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Algebraic solutions of the Lamé equation, revisited

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Abstract

A minor error in the necessary conditions for the algebraic form of the Lamé equation to have a finite projective monodromy group, and hence for it to have only algebraic solutions, is pointed out (see Baldassarri, *J. Differential Equations* 41 (1) (1981) 44). It is shown that if the group is the octahedral group S_4, then the degree parameter of the equation may differ by $\pm 1/6$ from an integer; this possibility was missed. The omission affects a recent result on the monodromy of the Weierstrass form of the Lamé equation (see Churchill, *J. Symbolic Comput.* 28 (4–5) (1999) 521). The Weierstrass form, which is a differential equation on an elliptic curve, may have, after all, an octahedral projective monodromy group.

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1. Introduction

The Lamé equation is a second-order Fuchsian differential equation. It may be written $L_{\ell,B}u = 0$, where $L_{\ell,B}$ is the Lamé operator with complex parameters ℓ and B. The first, the so-called degree parameter, is often denoted n, but the notation ℓ is

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used here, to hint at connections with Lie group representation theory. B is an accessory parameter, which in many applications plays the role of an eigenvalue.

The Lamé equation arose in a classical setting: the solution of Laplace's equation in ellipsoidal coordinates by separation of variables. In that context, its solutions include the ellipsoidal harmonics. In classical treatments, ℓ is accordingly an integer, or perhaps a half-odd-integer [17, Chapter XXIII]. The latter case arises in a more complicated separation of variables problem (see [7, Section 15.1.3; 14, Chapter IX, Ex. 4]). In modern applications, ℓ may vary continuously. For example, the Lamé equation with $\ell \in [0,2]$ has been used to compute the Hubble distance–redshift relation in inhomogeneous, spatially flat cosmologies [9]. In that application, $\{\ell(\ell+1)/6\} \in [0,1]$ is the fraction of inhomogeneous matter in the universe that is 'dark', i.e., excluded from observation.

Actually, several distinct equations are referred to in the literature as the Lamé equation. We initially consider the algebraic form, rather than the Weierstrass or the Jacobi form. The algebraic form is defined on the complex projective line $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$, with

$$L_{\ell,B} \stackrel{\text{def}}{=} D^2 + \frac{1}{2}\sum_{i=1}^{3}\frac{1}{x-e_i}D - \frac{\ell(\ell+1)x+B}{4\prod_{i=1}^{3}(x-e_i)}, \tag{1.1}$$

where $D \stackrel{\text{def}}{=} d/dx$. Here $\ell, B, e_1, e_2, e_3 \in \mathbf{C}$, the e_i are distinct, and by convention, $e_1 + e_2 + e_3 = 0$. The equation $L_{\ell,B}u = 0$ has four regular singular points, three of which (e_1, e_2, e_3) have characteristic exponents $0, 1/2$, and one of which (∞) has exponents $-\ell/2,\ (\ell+1)/2$. So the algebraic-form Lamé equation is a special case of the Heun equation, which is the general second-order Fuchsian equation on $\mathbf{P}^1(\mathbf{C})$ with four singular points.

Via the map $(x,y) \mapsto x$, the line $\mathbf{P}^1(\mathbf{C})$ is doubly covered by the elliptic curve $y^2 = 4x^3 - g_2x - g_3$, where the invariants $g_2, g_3 \in \mathbf{C}$, at least one of which is nonzero, are defined by $4x^3 - g_2x - g_3 \equiv 4\prod_{i=1}^{3}(x-e_i)$. This curve will be denoted E_{g_2,g_3}. $L_{\ell,B}$ can be pulled back to a differential operator L_{ℓ,B,g_2,g_3} that acts on E_{g_2,g_3}. The pullback has only one singular point, namely $(x,y) = (\infty,\infty)$, which from a geometric point of view is why the Lamé case of the Heun equation is important. The pulled-back equation $L_{\ell,B,g_2,g_3}u = 0$ on E_{g_2,g_3} is the Weierstrass form, which is discussed in Section 5. Indirect references to the elliptic curve interpretation occur elsewhere, since when studying $L_{\ell,B}$, we classify various situations by supplying the corresponding value of J, Klein's modular function (also known as Klein's absolute invariant). Recall that $J \stackrel{\text{def}}{=} g_2^3/\Delta \in \mathbf{C}$, where $\Delta \stackrel{\text{def}}{=} g_2^3 - 27g_3^2 \neq 0$ is the modular discriminant. Iff two elliptic curves specified by g_2, g_3 have the same value of J, they are birationally equivalent, e.g., homeomorphic as complex manifolds [6, Section 5.3].

The determination of all quadruples ℓ, B, g_2, g_3 for which $L_{\ell,B}$ has only algebraic functions in its kernel is an unsolved problem. The nonclassical case $2\ell \notin \mathbf{Z}$ of this problem is perhaps the most tractable. Singer [16] and Morales-Ruiz and Simó

[12, Lemma 1] mention an unpublished result of Dwork that for any fixed ℓ for which $2\ell \notin \mathbf{Z}$, if e_1 is fixed, then there are only a finite number of pairs e_3, B for which all solutions of $L_{\ell,B}u = 0$ are algebraic. In essence, for each $\ell \notin (1/2)\mathbf{Z}$ there are only a finite number of 'algebraic' pairs J, B; though for this statement to make sense, B would need to be redefined in a scale-invariant way, constant on each elliptic curve isomorphism class.

The difficulty of finding all ℓ, B, g_2, g_3 for which the Lamé equation $L_{\ell,B}u = 0$ has only algebraic solutions contrasts with the classical solution of the corresponding problem for the hypergeometric equation $L_{\lambda,\mu,\nu}v = 0$, the canonical second-order Fuchsian equation on $\mathbf{P}^1(\mathbf{C})$ with three singular points. Here

$$L_{\lambda,\mu,\nu} \stackrel{\text{def}}{=} \frac{d^2}{dz^2} + \frac{1-\lambda^2}{4z^2} + \frac{1-\mu^2}{4(z-1)^2} + \frac{\lambda^2+\mu^2-1-\nu^2}{4z(z-1)} \tag{1.2}$$

is the (normal-form) hypergeometric operator with exponent differences $\lambda, \mu, \nu \in \mathbf{C}$, and the singular points on $\mathbf{P}^1(\mathbf{C})$ (coordinatized by z) are $z = 0, 1, \infty$. It is a classical result of Schwarz that if $\lambda, \mu, \nu \notin \mathbf{Z}$, then $L_{\lambda,\mu,\nu}v = 0$ will have only algebraic solutions iff a suitably normalized version of λ, μ, ν (regarded as an unordered triple) appears on a certain list. This is the famous 'Schwarz list', which has 15 entries, numbered I–XV (see [7, Section 2.7.2; 13; 14, Section 30]). The case when one of λ, μ, ν is an integer is degenerate, and can be handled by other means (it has its own list). To each list entry there corresponds a finite group, to which the projective monodromy group $G(L_{\lambda,\mu,\nu})$, which will be a finite subgroup of the Möbius group $PGL(2,\mathbf{C})$, is necessarily isomorphic. The possible groups are cyclic (C_n, $n \geqslant 1$), dihedral (D_n, $n \geqslant 2$), tetrahedral (A_4), octahedral (S_4), and icosahedral (A_5).

Klein's theory of pullbacks of Fuchsian operators grew out of Schwarz's classification theory. Associated to any second-order Fuchsian operator F on an algebraic curve over $\mathbf{C}$ is a projective monodromy group $G(F) \leqslant PGL(2,\mathbf{C})$. Klein showed that $G(F)$ will be finite, which is almost enough to ensure that $Fu = 0$ has only algebraic solutions, iff F is a (weak) pullback from $\mathbf{P}^1(\mathbf{C})$ of some $L_{\lambda,\mu,\nu}$, where λ, μ, ν belongs to a small sublist, called the 'basic Schwarz list'. (Other list entries can be omitted since they are redundant: they themselves correspond to pullbacks.) Necessarily $G(F) \leqslant G(L_{\lambda,\mu,\nu})$; and in fact, there is at least one λ, μ, ν on the basic Schwarz list, with corresponding pullback, such that $G(F) = G(L_{\lambda,\mu,\nu})$. If the pullback is known explicitly, $G(F)$ may readily be computed, and the solutions of $Fu = 0$ may be computed too. All solutions will be algebraic, provided the Wronskian of F is algebraic. The proofs of Klein were modernized by Baldassarri and Dwork in [1,3].

