_Y)_{\text{odd}} \leq \text{cat}_0(Y). \]

This observation inspires us to formulate the following new conjecture.

Conjecture 13. Let \( X \) be a simply connected space of finite type. If \( \text{cat}_0(X) < \infty \), then \( \dim(\text{Rad } L_Y)_{\text{odd}} \leq \text{cat}_0 X \).

To conclude this section, we provide an example of a hyperbolic space with nonzero radical satisfying Félix’s conjecture.

Example 14. Consider the following fibre product:

\[
\begin{array}{ccc}
S^4 & \longrightarrow & S^4 \\
\downarrow & & \downarrow i \\
Y & \longrightarrow & E \\
\downarrow p & & \downarrow p \\
X = (S^2 \vee S^3) \times S^5 & \longrightarrow & S^3 \times S^5
\end{array}
\]

where the model of the fibration \( p \) is given by

\[ (A(x_3, y_5), 0) \rightarrow (A(x_3, y_5, u_4, v_7), D) \rightarrow (A(u_4, v_7), D) \]

with \( D(u_4) = 0, D(v_7) = u_4^2 - x_3y_5 \). The indices denote the respective degree of the generators. Moreover,

\[
\begin{align*}
& f^\ast(x_3) = x_3', \text{ generator of } H^3(S^3; \mathbb{Q}) \\
& f^\ast(y_5) = y_5', \text{ generator of } H^5(S^5; \mathbb{Q})
\end{align*}
\]

Then \( Y \) satisfies Félix’s conjecture.

Using the Serre spectral sequence, we see that the rational cohomology of \( Y \) is of finite dimension, so \( \text{cat}_0(Y) \) and depth \( L_Y \) are finite. Moreover, \( \text{cat}_0(S^4) \) is also finite. There are no odd Engel elements in \( L_X \), since \( L_X = L(a_1, b_2) \times L(c_4) \), where \( L \) denotes the free Lie algebra functor. By Proposition 6,

\[ 1 = \dim \left( \text{Rad } L_{S^4} \right)_{\text{odd}} = \dim \left( \text{Rad } L_Y \right)_{\text{odd}}. \]

As \( Y \) is a coformal space of finite category, Proposition 9 implies that

\[ 1 = \dim \left( \text{Rad } L_Y \right)_{\text{odd}} \leq \dim \left( \text{Rad } L_Y \right)_{\text{even}}. \]

By Proposition 20, \( \text{cat}_0(Y) \leq 2 \cdot 3 - 1 = 5 \), and so

\[ \dim \left( \text{Rad } L_Y \right)_{\text{even}} < \text{depth } L_Y \leq 5. \]

The essential difference between this example and Counter-Example 11 is that here the space \( Y \) is coformal. Moreover, in the preceding fibre product, the generator of \( \pi_5(X) \otimes \mathbb{Q} \) is not identified with an element of the free Lie algebra \( \pi_5(S^2 \vee S^3) \otimes \mathbb{Q} \). ∎

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Appendix. Our rational homotopy toolbox

We refer the reader to [7] for a detailed introduction to the Sullivan and Quillen models of rational homotopy theory. We recall here only the specific notions and results used in this paper.

A link between the Sullivan model of a space and its homotopy Lie algebra is provided by the following functor.

Definition 15 ([2]). The cochain algebra on a graded Lie algebra \( L \), denoted \( C^\ast(L) \), is the commutative graded Lie algebra \( (A^\ast L^\ast, d_L) \) with

\[ (d_L s^{-1} f; sx, sy) = (-1)^{|x|} (f; [x, y]), \]

where \( L^\ast \) is the \( \mathbb{Q} \)-linear dual of \( L \).
Recall that the Lusternik–Schnirelmann category \( (L\text{-}S \text{cat}) \) of a space \( X \), denoted \( \text{cat}(X) \), is the least integer \( n \) such that \( X \) can be covered by \( n + 1 \) open sets that are all contractible in \( X \). The nilpotence index of the cohomology ring of \( X \) gives a lower bound on \( \text{cat}(X) \).

**Proposition 16** ([2]). Let \( X \) be a topological space, \( A \) a ring, and \( H^*(X; A) \) the reduced cohomology of \( X \) with coefficients in \( A \). If \( \text{cat}(X) = n \), then every product of \( n + 1 \) classes in \( H^*(X; A) \) is zero.

Let \( X \) be a simply connected space of finite type. In rational homotopy theory, we are particularly concerned with the rational Lusternik–Schnirelmann category of \( X \), denoted \( \text{cat}_0(X) \), which is the \( L\text{-}S \) category of its rationalization \( X_0 \):

\[
\text{cat}_0(X) = \text{cat}(X_0).
\]

There is a well-known upper bound on the rational \( L\text{-}S \) category in terms of the *formal dimension* of \( H^*(X; \mathbb{Q}) \), which is the greatest \( n \) such that \( H^n(X; \mathbb{Q}) \) is different from zero.

**Proposition 17** ([2]). Let \( X \) be an \((r - 1)\)-connected space with finite rational cohomology. If the formal dimension of \( X \) is \( n \), then

\[
\text{cat}_0(X) \leq n/r.
\]

The following computation of \( L\text{-}S \) category proves useful in this paper.