In a remarkable paper, Baldassarri [2] applied Klein's theory to the Lamé equation. By determining necessary conditions for the existence of a pullback of $L_{\ell,B}$ from each possible $L_{\lambda,\mu,\nu}$, Baldassarri derived a necessary condition for $L_{\ell,B}u = 0$ to have only algebraic solutions, and also necessary conditions for $G(L_{\ell,B})$ to be each possible finite subgroup of the Möbius group. It is a classical result that $G(L_{\ell,B})$ is never cyclic, and can be dihedral only if $2\ell \in \mathbf{Z}$. Moreover, in the nonclassical case

$2\ell \notin \mathbf{Z}$, it cannot be dihedral. Baldassarri showed that if $2\ell \notin \mathbf{Z}$, all solutions of $L_{\ell,B}u = 0$ can be algebraic only if one of $\ell \pm 1/10$, $\ell \pm 1/6$, $\ell \pm 1/4$, $\ell \pm 3/10$ is an integer. Moreover, $G(L_{\ell,B})$ cannot be tetrahedral, so if it is finite, it must be octahedral or icosahedral.

Unfortunately, [2] errs in its treatment of the octahedral case. In Theorem 3.1, we restate the conditions of [2] with the following correction: For $G(L_{\ell,B})$ to be octahedral, it is necessary that one of the four numbers $\ell \pm 1/4$, $\ell \pm 1/6$ be an integer, but not that one of $\ell \pm 1/4$ be an integer. We discovered the need for this correction while examining the implications for Lamé monodromy of [11], which in effect classifies all *strong* pullbacks of the hypergeometric to the Heun equation. Pulling back 'algebraic' $L_{\lambda,\mu,\nu}$ via the quadratic and cubic cyclic maps treated in [11] yields useful examples of Lamé operators with only algebraic functions in their kernels, including a counterexample to the necessary condition of [2]. The counterexample appears in Proposition 3.4, and explicit formulae for the solutions of a number of interesting Lamé equations with projectively finite monodromy are given in Section 4.

The corrected necessary condition for $G(L_{\ell,B})$ to be octahedral overlaps with the necessary condition that it be icosahedral, which is that one of the six numbers $\ell \pm 1/10$, $\ell \pm 1/6$, $\ell \pm 3/10$ be an integer. For example, $\ell = 1/6$ is both an octahedral and an icosahedral alternative. It follows from Propositions 3.4 and 3.5 that there are Lamé operators with $\ell = 1/6$ of both the octahedral and icosahedral types. This implies that in the nonclassical $2\ell \notin \mathbf{Z}$ case, finite projective monodromy is not determined uniquely by ℓ.

Churchill [4] studied the monodromy of the Weierstrass-form Lamé equation $L_{\ell,B,g_2,g_3}u = 0$ on the elliptic curve E_{g_2,g_3}, and employed the results of [2] to derive similar results on the projective monodromy group $G(L_{\ell,B,g_2,g_3})$. In particular, he deduced that it cannot be octahedral. Unfortunately, this deduction is invalidated by the error in [2] and the consequent nonuniqueness. In Section 5, we provide details, including Theorem 5.1, a corrected theorem on $G(L_{\ell,B,g_2,g_3})$ and its relation to $G(L_{\ell,B})$. We also give an example of an equation $L_{\ell,B,g_2,g_3}u = 0$ with octahedral projective monodromy.

2. Preliminaries

The following definitions and results are fairly standard [1,3], but are included to make this paper self-contained. Suppose C is a nonsingular algebraic curve over $\mathbf{C}$ with function field $K/\mathbf{C}$, and that D is a nontrivial derivation of $K/\mathbf{C}$. (For example, $C = \mathbf{P}^1(\mathbf{C})$, with $K = \mathbf{C}(x)$, the field of rational functions, and D the usual derivation d/dx.) Consider the monic second-order operator

$$L = D^2 + \mathscr{A} \cdot D + \mathscr{B}, \tag{2.1}$$

where $\mathscr{A}, \mathscr{B} \in K$. Let $\{P_1, \ldots, P_r\}$ be its set of singular points, which comprises the poles of $\mathscr{A}$ and $\mathscr{B}$, and possibly the point at infinity; and let P be an ordinary point. A $GL(2, \mathbf{C})$ monodromy representation of the fundamental group of the punctured curve, $\pi_1(C\backslash\{P_1,\dots,P_r\};P)$, is obtained by analytically continuing any two linearly independent function elements u_1,u_2 around closed paths that issue from P. Its image in $GL(2,\mathbf{C})$ is the monodromy group of L (its isomorphism class is independent of the choice of u_1,u_2 and P). The image of the monodromy group in $PGL(2,\mathbf{C})$, obtained by quotienting out its intersection with $\mathbf{C}\backslash\{0\}$, is the projective monodromy group $G(L)$, the group of monodromies of the ratio u_2/u_1.

Iff $G(L)$ is finite, any ratio of independent solutions of $Lu=0$ will be algebraic over K, with Galois group $G(L)$. Let τ be such a ratio. By calculation, if $\mathscr{A}=0$, then $u_1 \stackrel{\mathrm{def}}{=} 1/\sqrt{D\tau}$ and $u_2 \stackrel{\mathrm{def}}{=} \tau/\sqrt{D\tau}$ will satisfy $Lu_i=0$. Moreover, these u_1,u_2 are independent. So if $\mathscr{A}=0$, all solutions of $Lu=0$ are algebraic over K iff $G(L)$ is finite. $\mathscr{A}=0$ can be weakened to the condition that the Wronskian $w=w(L)$, defined locally on C by $Dw+\mathscr{A}\cdot w=0$, be algebraic over K. This is because

$$\begin{aligned}\hat{L} &= 1/\sqrt{w}\circ L\circ\sqrt{w}\\ &= D^2 - D\mathscr{A}/2 - \mathscr{A}^2/4+\mathscr{B}\end{aligned} \tag{2.2}$$

equals $D^2+\hat{\mathscr{A}}\cdot D+\hat{\mathscr{B}}$ with $\hat{\mathscr{A}}=0$, i.e., is of 'normal form'. The groups $G(L)$ and $G(\hat{L})$ are isomorphic, and $Lu=0$ iff $\hat{L}(u/\sqrt{w})=0$. That is, the solution space of $Lu=0$ is spanned by $\sqrt{w}/\sqrt{D\hat{\tau}}$ and $\sqrt{w}\hat{\tau}/\sqrt{D\hat{\tau}}$, where $\hat{\tau}$ is any ratio of solutions of $\hat{L}\hat{u}=0$; $\hat{\tau}$ is algebraic iff $G(L)$ is finite. So if $w(L)$ is algebraic, $Lu=0$ has a full set of algebraic solutions iff $G(L)$ is finite.

Let $\xi: C\to C'$ be a rational map of algebraic curves, where C' is another nonsingular algebraic curve over $\mathbf{C}$, with its own function field $K'/\mathbf{C}$ and nontrivial derivation D'. If L is as in (2.1), and $L'=(D')^2+\mathscr{A}'\cdot D'+\mathscr{B}'$, with $\mathscr{A}',\mathscr{B}'\in K'$, is a similar monic second-order operator on C', then L is said to be a strong pullback of L' if there are independent solutions u_1,u_2 and u_1',u_2' of L,L', respectively, such that $u_i=u_i'\circ\xi$. For example, if x is the coordinate on C and $C'=\mathbf{P}^1(\mathbf{C})$ is coordinatized by z, so that $z=\xi(x)$ is a rational function on C, and $L'=D_z^2+\mathscr{B}'$, then the strong pullback of L' is

$$(d\xi/dx)^2\left[\frac{d^2}{d\xi^2}+\mathscr{B}'(\xi)\right]=\frac{d^2}{dx^2}-\frac{d^2\xi/dx^2}{d\xi/dx}\frac{d}{dx}+(d\xi/dx)^2\mathscr{B}'(\xi(x)), \tag{2.3}$$

where the prefactor $(d\xi/dx)^2$ ensures monicity.

If L,M are monic second-order operators on C (resp. C'), L is said to be projectively equivalent to M (written $L\sim M$) iff any ratio of independent solutions of $Mu=0$ is a ratio of solutions of $Lu=0$, i.e., iff $M=h^{-1}\circ L\circ h$ for some $h\in K$ (resp. K'). Note that if M is normal-form, then $h\propto\sqrt{w(L)}$ as in (2.2), and M is uniquely determined by L.

If L,L' are monic second-order operators on C,C', L is said to be a *weak* pullback of L' (hereafter, a pullback) if there are $\tau\in K$, $\tau'\in K'$, ratios of independent solutions of $Lu=0$, $L'u'=0$, with $\tau=\tau'\circ\xi$. That is, $L\sim M$ and $L'\sim M'$, with M a strong pullback of M'. Pullbacks are not unique, though there is a unique normal-form pullback.

Lemma 2.1. *$L = D^2 + \mathscr{A} \cdot D + \mathscr{B}$ on C is a pullback by $\xi : C \to \mathbf{P}^1(\mathbf{C})$ of the normal-form operator $L' = D_z^2 + \mathscr{B}'$ on $\mathbf{P}^1(\mathbf{C})$ iff*

$$
\begin{aligned}
&- D\mathscr{A}/2 - \mathscr{A}^2/4 + \mathscr{B} \\
&\quad = \frac{1}{2}\frac{d}{dx}\left(\frac{d^2\xi/dx^2}{d\xi/dx}\right) - \frac{1}{4}\left(\frac{d^2\xi/dx^2}{d\xi/dx}\right)^2 + (d\xi/dx)^2 B'(\xi(x)).
\end{aligned}
\tag{2.4}
$$

If this is the case, the solution space of $Lu = 0$ is spanned by

$$
\frac{\sqrt{w(L)}}{\sqrt{D(\tau' \circ \xi)}}, \quad \frac{\sqrt{w(L)}(\tau' \circ \xi)}{\sqrt{D(\tau' \circ \xi)}},
\tag{2.5}
$$

where τ' is any ratio of independent solutions of $L'u' = 0$.

Proof. The strong pullback of L' is given by (2.3), and according to formula (2.2), the unique normal-form weak pullback of L' will be $D^2 + \hat{\mathscr{B}}$, where $\hat{\mathscr{B}}$ is defined as the right-hand side of (2.4). But as computed in (2.2), an operator of the form $D^2 + \hat{\mathscr{B}}$ is projectively equivalent to L iff $\hat{\mathscr{B}} = -D\mathscr{A}/2 - \mathscr{A}^2/4 + \mathscr{B}$. The final statement follows from the above remarks about the solution space of $Lu = 0$ in relation to that of $(D^2 + \hat{\mathscr{B}})\hat{u} = 0$. $\square$

We now specialize to operators $F = D^2 + \mathscr{A} \cdot D + \mathscr{B}$ on C that are Fuchsian, i.e., have two characteristic exponents $\alpha_{i,1}, \alpha_{i,2} \in \mathbf{C}$ (which may be the same) at each singular point P_i. If $\alpha_{i,1} - \alpha_{i,2} \notin \mathbf{Z}$, this means $Fy = 0$ has solutions $y_{i,j},\ j = 1, 2$, at P_i that are of the form $t^{\alpha_{i,j}}$ times an invertible function of t, where t is a local uniformizing parameter (if $\alpha_{i,1} - \alpha_{i,2} \in \mathbf{Z}$, one solution may be logarithmic). The exponent differences $\rho(F, P_i) = \alpha_{i,1} - \alpha_{i,2}$ are defined up to sign; when $\rho \in \mathbf{R}$, the convention $\rho \geqslant 0$ will be adhered to. At each ordinary point, the exponents are $0, 1$, so the exponent difference is unity.

Theorem 2.2 is Klein's pullback theorem, taken from [1, Theorem 1.8]. The auxiliary Table 1 is the basic Schwarz list of exponent differences λ, μ, ν. The hypergeometric operator $L_{\lambda,\mu,\nu}$ corresponding to each row has a full set of algebraic solutions, and there is a ratio τ' of solutions which, as an algebraic function from $z \in \mathbf{P}^1(\mathbf{C})$ to $\tau' \in \mathbf{P}^1(\mathbf{C})$, is the inverse of a single-valued, i.e., rational, 'polyhedral function' $z = z(\tau')$. These are tabulated in the final column, which is adapted from [14, Section 31; 15, Section 14.3]. Each is automorphic under the corresponding finite subgroup of the Möbius group.

Theorem 2.2. *Let $F = D^2 + \mathscr{A} \cdot D + \mathscr{B}$ be a Fuchsian operator on C, with $\mathscr{A}, \mathscr{B} \in K$, and suppose that $G(F)$ is finite. There is a unique λ, μ, ν on the basic Schwarz list such*

Table 1
The basic Schwarz list

Case	λ, μ, ν	Group	Solution ratio inverse, $z = z(w)$
—	$1/n, 1, 1/n$	C_n	w^n
I	$1/2, 1/2, 1/n$	D_n	$\dfrac{(w^n+1)^2}{4w^n}$
II	$1/2, 1/3, 1/3$	A_4	$\dfrac{12\sqrt{-3}\,w^2(w^4-1)^2}{(w^4+2\sqrt{-3}w^2+1)^3}$
IV	$1/2, 1/3, 1/4$	S_4	$\dfrac{-(w^{12}-33w^8-33w^4+1)^2}{108\,w^4(w^4-1)^4}$
VI	$1/2, 1/3, 1/5$	A_5	$\dfrac{[w^{30}+522(w^{25}-w^5)-10005(w^{20}+w^{10})+1]^2}{1728w^5(w^{10}+11w^5-1)^5}$

that $G(F)$ is isomorphic to $G(L_{\lambda,\mu,\nu})$ and F is a pullback of $L_{\lambda,\mu,\nu}$ by some rational map $\xi : C \to \mathbf{P}^1(\mathbf{C})$, where ξ is unramified over $\mathbf{P}^1(\mathbf{C})\backslash\{0, 1, \infty\}$. Moreover, if for any λ, μ, ν on the list, F is a pullback of $L_{\lambda,\mu,\nu}$, then $G(F)$ is isomorphic to a subgroup of $G(L_{\lambda,\mu,\nu})$.

If $G(F)$ is finite and the Wronskian $w(F)$ is algebraic, $Fu = 0$ will have a full set of algebraic solutions; and if λ, μ, ν and the pullback map $\xi : C \to \mathbf{P}^1(\mathbf{C})$, which are guaranteed to exist by Theorem 2.2, are known, a basis for the solution space of $Fu = 0$ may be computed from (2.5), in which $\tau' = \tau'(z)$ is the inverse of the polyhedral function in the final column of the table.

It is worth noting that as algebraic functions, the possible τ' are quite special. Each is ramified over $z = 0, 1, \infty$, at most, and the ramification order of each of the points in $(\tau')^{-1}(0)$, $(\tau')^{-1}(1)$, $(\tau')^{-1}(\infty)$ is $1/\lambda - 1$, $1/\mu - 1$, $1/\nu - 1$, respectively. That is, if m denotes the mapping degree of $z = z(\tau')$, i.e., $m = |G(L_{\lambda,\mu,\nu})|$, the projective monodromy of $L_{\lambda,\mu,\nu}v = 0$ around each of the singular points $z = 0, 1, \infty$ is always a restricted sort of permutation of the m branches of τ', the cycle decomposition of which comprises, respectively, λm cycles of length $1/\lambda$, μm cycles of length $1/\mu$, and νm cycles of length $1/\nu$.