**Proposition 18** ([3]). Let \( X \) and \( Y \) be two simply connected spaces of finite type. Then

\[
\text{cat}_0(X \vee Y) = \max(\text{cat}_0(X), \text{cat}_0(Y)).
\]

Another advantage of \( \text{cat}_0 \) is that we can compare two spaces that are the source and the target of a map, as the “Mapping Theorem” states below.

**Theorem 19** ([3]). Let \( f : X \to Y \) be a continuous map between two simply connected spaces of finite type. If

\[
\pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}
\]

is injective, then \( \text{cat}_0(X) \leq \text{cat}_0(Y) \).

For certain fibrations, we can bound the category of the total space.

**Proposition 20** ([2]). If \( F \to E \to B \) is a fibration of simply connected \( CW \)-complexes of finite type, then

\[
\text{cat}_0(E) + 1 \leq (\text{cat}_0(F) + 1)(\text{cat}_0(B) + 1).
\]

Another important homotopy invariant related to the rational Lusternik–Schnirelmann category of a space is the depth of its homotopy Lie algebra.

**Definition 21** ([2]). Let \( L \) be a graded Lie algebra. We define the depth of \( L \) by

\[
\text{depth } L = \inf \{ n \mid \text{Ext}^n_{\mathbb{Q}}(\mathbb{Q}, UL) \neq 0 \}.
\]

There is a beautiful theorem that relates the depth of the homotopy Lie algebra of a space to its rational \( L\text{-}S \)-category. To state the theorem, we need one more definition.

**Definition 22.** Let \( X \) be a simply connected space of finite type, and \((AV, d)\) its minimal Sullivan model. The *coformal space* associated to \( X \), denoted \( \tilde{X} \), is the simply connected space of finite type whose minimal Sullivan model is \((AV, d_2)\), where \( d_2 \) is the quadratic part of \( d \).

**Theorem 23** ([4]). If \( X \) is a simply connected space of finite type, then

1. \( \text{depth } L_X \leq \text{cat}_0(X) \leq \text{cat}_0(\tilde{X}) \); and
2. if \( \text{depth } L_X = \text{cat}_0(X) \), then \( \text{cat}_0(X) = \text{cat}_0(\tilde{X}) \).

Another rational homotopy invariant we use in this article is the graded Gottlieb group of a rational space \( X \).

**Definition 24** ([2]). Let \( X \) be a simply connected space of finite type. The \( p \)th *Gottlieb group* of \( X \), denoted \( G_p(X) \), is the subgroup of \( \pi_*(X) \) consisting of the classes represented by maps \( \alpha : S^p \to X \) such that the map \( \alpha \vee id : S^p \vee X \to X \) can be extended over \( S^p \times X \), up to homotopy.

**Proposition 25** ([11,3]). Let \( X \) be a simply connected space of finite type. If \( \text{cat}_0X \leq n \), then

1. \( G_{\text{even}}(X) \otimes \mathbb{Q} = 0 \); and
2. \( \dim G_{\text{even}}(X) \otimes \mathbb{Q} \leq n \).

**Proposition 26** ([1]). Let \( X \) be a simply connected space of finite type. Then, under the isomorphism induced by the adjunction, the elements \( G_{n+1}(X) \otimes \mathbb{Q} \) are in the centre of \( L_X \).
Proposition 27 ([2]). Let $F \to E \to B$ be a fibration of simply connected spaces of finite type. Then the image of the connecting map of the homotopy long exact sequence $\delta : \pi_* (B) \to \pi_{* - 1} (F)$ is contained in $G_{* - 1} (F)$.

The last tool from rational homotopy theory that we need in this paper is the evaluation map.

Definition 28 ([6]). Let $(A, d_A)$ be an augmented differential algebra. Let $(P, d_P) \xrightarrow{\sim} (Q, 0)$ be a semi-free $(A, d_A)$-resolution of $Q$. The evaluation map of $(A, d_A)$ is a linear map:

\[ ev_A : \text{Ext}_{(A, d_A)} (\mathbb{Q}, (A, d_A)) \to H^* (A, d_A), \]

which, to an element $[f] \in \text{Ext}_{(A, d_A)} (\mathbb{Q}, (A, d_A))$ represented by a cycle $f : (P, d_P) \to (A, d_A)$, assigns the class $[f(p)] \in H^* (A, d_A)$, where $p$ is a cycle of $(P, d_P)$ representing $1$.

The evaluation map of a topological space $X$, denoted $ev_X$, is the evaluation map of the rational cochain complex $C^* (X, \mathbb{Q})$.

Theorem 29 ([9]). Let $X$ be a simply connected space of finite type, with $\pi_* (X) \otimes \mathbb{Q}$ of finite dimension. Then the following statements are equivalent.

1. $H^* (X; \mathbb{Q})$ is of finite dimension.
2. $ev_X$ is different from zero.

Theorem 30 ([9]). Let $F \to E \to B$ be a fibration of simply connected spaces of finite type.

- If $H^* (F; \mathbb{Q})$ is of finite dimension, then $ev_B \neq 0$ implies that $ev_E \neq 0$.
- If $\pi_* (F) \otimes \mathbb{Q}$ is of finite dimension and $\pi_* (\rho) \otimes \mathbb{Q}$ is surjective, then $ev_F \neq 0$ implies that $ev_E \neq 0$.

We refer the reader to [6,9] for further details.

References