Lemma 2.3. *Suppose the Fuchsian operator $F = D^2 + \mathscr{A} \cdot D + \mathscr{B}$ is a pullback of $L_{\lambda,\mu,\nu}$ via $\xi : C \to \mathbf{P}^1(\mathbf{C})$. The exponent difference $\rho(F, P)$ at any $P \in C$ equals h times the exponent difference $\rho(L_{\lambda,\mu,\nu}, \xi(P))$, if h is the multiplicity with which P is mapped to $\xi(P)$, i.e.,* 1 *plus the ramification order of ξ at P.*

Proof. Consider the series expansions of solution ratios τ, τ' of $Fy = 0$, $L_{\lambda,\mu,\nu}y = 0$ at $P, \xi(P)$, respectively. Each is of the form t^ρ times an invertible function of t, where t is a uniformizing parameter; and locally, $\xi(t) \sim t^h$. $\square$

Lemma 2.3 constrains the Fuchsian operators F to which Theorem 2.2 can be applied, i.e., F for which $G(F)$ is finite. For example, there must be a row of Table 1 such that each of the singular point exponent differences $\{\rho(F, P_i)\}_{i=1}^r$ is an integer multiple of one of the corresponding λ, μ, ν. It also constrains the monodromy at each P_i. Suppose without loss of generality that $\xi(P_i) = 0$. The projective monodromy of $L_{\lambda,\mu,\nu}v = 0$ around $z = 0$ permutes the m branches of τ', and the cycle decomposition of the permutation comprises λm cycles of length $1/\lambda$. So the projective monodromy of $Fu = 0$ around P_i must be isomorphic to an integer power of such a permutation. Together with the fact that $G(F)$, the group of permutations of the branches of τ which is generated by these monodromies, must be identical to the Galois group of τ over K (rather than being a proper subset of it), this imposes substantial constraints.

The following lemma will be used in the next section.

Lemma 2.4. *If $L_{\ell,B}$ is an algebraic-form Lamé operator with finite projective monodromy group, so that it is a pullback of some $L_{\lambda,\mu,\nu}$ on the basic Schwarz list by a rational map $\xi : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$ of the sort guaranteed to exist by Theorem 2.2, then provided $\ell + 1/2 \notin \mathbf{Z}$, ξ must map the set of singular points $\{e_1, e_2, e_3, \infty\}$ into $\{0, 1, \infty\}$.*

Proof. The only ramification points of ξ are above $z = 0, 1, \infty$. So if $\xi(P) \notin \{0, 1, \infty\}$, $\rho(L_{\ell,B}, P) = \rho(L_{\lambda,\mu,\nu}, \xi(P)) = 1$ by Lemma 2.3. Since $\rho(L_{\ell,B}, e_i) = 1/2$ and $\rho(L_{\ell,B}, \infty) = \pm(\ell + 1/2)$, the claim follows. $\square$

3. Key results

Theorem 3.1. *The equation $L_{\ell,B}u = 0$ on $\mathbf{P}^1(\mathbf{C})$ has a full set of algebraic solutions, i.e., solutions algebraic over $\mathbf{C}(x)$, iff $G(L_{\ell,B})$ is finite. In the nonclassical case $2\ell \notin \mathbf{Z}$, $G(L_{\ell,B})$ is finite iff it is octahedral (i.e., isomorphic to S_4), in which case ℓ must equal $n \pm 1/6$ or $n \pm 1/4$, with n an integer; or icosahedral (i.e., isomorphic to A_5), in which case ℓ must equal $n \pm 1/10$, $n \pm 1/6$, or $n \pm 3/10$, with n an integer.*

Proof. The Wronskian $w(L_{\ell,B})$ equals $\prod_{i=1}^3 (x - e_i)^{-1/2}$, which is algebraic; so $L_{\ell,B}u = 0$ having a full set of algebraic solutions is equivalent to finiteness of $G(L_{\ell,B})$. The necessary conditions on ℓ come from conditions imposed by Lemma 2.3 on pullbacks of $L_{\ell,B}$ from $L_{\lambda,\mu,\nu}$ on the basic Schwarz list, since such a pullback is guaranteed to exist by Theorem 2.2. As the final sentence of that theorem acknowledges, a Fuchsian operator F can be a pullback of $L_{\lambda,\mu,\nu}$ with $G(F)$ isomorphic to a proper subgroup of $G(L_{\lambda,\mu,\nu})$, rather to $G(L_{\lambda,\mu,\nu})$. To compensate, one must consider the various $L_{\lambda,\mu,\nu}$ 'in order'. The rows of Table 1 are ordered so that if G_2 appears in a later row than G_1, then G_2 is not isomorphic to a subgroup of G_1.

The analysis begins with the tetrahedral row, since it is a classical result that if $2\ell \notin \mathbf{Z}$, $G(L_{\ell,B})$ cannot be cyclic or dihedral. If $G(L_{\ell,B})$ is tetrahedral, $L_{\ell,B}$ must be a pullback of $L_{1/2,1/3,1/3}$. The operator $L_{\ell,B}$ has exponent differences $1/2,\ 1/2,\ 1/2,\ \pm(\ell+1/2)$ at $x=e_1,e_2,e_3,\infty$, respectively. It follows from Lemmas 2.4 and 2.3 that ξ must map e_1,e_2,e_3 to 0. Also, it must map ∞ to 1 or ∞. Hence $\xi^{-1}(1)$, resp. $\xi^{-1}(\infty)$, must comprise only ordinary points with exponent differences equal to unity. By Lemma 2.3, each point in $\xi^{-1}(1)$, resp. $\xi^{-1}(\infty)$, must be mapped triply to ∞. So $3|\deg\xi$. This can be combined with the prediction of the 'degree formula' of Baldassarri and Dwork [3, Lemma 1.5], which is derived from the Hurwitz genus formula. If F, a second-order Fuchsian operator on an algebraic curve C with genus g, has exponent differences $\{\rho_i\}$ and is a pullback by a rational function ξ from F', a Fuchsian operator on $\mathbf{P}^1(\mathbf{C})$ with exponent differences $\{\rho_i'\}$, then

$$\left[2-2g+\sum_i\ (\rho_i-1)\right]=(\deg\xi)\left[2+\sum_i\ (\rho_i'-1)\right]. \tag{3.1}$$

The degree formula (3.1) yields $\pm(\ell+1/2)-1/2=(\deg\xi)/6$ when applied to $F=L_{\ell,B}$, $F'=L_{1/2,1/3,1/3}$. In conjunction with $3|\deg\xi$, this contradicts $2\ell\notin\mathbf{Z}$. [This ruling out of the tetrahedral alternative is taken from [2, Proposition 3.1].]

If $G(L_{\ell,B})$ is octahedral, $L_{\ell,B}$ must be a pullback of $L_{1/2,1/3,1/4}$. The point $x=\infty$ cannot be mapped to 0, since by Lemma 2.3 that would imply that $\ell+1/2$ is an integer multiple of $1/2$, which is a contradiction. However, it can be mapped to 1, in which case $\ell+1/2$ must be an integer multiple of $1/3$, or to ∞, in which case $\ell+1/2$ must be an integer multiple of $1/4$. That is, ℓ must equal $n\pm1/6$ or $n\pm1/4$, with n an integer. [The possibility that $\xi(\infty)=1$ was erroneously ruled out in [2, Section 3], by an argument based on the incorrect assumption that $\xi(e_i)$ must equal 0 for all i.]

If $G(L_{\ell,B})$ is icosahedral, $L_{\ell,B}$ must be a pullback of $L_{1/2,1/3,1/5}$. As in the octahedral case, $x=\infty$ cannot be mapped to 0. It can be mapped to 1, in which case $\ell+1/2$ must be an integer multiple of $1/3$, or to ∞, in which case $\ell+1/2$ must be an integer multiple of $1/5$. That is, ℓ must equal $n\pm1/6$, with n an integer, or $n\pm1/10$ or $n\pm3/10$, with n an integer. □

According to Propositions 3.4 and 3.5 below, the five alternatives listed in Theorem 3.1 can each be realized.

Definition 3.2. The harmonic case is the case when $J=1$, i.e., when $g_3=0$, so that the unordered set $\{e_1,e_2,e_3\}$ comprises three equally spaced collinear points in $\mathbf{C}$, i.e., is of the form $\alpha\{-1,0,1\}$. The equianharmonic case is the case when $J=0$, i.e., when $g_2=0$, so that $\{e_1,e_2,e_3\}$ is the vertex set of an equilateral triangle in $\mathbf{C}$, i.e., is of the form $\alpha\{1,\omega,\omega^2\}$ with $\omega^3=1$. In both cases, $\alpha\neq0$ is arbitrary.

Lemma 3.3. *In the harmonic case, $L_{\ell,0}$ is a pullback of $L_{1/2,(2\ell+1)/4,1/4}$, and in the equianharmonic case, $L_{\ell,0}$ is a pullback of $L_{1/2,1/3,(2\ell+1)/6}$. Here $\ell\in\mathbf{C}$ is arbitrary. These pullbacks are via maps ξ which up to composition with Möbius transformations are of the cyclic form $\xi(x)=x^k$, where $k=2,3$, respectively.*

Proof. The map $\xi_2(x) \stackrel{\text{def}}{=} x^2$ takes $x = 0, \infty$ to $0, \infty$, each with multiplicity 2, and $x = \pm 1$ to 1 with multiplicity 1. By the theory of Fuchsian differential operators, any pullback of $L_{\lambda,\mu,\nu}$ via ξ_2 will be a Fuchsian operator with $-1, 0, 1, \infty$ as its only possible singular points. By Lemma 2.3, the respective exponent differences will be $\mu, 2\lambda, \mu, 2\nu$. If $\lambda, \mu, \nu = 1/4, 1/2, (2\ell + 1)/4$, the singular point locations and exponent differences will be identical to those of $L_{\ell,B}$ (harmonic case). Similarly, any pullback of $L_{\lambda,\mu,\nu}$ via $\xi_3(x) \stackrel{\text{def}}{=} x^3$ will have singular points $0, 1, \omega, \omega^2, \infty$, with exponent differences $3\lambda, \mu, \mu, \mu, 3\nu$. If $\lambda, \mu, \nu = 1/3, 1/2, (2\ell + 1)/6$, the point $x = 0$ will become an ordinary point, and the singular point locations and exponent differences will be identical to those of $L_{\ell,B}$ (equianharmonic case).

The value for the accessory parameter B of the pullback can be shown to be zero in both cases. This follows from Lemma 2.1, since in both cases a computation (omitted here) yields equal values for the left-hand and right-hand sides of (2.4), irrespective of ℓ, iff B is set equal to zero. It also follows from a theorem of [11], which determines the values of the accessory parameter and exponent parameters for which Heun operators are strong pullbacks of $L_{\lambda,\mu,\nu}$.

The permutation of $1/4, 1/2, (2\ell + 1)/4$ into $1/2, (2\ell + 1)/4, 1/4$, as required by the statement of the lemma, is accomplished by choosing $\xi = M \circ \xi_2$, where $M(z) = (z - 1)/z$ is the Möbius transformation that maps $0, 1, \infty$ to $\infty, 0, 1$. So in the harmonic case, $\xi(x) = (x^2 - 1)/x^2$. Similarly, the permutation of $1/3, 1/2, (2\ell + 1)/6$ into $1/2, 1/3, (2\ell + 1)/6$ is accomplished by composing ξ_3 with the map $z \mapsto 1 - z$. So in the equianharmonic case, $\xi(x) = 1 - x^3$. $\square$

It should be noted that cyclic pullbacks of hypergeometric operators have been studied or applied by several other authors. In the harmonic case, Ivanov [8] discovered that the Jacobi form of the Lamé equation can be reduced to the hypergeometric equation, via a quadratic transformation analogous to $\xi(x) = x^2$. In the equianharmonic case, Clarkson and Olver [5] discovered that the Weierstrass form of the Lamé equation can be similarly reduced, via a cubic transformation analogous to $\xi(x) = x^3$. Our efforts to understand their results led to [11], and ultimately to this paper. Recently, the Clarkson–Olver transformation has been applied by Kantowski and Thomas [9, Eq. (12)].

Proposition 3.4. *Let n denote an integer.*

(1) *In the harmonic case* $(J = 1)$, $G(L_{n\pm 1/6,0})$ *is octahedral if* $n \equiv 0 \pmod 2$, *resp.* $n \equiv 1 \pmod 2$.
(2) *In the equianharmonic case* $(J = 0)$:
   (a) $G(L_{n\pm 1/4,0})$ *is octahedral if* $n \equiv 0 \pmod 3$, *resp.* $n \equiv 2 \pmod 3$.
   (b) $G(L_{n\pm 1/10,0})$ *is icosahedral if* $n \equiv 0 \pmod 3$, *resp.* $n \equiv 2 \pmod 3$.
   (c) $G(L_{n\pm 3/10,0})$ *is icosahedral if* $n \equiv 1 \pmod 3$.

Proof. This follows from Lemma 3.3, together with Schwarz's classical characterization of the λ, μ, ν for which $G(L_{\lambda,\mu,\nu})$ is finite. If the unordered triple λ, μ, ν appears on the full Schwarz list, then $G(L_{\lambda,\mu,\nu})$ will be finite, and the same will be true if a normalized version of λ, μ, ν appears there. Normalization is performed by replacing λ, μ, ν by $a \pm \lambda, b \pm \mu, c \pm \nu$, where a, b, c are any integers whose sum is even (see [7, Section 2.7.2; 14, Section 28]). Preservation of algebraicity can be verified from the Gauss contiguity relations, which solutions of hypergeometric equations must satisfy.

The full list includes Cases I, II, IV, VI of Table 1, and also, among others, the icosahedral Case XIV, for which $\lambda, \mu, \nu = 1/2, 2/5, 1/3$ (see [7, Section 2.7.2; 14, Section 30]). By choosing appropriate integers a, b, c (and interchanging the μ, ν of Case XIV), it is readily verified that $G(L_{\lambda,\mu,\nu})$ is isomorphic to

(1) S_4 if $\lambda, \mu, \nu = 1/2, k, 1/4$ with $k \in \mathbf{Z} \pm 1/3$ [Case IV].
(2)
   (a) S_4 if $\lambda, \mu, \nu = 1/2, 1/3, k$ with $k \in \mathbf{Z} \pm 1/4$ [Case IV].
   (b) A_5 if $\lambda, \mu, \nu = 1/2, 1/3, k$ with $k \in \mathbf{Z} \pm 1/5$ [Case VI].
   (c) A_5 if $\lambda, \mu, \nu = 1/2, 1/3, k$ with $k \in \mathbf{Z} \pm 2/5$ [Case XIV].

By Lemma 3.3, each of these $L_{\lambda,\mu,\nu}$ can be pulled back to a Lamé operator of the form $L_{\ell,0}$, with ℓ determined by $(2\ell + 1)/4 = k$ (Case 1), or by $(2\ell + 1)/6 = k$ (Cases 2(a)–2(c)). The operators $L_{\ell,0}$ of the proposition are a proper subset: the ones for which $2\ell \notin \mathbf{Z}$. The reason for imposing this additional restriction is that if $2\ell \notin \mathbf{Z}$, $G(L_{\ell,0})$ is guaranteed to be isomorphic to $G(L_{\lambda,\mu,\nu})$, rather than to a proper subgroup. That is because, by Theorem 3.1, the only possible groups are S_4 and A_5, and neither is a subgroup of the other. $\square$

Case 1 of Proposition 3.4 provides a counterexample to the necessary condition of [2]. It should be mentioned that Case 2(b) is actually a generalization of another result of [2], which is that in the equianharmonic case, $L_{1/10,0}$ can be pulled back from $L_{1/2,1/3,1/4}$ via a degree-3 cyclic map. In fact, Baldassarri was the first to see the relevance of degree-3 cyclic maps in this context.

The following proposition shows that the remaining alternative of Theorem 3.1, which Proposition 3.4 did not cover, can also be realized. Unlike Proposition 3.4, it is specific to a single value of ℓ, and also to a nonzero value of the accessory parameter B.

Proposition 3.5. *Suppose that* $J = -80$, *i.e.*, $g_2 = 80\alpha^2/3$ *and* $g_3 = -80\alpha^3/3$ *for some* $\alpha \neq 0$; *equivalently, that* e_1, e_2, e_3 *are the roots of* $3x^3 - 20x + 20$, *multiplied by some* $\alpha \neq 0$. *Then* $G(L_{1/6,-\alpha/9})$ *is icosahedral.*

Proof. This $\ell = 1/6$ example was constructed by a method suggested by the proof of Lemma 3.3. The method proceeds as follows. The first step is to find a rational $\xi(x)$, unramified over $\mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$, such that the pullback of $L_{1/2,1/3,1/5}$ has the same exponent differences as any $L_{1/6,B}$. The three singular points of the pullback that have exponent difference $1/2$ are taken to be e_1, e_2, e_3. The second step is to use the formula (2.4) of Lemma 2.1 to compute the unique B for which $L_{1/6,B}$ with this choice of e_1, e_2, e_3 is, in fact, the pullback.

It was noted in the proof of Theorem 3.1 that in the $\ell \in \mathbf{Z} \pm 1/6$ icosahedral case, ξ must map the singular point $x = \infty$ to $z = 1$. Since $x = \infty$ has exponent difference $\pm(\ell + 1/2) = \pm 2/3$, the mapping must have multiplicity 2. In the same way, it follows that ξ must map each e_i to $z = 0$ with multiplicity 1. The function ξ is characterized by the points in $\xi^{-1}(\{0, 1, \infty\})$ and the multiplicities with which they are mapped. Suppose that $\xi^{-1}(0)$ includes n_0 ordinary points, besides e_1, e_2, e_3; that $\xi^{-1}(1)$ includes n_1 ordinary points, besides ∞; and that $\xi^{-1}(\infty)$ includes n_∞ ordinary points. By Lemma 2.3, ξ must map each of the n_0, n_1, n_∞ ordinary points with multiplicity $2, 3, 5$, respectively. The integers n_0, n_1, n_∞ must satisfy

$$3 + 2n_0 = 2 + 3n_1 = 5n_\infty = \deg \xi, \tag{3.2}$$

$$(3 + n_0) + (1 + n_1) + n_\infty = 2 + \deg \xi. \tag{3.3}$$

Here (3.2) is the degree condition. Eq. (3.3) is a consequence of the Hurwitz formula, according to which any rational map $\xi : C \to \mathbf{P}^1(\mathbf{C})$ from a nonsingular algebraic curve C of genus g to $\mathbf{P}^1(\mathbf{C})$ that is unramified above $\mathbf{P}^1(\mathbf{C}) \backslash \{P_1, \dots, P_r\}$ satisfies $|\xi^{-1}(\{P_1, \dots, P_r\})| = 2 - 2g + (r-2)\deg \xi$.

The only solution of (3.2) and (3.3) is $n_0 = n_1 = n_\infty = 1$, with $\deg \xi = 5$. So any function ξ by which an operator of the form $L_{1/6,B}$ can be pulled back from $L_{1/2,1/3,1/5}$ must be of the form

$$\xi(x) = \frac{(x - C_1)(x - C_2)(x - C_3)(x - C_4)^2}{(x - C_5)^5} = 1 - \frac{C_6(x - C_7)^3}{(x - C_5)^5} \tag{3.4}$$

for certain $C_1, \dots, C_7 \in \mathbf{C}$, where C_1, C_2, C_3 are to be identified with e_1, e_2, e_3. Solutions of (3.4) may be constructed by elimination theory. Imposing the condition $e_1 + e_2 + e_3 = 0$ yields an essentially unique solution, namely

$$\xi(x) = \frac{(3x^3 - 20x + 20)(2x - 5)^2}{12(x - 1)^5} = 1 - \frac{(5x - 8)^3}{12(x - 1)^5}, \tag{3.5}$$

which requires $g_2 = 80/3$ and $g_3 = -80/3$. On the right-hand side of (3.5), x may be replaced by x/α for any $\alpha \in \mathbf{C} \backslash \{0\}$. It follows by substituting (3.5) into (2.4), and some algebraic manipulation, that $L_{1/6,B}$ will be a pullback iff $B = -\alpha/9$. The α-dependence is due to B not being scale-invariant. $\square$

Corollary 3.6. *In the nonclassical case $2\ell \notin \mathbf{Z}$, finite projective monodromy of the Lamé equation is not uniquely determined by ℓ.*

Proof. By Propositions 3.4 and 3.5, $G(L_{1/6,B})$ is octahedral when $J=1$ and icosahedral when $J=-80$, if in each case, B is appropriately chosen. $\square$

4. Explicit formulas

In practical applications of the Lamé equation, such as the astrophysical application of [9], it is useful to have explicit formulas for the algebraic solutions, if any. The five cases of the following proposition, which correspond to the four cases of Proposition 3.4 and to Proposition 3.5, should serve as examples.

Proposition 4.1. *Let $\tau = \tau(x)$, an algebraic complex-valued function of a complex argument, be defined as follows:*

(1) *In the harmonic case $\{e_1, e_2, e_3\} = \{-1, 0, 1\}$, if $\ell = 1/6$ and $B = 0$, let τ be defined by*

$$\frac{-(\tau^{12} - 33\tau^8 - 33\tau^4 + 1)^2}{108\,\tau^4(\tau^4-1)^4} = \frac{x^2-1}{x^2}.$$

(2) *In the equianharmonic case $\{e_1, e_2, e_3\} = \{1, \omega, \omega^2\}$:*

(a) *if $\ell = 1/4$ and $B = 0$, let τ be defined by*

$$\frac{-(\tau^{12} - 33\tau^8 - 33\tau^4 + 1)^2}{108\tau^4(\tau^4-1)^4} = 1 - x^3;$$

(b) *if $\ell = 1/10$ and $B = 0$, let τ be defined by*

$$\frac{[\tau^{30} + 522(\tau^{25} - \tau^5) - 10005(\tau^{20} + \tau^{10}) + 1]^2}{1728\,\tau^5(\tau^{10} + 11\tau^5 - 1)^5} = 1 - x^3;$$

(c) *if $\ell = 7/10$ and $B = 0$, let τ be defined by*

$$\begin{aligned}&\frac{[\tau^{30} + 522(\tau^{25} - \tau^5) - 10005(\tau^{20} + \tau^{10}) + 1]^2}{1728\,\tau^5(\tau^{10} + 11\tau^5 - 1)^5}\\ &\quad = \frac{s(157464s^3 - 352107s^2 + 708750s - 546875)^2}{(189s-125)^5},\end{aligned}$$

where s signifies $1 - x^3$.

(3) *In the case when* e_1, e_2, e_3 *are the roots of* $3x^3 - 20x + 20$, *if* $\ell = 1/6$ *and* $B = -1/9$, *let* τ *be defined by*

$$\frac{[\tau^{30} + 522(\tau^{25} - \tau^5) - 10005(\tau^{20} + \tau^{10}) + 1]^2}{1728\,\tau^5(\tau^{10} + 11\tau^5 - 1)^5} = \frac{(3x^3 - 20x + 20)(2x-5)^2}{12(x-1)^5}.$$

In each of these five cases, the Lamé equation $L_{\ell,B}u = 0$ *has a full set of algebraic solutions. Its solution space is spanned by*

$$\left[\prod_{i=1}^{3}(x - e_i)\right]^{-1/4}\left\{\frac{1}{\sqrt{d\tau/dx}}, \frac{\tau}{\sqrt{d\tau/dx}}\right\}, \tag{4.1}$$

where τ *is case-specific. In cases* 1 *and* 2(a), *the projective monodromy group* $G(L_{\ell,B})$, *i.e., the Galois group of* τ *over* $\mathbf{C}(x)$, *is octahedral, and in cases* 2(b), 2(c), *and* 3, *it is icosahedral.*

Proof. The solution space (4.1) is of the form specified by Lemma 2.1 in (2.5). In each case, τ is defined so that $\tau = \tau' \circ \xi$, where ξ is the rational function by which $L_{\ell,B}$ is pulled back from some $L_{\lambda,\mu,\nu}$, and τ' is a ratio of solutions of $L_{\lambda,\mu,\nu}v = 0$. In all cases except 2(c), the right-hand side of the defining equation is $z = \xi(x)$, as supplied in the proof of Lemma 3.3 or the proof of Proposition 3.5, and the left-hand side is the appropriate polyhedral function, as supplied in the final column of Table 1, applied to τ.

Case 2(c) is special. As was sketched in the proof of Proposition 3.4, $L_{7/10,0}$ is the pullback via $\xi(x) = 1 - x^3$ of $L_{1/2,1/3,2/5}$, which is Schwarz's Case XIV (modulo the interchange of μ, ν). Case XIV is not on the basic Schwarz list, and in fact, it is not the case that a ratio $\tau' = \tau'(z)$ of independent solutions of $L_{1/2,1/3,2/5}v = 0$ is the inverse of a rational function. However, Case XIV is itself a pullback of the basic icosahedral Case VI. So one can choose $\tau' = \tau'' \circ \xi'$, where ξ' is the rational function by which $L_{1/2,1/3,2/5}$ is pulled back from $L_{1/2,1/3,1/5}$, and τ'' is a ratio of solutions of $L_{1/2,1/3,1/5}v = 0$, the inverse of which is listed in Table 1. The formula in Case 2(c) defines τ so that $\tau = \tau' \circ \xi = \tau'' \circ \xi' \circ \xi$.

A rational map $\bar{\xi}$ equivalent to ξ' was worked out by Klein in 1877, in a paper in which he completed the reduction of the Schwarz list to the basic Schwarz list [10, Section 10]. His formula was

$$\bar{\xi}(s) = 1 - \frac{(64s + 189)(64s^2 + 133s + 49)^3}{7^7 \cdot 27 \cdot (s+1)^2}, \tag{4.2}$$

which maps $s = 0, -189/64, -1$, respectively to $\bar{\xi}(s) = 0, 1, \infty$. For our purposes, this morphism must be composed with a Möbius transformation. Composing with

$M(s) = 189s/(125 - 189s)$, which takes $s = 0, 1, \infty$ to $M(s) = 0, -189/64, -1$, yields

$$(\bar{\xi}\circ M)(s) = \frac{s(157464s^3 - 352107s^2 + 708750s - 546875)^2}{(189s - 125)^5} \tag{4.3}$$

as the rational map $\xi' = \xi'(s)$ by which $L_{1/2,1/3,2/5}$ is pulled back from $L_{1/2,1/3,1/5}$. This map appears on the right-hand side in Case 2(c). $\square$

5. The Weierstrass form

In classical treatments [17], the Weierstrass-form Lamé equation is regarded as an equation on $\mathbf{C}$, of the form

$$\frac{d^2u}{dt^2} - [\ell(\ell+1)\wp(t) + B]u = 0. \tag{5.1}$$

Here $\wp: \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ is the Weierstrass $\wp$-function corresponding to some period lattice $\mathscr{L} = \omega_1\mathbf{Z} + \omega_2\mathbf{Z} \subset \mathbf{C}$, with ω_1, ω_2 independent over $\mathbf{R}$; i.e., $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for some $g_2, g_3 \in \mathbf{C}$ for which $\Delta = g_2^3 - 27g_3^2 \neq 0$. Eq. (5.1) is a Schrödinger equation with an elliptic potential, extended to the complex domain. The algebraic Lamé equation $L_{\ell,B}u = 0$ can be obtained from (5.1) by the substitution $x = \wp(t)$. That is, (5.1) is the strong pullback to $\mathbf{C}$ of $L_{\ell,B}u = 0$ by $\wp$.

Another interpretation is possible. The map $\wp: \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ is the composition of two maps, $\phi: \mathbf{C} \to E_{g_2,g_3}$ and $\pi: E_{g_2,g_3} \to \mathbf{P}^1(\mathbf{C})$. Here E_{g_2,g_3} is the elliptic curve specified by $y^2 = 4x^3 - g_2x - g_3$, and the maps ϕ and π are defined by $\phi(t) = (\wp(t), \wp'(t))$ and $\pi(x,y) = x$. E_{g_2,g_3} is homeomorphic to a torus, and the projection π is a double cover of $\mathbf{P}^1(\mathbf{C})$ by E_{g_2,g_3}. From an algebraic–geometric point of view, it is more reasonable to pull the algebraic-form Lamé equation back to E_{g_2,g_3} via π, than to $\mathbf{C}$ via $\pi\circ\phi$. We call the resulting equation on E_{g_2,g_3} the Weierstrass-form Lamé equation, and write it $L_{\ell,B,g_2,g_3}u = 0$. By examination, the operator L_{ℓ,B,g_2,g_3} has only one singular point, namely the point O, i.e., $(x,y) = (\infty, \infty)$, where its characteristic exponents are $-\ell, \ell+1$. We shall informally regard E_{g_2,g_3} as a subset of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$, coordinatized by (x,y), although in a more careful treatment E_{g_2,g_3} would be defined as the projective curve $y^2z = 4x^3 - g_2xz^2 - g_3z^3$ in $\mathbf{P}^2(\mathbf{C})$, equipped with homogeneous coordinates (x,y,z).

The pullback theory of Section 2 applies when the algebraic curve C equals E_{g_2,g_3}, just as it applied when $C = \mathbf{P}^1(\mathbf{C})$ and $K = \mathbf{C}(x)$. The function field $\tilde{K}$ on E_{g_2,g_3} is $\mathbf{C}(x,y) \supset \mathbf{C}(x)$, a degree-2 extension, and the derivation $D = d/dx$ extends in the obvious way to $\tilde{K}$, via $Dy \stackrel{\text{def}}{=} (12x^2 - g_2)/2y$. With these choices, L_{ℓ,B,g_2,g_3} is of the form $D^2 + \tilde{\mathscr{A}} \cdot D + \tilde{\mathscr{B}}$, for $\tilde{\mathscr{A}}, \tilde{\mathscr{B}} \in K \subset \tilde{K}$. Since the Wronskian is algebraic, the projective monodromy group $G(L_{\ell,B,g_2,g_3})$ is finite iff a ratio of solutions $\tilde{\tau}$ of $L_{\ell,B,g_2,g_3}u = 0$ on E_{g_2,g_3} is algebraic over $\mathbf{C}(x,y)$. This is equivalent to $G(L_{\ell,B})$ being finite, which occurs iff a ratio of solutions τ of $L_{\ell,B}u = 0$ on $\mathbf{P}^1(\mathbf{C})$ is algebraic over $\mathbf{C}(x)$. The equivalence is due to the (local) pullback property $\tilde{\tau} = \tau \circ \pi$, which implies that the two sorts of algebraicity are equivalent. But the two groups may not be isomorphic. This is because $\tilde{\tau}$ may have lower degree over $\mathbf{C}(x,y)$ than τ has over $\mathbf{C}(x)$.

In a thought-provoking paper, Churchill [4] examined the implications of finite group theory for the monodromy of L_{ℓ,B,g_2,g_3}, as well as for the monodromy of the hypergeometric operator $L_{\lambda,\mu,\nu}$. The (projective) monodromy group of $L_{\lambda,\mu,\nu}$ is doubly generated: it is generated by the images of loops around any two of the singular points $z = 0, 1, \infty$. Similarly, since E_{g_2,g_3} is homeomorphic to a torus, the (projective) monodromy group of L_{ℓ,B,g_2,g_3} is generated by the images of only two loops. But the conjugacy classes of the two monodromy (resp. projective monodromy) generators in $GL(2,\mathbf{C})$ (resp. $PGL(2,\mathbf{C})$) are determined by the characteristic exponents of the singular point(s). This constrains what, up to isomorphism, the monodromy group (resp. projective monodromy group) may be, in the case when it is finite.

In this way, Churchill was able to obtain significant results on the projective monodromy of L_{ℓ,B,g_2,g_3} without using pullbacks. Like $G(L_{\ell,B})$, $G(L_{\ell,B,g_2,g_3})$ can never be cyclic, and can be dihedral only if $2\ell \in \mathbf{Z}$. Moreover, in the nonclassical case $2\ell \notin \mathbf{Z}$, it cannot be dihedral. He showed that in the nonclassical case, $G(L_{\ell,B,g_2,g_3})$ can be tetrahedral only if $\ell \in \mathbf{Z} \pm 1/4$, and can be octahedral or icosahedral only if $\ell \in \mathbf{Z} \pm 1/10$, $\ell \in \mathbf{Z} \pm 1/6$, or $\ell \in \mathbf{Z} \pm 3/10$.

Necessarily $G(L_{\ell,B,g_2,g_3}) \trianglelefteq G(L_{\ell,B})$ [4], so conditions on $G(L_{\ell,B,g_2,g_3})$ yield conditions on $G(L_{\ell,B})$, and vice versa. By combining his results with those of [2], Churchill was able to deduce that $G(L_{\ell,B,g_2,g_3})$ cannot be octahedral. Unfortunately, this is incorrect: the proofs of Theorem 5.3 and Corollary 5.4 of [4], which include this assertion, rely crucially on the incorrect result of [2] that $G(L_{\ell,B})$ can be octahedral only if $\ell \in \mathbf{Z} \pm 1/4$.

Theorem 5.1 is a characterization of $G(L_{\ell,B,g_2,g_3})$ as well as $G(L_{\ell,B})$, which is obtained from pullback theory alone. This theorem builds on and subsumes Theorem 3.1.

Theorem 5.1. *The equation* $L_{\ell,B,g_2,g_3}u = 0$ on E_{g_2,g_3} *has a full set of algebraic solutions iff* $G(L_{\ell,B,g_2,g_3})$ *is finite, which is equivalent to* $L_{\ell,B}u = 0$ *on* $\mathbf{P}^1(\mathbf{C})$ *having a full set of algebraic solutions, and to* $G(L_{\ell,B})$ *being finite. In the nonclassical case* $2\ell \notin \mathbf{Z}$, *the following are the only ways this can occur.*

(1) $G(L_{\ell,B,g_2,g_3})$ *is tetrahedral and* $G(L_{\ell,B})$ *is octahedral; in which case* ℓ *must equal* $n \pm 1/4$, *with n an integer.*
(2) $G(L_{\ell,B,g_2,g_3})$ *is octahedral and* $G(L_{\ell,B})$ *is octahedral; in which case* ℓ *must equal* $n \pm 1/6$, *with n an integer.*
(3) $G(L_{\ell,B,g_2,g_3})$ *is icosahedral and* $G(L_{\ell,B})$ *is icosahedral; in which case* ℓ *must equal* $n \pm 1/10$, $n \pm 1/6$, *or* $n \pm 3/10$, *with n an integer.*

All five of the preceding alternatives can be realized.

Proof. The first sentence has already been proved. The proof of the necessary conditions on ℓ for $G(L_{\ell,B,g_2,g_3})$ to be each possible finite group is similar to the proof of Theorem 3.1: it runs down the rows of the basic Schwarz list, beginning with the tetrahedral. Before beginning the proof, note that the pullback function ξ guaranteed to exist by Theorem 2.2 will map the lone singular point O of L_{ℓ,B,g_2,g_3} to one of $\{0,1,\infty\}$, since the exponent difference at O is $\pm(2\ell+1)$, and an analogue of Lemma 2.4 holds. Also, note that the degree formula (3.1) of Baldassarri and Dwork yields $\pm\ell = (\deg\xi)/3$ when applied to $F = L_{\ell,B,g_2,g_3}$ and $F' = L_{1/2,1/3,1/3}$, and $\pm\ell = (\deg\xi)/6$ when applied to $F = L_{\ell,B,g_2,g_3}$ and $F' = L_{1/2,1/3,1/4}$, since E_{g_2,g_3}, being elliptic, has genus $g = 1$.

If $G(L_{\ell,B,g_2,g_3})$ is tetrahedral, L_{ℓ,B,g_2,g_3} must be a pullback of $L_{1/2,1/3,1/3}$. By Lemma 2.3, if $\xi(O)=0$ then $2\ell+1$ is an integer multiple of $1/2$, i.e., $\ell \in \mathbf{Z}\pm 1/4$. The possibilities $\xi(O)=1,\infty$ can be ruled out, since they would imply respectively that $\xi^{-1}(\infty), \xi^{-1}(1)$ consists of ordinary points, each mapped with multiplicity 3. Either would imply $3|\deg\xi$, which with $\pm\ell = (\deg\xi)/3$ would contradict $2\ell \notin \mathbf{Z}$.

If $G(L_{\ell,B,g_2,g_3})$ is octahedral, L_{ℓ,B,g_2,g_3} must be a pullback of $L_{1/2,1/3,1/4}$. By Lemma 2.3, if $\xi(O)=1$ then $2\ell+1$ is an integer multiple of $1/3$, i.e., $\ell \in \mathbf{Z}\pm 1/6$. The possibilities $\xi(O)=0,\infty$ can be ruled out. If $\xi(O)=0$ then $\xi^{-1}(1), \xi^{-1}(\infty)$ consist of ordinary points, each mapped with multiplicity 3,4, respectively. This would imply $3|\deg\xi$ and $4|\deg\xi$, hence $12|\deg\xi$; which with $\pm\ell = (\deg\xi)/6$ would contradict $2\ell \notin \mathbf{Z}$. $\xi(O)=\infty$ is ruled out similarly.

If $G(L_{\ell,B,g_2,g_3})$ is icosahedral, L_{ℓ,B,g_2,g_3} must be a pullback of $L_{1/2,1/3,1/5}$. By Lemma 2.3, if $\xi(O)=1$ then $2\ell+1$ is an integer multiple of $1/3$, i.e., $\ell \in \mathbf{Z}\pm 1/6$, and if $\xi(O)=\infty$ then $2\ell+1$ is an integer multiple of $1/5$, i.e., $\ell \in \mathbf{Z}\pm 1/10$ or $\ell \in \mathbf{Z}\pm 3/10$. The possibility $\xi(O)=0$ can be ruled out, since it would imply that $2\ell+1$ is an integer multiple of $1/2$, i.e., $\ell \in \mathbf{Z}\pm 1/4$. The group $G(L_{\ell,B})$ is finite if G_{ℓ,B,g_2,g_3} is finite, so if $\xi(O)=0$, Theorem 3.1 implies that $G(L_{\ell,B})$ is octahedral. But $G(L_{\ell,B,g_2,g_3})$ must be isomorphic to a subgroup of $G(L_{\ell,B})$.

The classification scheme of the theorem results from combining the just-derived conditions on $G(L_{\ell,B,g_2,g_3})$ with the conditions of Theorem 3.1 on $G(L_{\ell,B})$. That $G(L_{\ell,B,g_2,g_3})$ octahedral implies $G(L_{\ell,B})$ octahedral is due to A_5 not having any normal S_4 subgroup. The realizability of all five alternatives was proved in Section 3 (it follows from Propositions 3.4 and 3.5). □

By Theorem 5.1, $G(L_{\ell,B})$ being octahedral does not uniquely determine the group $G(L_{\ell,B,g_2,g_3})$: it may be either octahedral or tetrahedral. The latter occurs when the extension $\mathbf{C}(x,y,\tilde{\tau})/\mathbf{C}(x,y)$ has lower degree than $\mathbf{C}(x,\tau)/\mathbf{C}(x)$. The two possibilities are exemplified by Cases 1 and 2(a) of Proposition 4.1, respectively, which have ℓ, B, g_2, g_3 equal to $1/6,0,4,0$ and $1/4,0,0,4$. A ratio $\tilde{\tau}$ of solutions of $L_{\ell,B,g_2,g_3}u=0$ is specified by

$$\frac{-(\tilde{\tau}^{12}-33\tilde{\tau}^8-33\tilde{\tau}^4+1)^2}{108\tilde{\tau}^4(\tilde{\tau}^4-1)^4} = \begin{cases} \dfrac{x^2-1}{x^2}, & \ell,B,g_2,g_3 = 1/6,0,4,0, \\ 1-x^3, & \ell,B,g_2,g_3 = 1/4,0,0,4, \end{cases} \tag{5.2}$$

since $\tilde{\tau} = \tau \circ \pi$. In the second case, $y^2 = 4x^3 - g_2 x - g_3 = 4x^3 - 4$, so $1 - x^3 = -y^2/4$, implying that the minimum polynomial of τ over $\mathbf{C}(x)$ is reducible over $\mathbf{C}(x, y)$. In fact, $\tilde{\tau}$ can be chosen to satisfy

$$\frac{\tilde{\tau}^{12} - 33\tilde{\tau}^8 - 33\tilde{\tau}^4 + 1}{(\sqrt{108}/2)\tilde{\tau}^2(\tilde{\tau}^4 - 1)^2} = \pm y, \tag{5.3}$$

where either sign is acceptable. Each sign yields a 12-branched algebraic function $\tilde{\tau}$ on the equianharmonic elliptic curve $E_{0,4}$ (with $J = 0$) that projects to τ, rather than yielding a 24-branched function on $E_{0,4}$. And $\mathbf{C}(x, y, \tilde{\tau})$ is obtained from $\mathbf{C}(x)$ via the tower $\mathbf{C}(x) \subset \mathbf{C}(x, y) \subset \mathbf{C}(x, y, \tilde{\tau})$, where the extensions are algebraic of degrees 2 and 12, respectively. The group $G(L_{1/4,0,0,4})$, which is the Galois group of $\tilde{\tau}$ over $\mathbf{C}(x, y)$, has order 12 and must be tetrahedral, i.e., isomorphic to A_4.

In general, this reduction may not occur. In the first case of (5.2), in which $y^2 = 4x^3 - 4x$, the analogous substitution does not lead to a reduction of the degree. The function $\tilde{\tau}$ on the harmonic elliptic curve $E_{4,0}$ (with $J = 1$) is 24-branched, like τ, the function to which it projects. So the group $G(L_{1/6,0,4,0})$ has order 24 and must be octahedral, i.e., isomorphic to S_4.

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